ISSN 0012-2661, Differential Equations, 2011, Vol. 47, No. 10, pp. 1414–1425. © Pleiades Publishing, Ltd., 2011.
Original Russian Text © I.T. Kiguradze, T.I. Kiguradze, 2011, published in Differentsial'nye Uravneniya, 2011, Vol. 47, No. 10, pp. 1400–1411.

ORDINARY DIFFERENTIAL EQUATIONS

Conditions for the Well-Posedness of Nonlocal Problems for Second-Order Linear Differential Equations

I. T. Kiguradze and T. I. Kiguradze

Razmadze Mathematical Institute, Tbilisi State University, Tbilisi, Georgia
Florida University of Technology, Melbourne, Florida, USA

Received April 7, 2011

Abstract—For second-order linear differential equations, we obtain sharp sufficient conditions for the well-posedness of nonlocal problems with functional and multipoint boundary conditions.

DOI: 10.1134/S0012266111100041

1. STATEMENT OF THE MAIN RESULTS

Let $-\infty < a < t_0 < b < +\infty$, let $r : [a,b] \to (0,+\infty)$ be a continuous function, and let p, $q : [a,b] \to \mathbb{R}$ be Lebesgue integrable functions. On the interval $[a,b]$, consider the differential equation

$$(r(t)u')' = p(t)u + q(t) \tag{1.1}$$

with nonlocal conditions of one of the following three types:

$$u(a) = l_1(u) + c_1, \qquad u(b) = l_2(u) + c_2, \tag{1.2}$$

$$u(a) = l_1(u) + c_1, \qquad r(b)u'(b) = l_2(ru') + c_2, \tag{1.3}$$

$$r(a)u'(a) = l_1(ru') + c_1, \qquad r(b)u'(b) = l_2(ru') + c_2, \tag{1.4}$$

where $l_1 : C([a,t_0]) \to \mathbb{R}$ and $l_2 : C([t_0,b]) \to \mathbb{R}$ are linear bounded functionals and $c_i \in \mathbb{R}$ $(i = 1,2)$. The multipoint boundary conditions

$$u(a) = \sum_{k=1}^{m} l_{1k}u(a_k) + c_1, \qquad u(b) = \sum_{k=1}^{m} l_{2k}u(b_k) + c_2, \tag{1.2$'$}$$

$$u(a) = \sum_{k=1}^{m} l_{1k}u(a_k) + c_1, \qquad r(b)u'(b) = \sum_{k=1}^{m} l_{2k}r(b_k)u'(b_k) + c_2, \tag{1.3$'$}$$

$$r(a)u'(a) = \sum_{k=1}^{m} l_{1k}r(a_k)u'(a_k) + c_1, \qquad r(b)u'(b) = \sum_{k=1}^{m} l_{2k}r(b_k)u'(b_k) + c_2, \tag{1.4$'$}$$

where $m \ge 1$, $l_{ik} \in \mathbb{R}$, and

$$a < a_m < \cdots < a_1 \le b_1 < \cdots < b_m < b, \tag{1.5}$$

are special cases of conditions (1.2)–(1.4).

For $l_i(u) \equiv 0$ $(i = 1,2)$, problems (1.1), (1.2), (1.1), (1.3), and (1.1), (1.4) are sufficiently well studied (see [1–9] and the bibliography therein).

I.T. Kiguradze and Lomtatidze [10] proved theorems of de la Vallée Poussin type containing in a sense sharp criteria for the unique solvability of problem (1.1), (1.2′) for the case in which $m = 1$, $l_{11} = 0$, and $l_{21} = 1$.

Il'in and Moiseev [11, 12] showed that if $l_{1k} = 0$, $l_{21}l_{2k} > 0$ $(k = 1,\dots,m)$, $\sum_{k=1}^{m} l_{2k} \le 1$, and $p(t) \ge 0$ for $a < t < b$ [respectively, $p(t) > 0$ for almost all $t \in (a,b)$], then problem (1.1), (1.2′) [respectively, problem (1.1), (1.3′)] has a unique solution.

Dovletov [13] proved the existence of a unique solution of problem (1.1), (1.2′) under the assumption that $p(t) \ge 0$ for $a < t < b$, $m = 2$, $l_{11} = l_{12} = 0$, and the constants l_{21} and l_{22} satisfy either the inequalities $l_{21} \ge 0$, $l_{22} \le 0$, and $l_{21} + l_{22} \le 1$ or the inequalities $l_{21} \le 0$ and $l_{22} \le 1$.

For $l_1(u) \equiv 0$, optimal sufficient conditions for the unique solvability of problems (1.1), (1.2) and (1.1), (1.3) were obtained by Lomtatidze [14, 15] and T.I. Kiguradze [16, 17]. Integral criteria for the unique solvability of such problems can be found in [18]. Nevertheless, neither of problems (1.1), (1.k) $(k = 2, 3, 4)$ has been completely studied for $l_i(u) \not\equiv 0$ $(i = 1, 2)$. The present paper is intended to bridge the gap. We obtain sharp conditions guaranteeing the well-posedness of problems (1.1), (1.k) and (1.1), (1.k'). These conditions are new even for $l_1(u) \equiv 0$ and $l_{1k} = 0$ $(k = 1,\dots,m)$.

Throughout the paper, we use the following notation and definitions.

If x and x_k $(k = 1, 2, \dots)$ are real numbers, then

$$[x]_+ = \frac{|x| + x}{2}, \qquad [x]_- = \frac{|x| - x}{2};$$
$$S_1(x_1) = [x_1]_+, \qquad S_k(x_1,\dots,x_k) = [x_k + S_{k-1}(x_1,\dots,x_{k-1})]_+ \qquad (k = 2, 3, \dots); \tag{1.6}$$

$C([t_1,t_2])$ and $C^1([t_1,t_2])$ are Banach spaces of continuous continuously differentiable functions $u : [t_1,t_2] \to \mathbb{R}$ with norms $\|u\|_C = \max\{|u(t)| : t_1 \le t \le t_2\}$ and $\|u\|_{C^1} = \|u\|_C + \|u'\|_C$; $\Lambda(t_1,t_2)$ is the space of linear bounded functionals $l : C([t_1,t_2]) \to \mathbb{R}$; $\Lambda^-(t_1,t_2)$ is the set of all $l \in \Lambda(t_1,t_2)$ such that $l(u) \le 0$ for an arbitrary nonnegative function $u \in C([t_1,t_2])$; $\Lambda^+_\tau(t_1,t_2)$, $\tau \in [t_1,t_2]$, is the set of all $l \in \Lambda(t_1,t_2)$ such that $l(u) > 0$ for an arbitrary function $u \in C([t_1,t_2])