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A. KHARAZISHVILI'S SOME RESULTS OF ON THE STRUCTURE OF PATHOLOGICAL FUNCTIONS

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Dedicated to Professor Alexander Kharazishvili on the occasion of his 70th birthday

Abstract. A brief survey of A. Kharazishvili's some works devoted to the real-valued functions with strange, pathological and paradoxical structural properties is presented. The presentation is primarily focused on the absolutely nonmeasurable functions, Sierpiński–Zygmund functions, sup-measurable and weakly sup-measurable functions of two real variables, and nonmeasurable functions of two real variables for which there exist both iterated integrals.

Professor Alexander B. Kharazishvili's scientific interests cover different fields of mathematics, primarily, the real analysis, measure theory, point set theory, and the geometry of Euclidean spaces. His works in mathematical analysis deal with the study of properties of various pathological (or paradoxical) real-valued functions of a real variable. In this direction, he has published several monographs in the worldwide known International Publishing Houses (see [33], [39], [43], [44]). In particular, his monograph [44] issued in 2000, 2006, 2017 is entirely devoted to those functions of real analysis that have strange structural properties from the viewpoints of continuity, monotonicity, differentiability and integrability. Although such functions have a very bad descriptive structure, quite often they turn out to be helpful for solving delicate questions and problems of mathematical analysis (see, e.g., [12], [15], [44]).

The present article may be considered as a short survey of certain Kharazishvili's results involving the above-mentioned topics.

1. It is well known that in the real analysis and, especially, in the differentiation theory, an important role is played by the notion of a differentiation system consisting of certain types of Lebesgue measurable sets. According to the classical Lebesgue theorem, if a system $\mathcal{L}$ of bounded Lebesgue measurable sets in the n-dimensional Euclidean space $\mathbf{R}^n$ is regular, then it is a differentiation system for the standard Lebesgue measure λ_n on the same space $\mathbf{R}^n$. The latter sentence means that for all absolutely continuous real-valued set functions with respect to λ_n, it becomes possible to reconstruct the Radon–Nikodym derivatives of such functions by using the standard differentiation process with respect to $\mathcal{L}$ (see, e.g., [10, 11], [46], [52]). However, this fundamental and useful result does not hold longer for some measures on $\mathbf{R}^n$ which properly extends λ_n. Indeed, for a certain extension μ of λ_n, Kharazishvili constructed a regular system $\mathcal{S}$ of bounded μ-measurable sets such that $\mathcal{S}$ is not a differentiation system for μ, i.e., there is a real-valued set function, absolutely continuous with respect to μ, for which the Radon-Nikodym derivative cannot be obtained by using the ordinary differentiation process with respect to $\mathcal{S}$. This result and some related ones were published in [24–26]).

2. In [26], [28], [29], [30] and [44], Kharazishvili considered logical aspects of the concept of generalized limits on the real line and also the concepts of generalized derivatives and generalized integrals. He indicated close connections of these concepts with the $\mathbf{ZF}+\mathbf{DC}$ set theory (where $\mathbf{DC}$ stands for the axiom of dependent choices) and established that:

(a) it is consistent with the $\mathbf{ZF}+\mathbf{DC}$ theory that these concepts cover only the first category subspaces of appropriate spaces;

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(b) it is consistent with the $\mathbf{ZF} + \mathbf{DC}$ theory that these concepts are always extendable to a wider subspaces.

The main technical tool for establishing the above results is a clever application of the Kuratowski–Ulam theorem to generalized limits, generalized derivatives and generalized integrals. It was also shown in the same works that the Banach–Steinhaus theorem (or, in another terminology, the principle of condensation of singularities) may be deduced from some special version of the Kuratowski–Ulam theorem. In this context, it should be mentioned that the most interesting situations occur when the generalized limits, derivatives or integrals are described by projective subsets of the appropriate Polish topological vector spaces (cf. [44, Chapter 22]). Kharazishvili's results on this topic were used and cited in [7], [16], [22], [48].

3. A function $f(x, y)$ of two real variables x and y is called sup-measurable if for every Lebesgue measurable function $\phi(x)$ of one real variable, the superposition $f(x, \phi(x))$ is also Lebesgue measurable. Using Luzin's classical C-property, it is not hard to show that in the above definition it suffices to require the Lebesgue measurability of $f(x, \phi(x))$ for only all continuous functions $\phi(x)$. Also, in the literature, there are some analogous versions of the sup-measurability of functions of two variables (cf. [4,5], [14], [21], [27], [44]); one of such versions is formulated in terms of functions having the Baire property.

Motivated by the theory of first-order ordinary differential equations, Kharazishvili introduced the notion of a weakly sup-measurable function of two variables. The definition of weakly sup-measurable functions $f(x, y)$ differs slightly from the definition of sup-measurable functions: it is required that the Lebesgue measurability of superpositions $f(x, \phi(x))$ should be valid for all those continuous functions $\phi(x)$, which are differentiable almost everywhere with respect to the Lebesgue measure $\lambda = \lambda_1$ on $\mathbf{R}$. Assuming some additional set-theoretic hypotheses, e.g., the Continuum Hypothesis (**CH**) or Martins Axiom (**MA**), and starting with the delicate properties of Jarniks continuous nowhere approximately differentiable function [23], Kharazishvili proved that there exist weakly sup-measurable functions, which are not sup-measurable. This result was published in his paper [31]. Also, it was shown in the same paper that there exists a first order ordinary differential equation

$$y' = f(x, y) \qquad ((x, y) \in \mathbf{R}^2),$$

whose right-hand side $f(x, y)$ is a weakly sup-measurable non-Lebesgue measurable function of two real variables and, for any initial condition $(x_0, y_0) \in \mathbf{R}^2$, this equation has a unique solution in the class of all locally absolutely continuous functions on $\mathbf{R}$. It should be especially emphasized that the above-mentioned result is a theorem of the **ZFC** set theory, i.e., it does not appeal to additional set-theoretical assumptions. In other words, there is a first-order ordinary differential equation $y' = f(x, y)$ in which the right-hand side $f(x, y)$ is very bad from the measurability viewpoint but, nevertheless, $f(x, y)$ turns out to be weakly sup-measurable and the corresponding Cauchy problem has a unique solution for any initial condition $(x_0, y_0) \in \mathbf{R}^2$.

In connection with the above results, there was formulated in [27] the question whether is it consistent with the **ZFC** theory that any sup-measurable function of two real variables is Lebesgue measurable. Roslanowski and Shelah constructed a model of **ZFC** in which the answer to this question is positive (see their joint article [52]).

In general, the topic connected with the sup-measurable and weakly sup-measurable functions turned out to be of interest for specialists in the real analysis. For this context, we refer the reader especially to the very recent paper: L. Bernal–Gonzalez, G. A. Munoz–Fernandez, D. L. Rodriguez–Vidanes, J. B. Seoane–Sepulveda, Algebraic genericity within the class of sup-measurable functions, Journal of Mathematical Analysis and Applications, v. 483, 2020.

Further, Kharazishvili investigated certain profound properties of a general superposition operator, he studied particularly generalized step functions with strange descriptive properties from the viewpoint of superposition operators (see [34], [36], [44]). The results obtained by Kharazishvili in this direction were cited in [2], [3], [4], [5], [8], [14], [52].

According to one old result of Sierpiński [55], there exists a real-valued Lebesgue measurable function g on $\mathbf{R}$ such that no Borel function on $\mathbf{R}$ majorizes g.

In [43], one can find another radically different proof of this statement and its further generalization. In order to formulate the generalized result, let us recall two notions.

A function from **R** into **R** is called a step-function if its range is at most countable.

A function from **R** into **R** is called universally measurable if it is measurable with respect to the completion of any σ-finite Borel measure on **R**.

Kharazishvili has proved in [43] that due to Martin's Axiom, there exists a universally measurable step-function $h : \mathbf{R} \to \mathbf{R}$ such that there is no Borel function ϕ above g and, simultaneously, there is no Borel function ψ below h.

Clearly, this statement is a strengthened form of Sierpiński's above-mentioned result. It should be noticed that for obtaining a stronger version of Sierpiński's result in terms of universally measurable functions, the usage of additional set-theoretical assumptions becomes necessary.

4. A series of scientific publications of Kharazishvili is devoted to the concept of absolute nonmeasurability of real-valued functions. In particular, this concept was introduced and thoroughly examined in his works [35], [39], [43], [44]. It makes sense to give a precise definition of this important concept.

Let E be an uncountable base set and $\mathcal{M}$ be a class of measures on E (in general, the measures from $\mathcal{M}$ are defined on different σ-algebras of subsets of E, but the case is not excluded when all members of $\mathcal{M}$ have the same domain).

A function $f : E \to \mathbf{R}$ is called absolutely nonmeasurable with respect to $\mathcal{M}$ if f turns out to be nonmeasurable with respect to every measure from $\mathcal{M}$.

The symbol $\mathcal{M}(E)$ denotes the class of all those measures μ on E which are nonzero, σ-finite and diffused (i.e., $\mu(\{x\}) = 0$ for each element $x \in E$).

Of course, the most interesting case from the viewpoint of real analysis is when $E = \mathbf{R}$. To illustrate the absolute nonmeasurability of functions, it seems reasonable to give a few examples about this concept.

Example 1. Let $\mathcal{M}$ be the class of all translation invariant measures on **R** which extend the Lebesgue measure λ and let V be a Vitali set in **R** (in other words, V is a selector of the quotient set $\mathbf{R}/\mathbf{Q}$, where **Q** denotes the field of all rational numbers). Let f be the characteristic function of V. It is well known that f is absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 2. Let $\mathcal{M}$ be the class of the completions of all nonzero σ-finite diffused Borel measures on **R** and let B be a Bernstein set in **R** (by the definition, B and $\mathbf{R} \setminus B$ contain no nonempty perfect sets). Let f denote the characteristic function of B. Then f turns out to be absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 3. According to Martin's Axiom, there exists an additive function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$. Actually, it was proved by Kharazishvili that the existence of such f follows from the existence of a generalized Luzin's set which is simultaneously a vector space over **Q** (see [35], [39], [43], [44]).

Kharazishvili obtained a characterization of absolutely nonmeasurable functions with respect to the class $\mathcal{M}(E)$ in terms of universal measure zero spaces.

Recall that a topological space T is a universal measure zero space (or an absolute null space) if there exists no nonzero σ-finite diffused Borel measure on T.

It turns out that for a function $f : E \to \mathbf{R}$, the following assertions are equivalent:

(1) f is absolutely nonmeasurable with respect to $\mathcal{M}(E)$;

(2) for each point $r \in \mathbf{R}$, the set $f^{-1}(r)$ is at most countable and the range of f is a universal measure zero subspace of **R**.

The proof of this equivalence can be found in [45]. It follows from the above characterization that the existence of absolutely nonmeasurable functions with respect to the class $\mathcal{M}(\mathbf{R})$ cannot be established within the **ZFC** set theory.

The equivalence between the assertions (1) and (2) has been applied many times by its author in the process of his studies of different types of pathological real-valued functions.

For instance, owing to the Continuum Hypothesis (**CH**), Kharazishvili has proved that there exists a large group of additive absolutely nonmeasurable functions acting from $\mathbf{R}$ into $\mathbf{R}$. More precisely, he established that assuming **CH**, there is a group $G \subset \mathbf{R}^{\mathbf{R}}$ such that:

(a) $\mathrm{card}(G) > \mathbf{c}$, where $\mathbf{c}$ denotes the cardinality of the continuum;

(b) all functions from G are additive;

(c) all functions from $G \setminus \{0\}$ are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

This result obtained by Kharazishvili can be found in [42]. By Martin's Axiom, he also proved that:

(d) every function from $\mathbf{R}$ into $\mathbf{R}$ is representable as a sum of two injective functions which are absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$;

(e) every additive function from $\mathbf{R}$ into $\mathbf{R}$ is representable as a sum of two injective additive functions which are absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$.

Furthermore, the concept of absolutely nonmeasurable functions was a starting point for obtaining a solution of one problem posed by Pelc and Prikry [49]. The method of Kharazishvili used by him for obtaining statement (e) is insomuch efficient that leads to a positive solution of the above-mentioned problem (see also his paper [32] considering some related questions). Recently, Zakrzewski [60] has introduced and studied the analogue of an absolute nonmeasurability in terms of the Baire property.

5. A cycle of Kharazishvili's publications deal with the Sierpiński–Zygmund functions. As was shown by Blumberg [9], for any function $f : \mathbf{R} \to \mathbf{R}$, there exists a dense subset X of $\mathbf{R}$ such that the restriction $f|X$ is continuous. In particular, the set X is countably infinite. On the other hand, Sierpiński and Zygmund have established in their celebrated paper [58] that there exists a function

$$f_{SZ} : \mathbf{R} \to \mathbf{R}$$

such that the restriction of f_{SZ} to any set $Y \subset \mathbf{R}$ of cardinality continuum is not continuous on Y. Consequently, if one assumes **CH**, then the restriction of f_{SZ} to any uncountable subset Y of $\mathbf{R}$ is not continuous on Y. So, under **CH**, the Sierpiński–Zygmund functions may be treated as totally discontinuous (i.e., discontinuous on all uncountable subsets of $\mathbf{R}$). On the other hand, as demonstrated by Shelah [54], there are models of the **ZFC** theory in which every function from $\mathbf{R}$ into $\mathbf{R}$ has a continuous restriction to some subset of $\mathbf{R}$ of the second category (which trivially is uncountable). A similar result was obtained in [52] for subsets of $\mathbf{R}$ having strictly positive outer Lebesgue measure. This circumstance shows that the existence of totally discontinuous functions from $\mathbf{R}$ into $\mathbf{R}$ cannot be established within the **ZFC** theory.

There are many works devoted to various extraordinary properties of Sierpiński–Zygmund functions (see, e. g., [6], [20], [47], [51]). An extensive survey of such a function is given in [13] with more or less complete list of references.

Naturally, Kharazishvili studied interrelations between Sierpiński–Zygmund type functions and absolutely nonmeasurable functions with respect to certain classes of measures on $\mathbf{R}$. It is known that every Sierpiński–Zygmund function is nonmeasurable with respect to the completion of any nonzero σ-finite diffused Borel measure on $\mathbf{R}$, i.e., every Sierpiński–Zygmund function is absolutely nonmeasurable with respect to the class of such measures and, consequently, every Sierpi'nski–Zygmund function is nonmeasurable in the Lebesgue sense. On the other hand, it was proved by Kharazishvili that there exists a translation invariant extension μ of the Lebesgue measure on $\mathbf{R}$ such that any Sierpiński–Zygmund function becomes measurable with respect to μ. In this connection, see his paper [38]. At the same time, he was able to establish that there exists an additive Sierpiński–Zygmund function, absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation invariant measures on $\mathbf{R}$ (see [37]).

It is not difficult to show that according to **CH**, every Sierpiński–Zygmund function is totally non-monotone, i.e., the restriction of such a function to any uncountable subset of $\mathbf{R}$ is not monotone. There arises the natural question whether any totally non-monotone function is a Sierpiński–Zygmund function. It turns out that the answer is no in certain models of the **ZFC** set theory. Namely, assuming the same **CH** and using some properties of the so-called Luzin's sets on $\mathbf{R}$ with the existence of continuous nowhere differentiable functions, it was established by Kharazishvili that there exists a totally non-monotone function from $\mathbf{R}$ into itself, which is not a Sierpiński–Zygmund function. This principal result shows a substantial difference between the Sierpiński–Zygmund functions and totally non-monotone functions. The result was first presented by the author at the Section of Functional Analysis of Ukrainian Mathematical Congress, (Kyiv, August 27-29, 2009). The title of his report was: "On continuous totally non-monotone functions". Later on, the same result in a more detailed form has been published in [41]. Also, Kharazishvili considered a stronger version of Sierpiński–Zygmund functions. A function $f : \mathbf{R} \to \mathbf{R}$ is called a Sierpiński–Zygmund function in the strong sense if for any set $X \subset \mathbf{R}$ with cardinality continuum, the restriction $f|X$ is not a Borel function on X. Kharazishvili proved that owing to **CH**, there exists a Sierpiński–Zygmund function which is not a Sierpiński–Zygmund function in the strong sense. When constructing such a function, he essentially used the result of Adian and Novikov [1] on the existence of real-valued semicontinuous functions on $\mathbf{R}$ which are not countably continuous. In addition, he observed that supposing Martin's Axiom, it can also be proved that the class of Sierpiński–Zygmund functions differs from the class of Sierpiński–Zygmund functions in the strong sense.

Finally, it should be especially mentioned that Kharazishvili considered a natural class of topologies on the real line $\mathbf{R}$ (more generally, on a set E of cardinality continuum) and proved the existence of a common Sierpiński–Zygmund function for this class (in this context, see, e.g., [39]).

6. If $f : [0,1]^2 \to \mathbf{R}$ is a function of two real variables, then even in the case of very bad descriptive properties of f it may happen that there exist two iterated integrals

$$\int_0^1 \Big(\int_0^1 f(x,y)d \Big) dy, \qquad \int_0^1 \Big(\int_0^1 f(x,y)dy \Big) dx$$

in the Riemann or in the Lebesgue sense. In particular, according to one old result of Sierpiński [57], there exists an injective function $\phi : [0,1] \to [0,1]$ such that the graph of ϕ is λ-massive in the unit square $[0,1]^2$, i.e., this graph meets every λ_2-measurable subset of $[0,1]^2$ with strictly positive λ_2-measure. Obviously, denoting by g the characteristic function of the graph of ϕ, one easily obtains the equalities

$$\int_0^1 \Big(\int_0^1 g(x,y)dx \Big) dy = \int_0^1 \Big(\int_0^1 g(x,y)dy \Big) dx = 0,$$

although g is not a λ_2-measurable function.

If the starting function f is integrable on $[0,1]^2$ in the Lebesgue sense, then no problems occur, because both iterated integrals for f do exist and they are equal to the two-dimensional Lebesgue integral of f.

If a Lebesgue measurable function $f : [0,1]^2 \to \mathbf{R}$ is not assumed to be Lebesgue integrable, then all of the following possibilities are realizable:

(a) none of the iterated integrals for f exists;
(b) one and only one of the iterated integrals does exist;
(c) both iterated integrals exist, but differ from each other;
(d) both iterated integrals exist and are equal to each other.

Moreover, in connection with case (d), G. Fichtenholz [17] constructed an example of a Lebesgue measurable non-integrable function $h : [0,1]^2 \to \mathbf{R}$ such that the equality

$$\int_a^b \Big(\int_c^d h(x,y)dy \Big) dx = \int_c^d \Big(\int_a^b h(x,y)dx \Big) dy$$

holds true for all rectangles $[a,b] \times [c,d] \subset [0,1]^2$.

In this context, it should be remarked that if $f : [0,1]^2 \to \mathbf{R}$ is a bounded function (not necessarily Lebesgue measurable) and both its iterated integrals exist in the Riemann sense, then they are equal to each other (see, e.g., [18]). This statement is compatible with Sierpiński's result [57].

For iterated integrals in the Lebesgue sense the situation is radically different. Sierpiński's famous theorem [56] states that by the Continuum Hypothesis there is a subset S of $[0,1]^2$ such that each set of the form

$$S \cap (\{x\} \times [0,1]) \qquad (x \in [0,1])$$

is at most countable, and each set of the form

$$S \cap ([0,1] \times \{y\}) \qquad (y \in [0,1])$$

is co-countable in $[0,1] \times \{y\}$. Denoting by f the characteristic function of S, it can easily be seen that both iterated integrals for f do exist, but differ from each other (one of them equals 0, while the other equals 1). In fact, Sierpiński established that the existence of S is equivalent to the Continuum Hypothesis. Later on, it was proved by Friedman [19] that an additional set-theoretical assumption is necessary for having iterated integrals with different values. A detailed survey of Sierpiński's theorem [56] and of its numerous consequences is presented in [59].

In the extensive work of Pkhakadze [50], the equality of the iterated integrals was thoroughly investigated from the viewpoint of structural properties of functions $f : [0,1]^2 \to \mathbf{R}$.

Let $\mathcal{F}$ denote the family of all those functions $f : [0,1]^2 \to \mathbf{R}$ for which both iterated integrals do exist and are equal to each other. It is not hard to check that $\mathcal{F}$ is a vector space over $\mathbf{R}$. Moreover, if one has a pointwise convergent sequence $\{f_n : n \in \mathbf{N}\}$ of functions from $\mathcal{F}$ and $|f_n| \leq \phi$ for some fixed nonnegative Lebesgue integrable function $\phi : [0,1]^2 \to \mathbf{R}$ and for all $n \in \mathbf{N}$, then the limit function

$$f = \lim_{n \to \infty} f_n$$

also belongs to $\mathcal{F}$. Pkhakadze has proved that if a set $Z \subset [0,1]^2$ is such that all its vertical and horizontal sections are closed, then the iterated integrals for the characteristic function of Z exist and are equal to each other, i.e., this function belongs to $\mathcal{F}$. This statement is again in coherence with Sierpiński's result [57]. In addition, assuming **CH**, Pkhakadze gave an example of a set $P \subset [0,1]^2$ whose all vertical sections are closed, all horizontal sections are of type F_σ and for which the iterated integrals do exist, but differ from each other. Consequently, the characteristic function of P does not belong to $\mathcal{F}$.

In [40], it was demonstrated that:

(i) by **CH**, there exists a non-negative bounded function $h \in \mathcal{F}$ such that the function h^2 does not belong to $\mathcal{F}$ (in other words, the vector space $\mathcal{F}$ is not an algebra);

(ii) by **CH**, there are two bounded functions $h_1 \in \mathcal{F}$ and $h_2 \in \mathcal{F}$ such that the function $\sup(h_1, h_2)$ does not belong to $\mathcal{F}$ (in other words, the vector space $\mathcal{F}$ is not a lattice).

In fact, Kharazishvili proved in [40] that according to **CH**, there are two sets $A \subset [0,1]^2$ and $B \subset [0,1]^2$ such that both characteristic functions of A and B belong to $\mathcal{F}$, but the characteristic function of $A \cup B$ does not belong to $\mathcal{F}$.

Remark. The authors began to write the present article ten years ago, intending to dedicate it to professor Alexander Kharazishvili on the occasion of his 60th birthday. For some reasons, the process of preparing the article was not finished in due time. Here we present a modified and expanded version of our previous unpublished survey of those results of A. Kharazishvili that are concerned with the structure of various types of pathological real-valued functions.

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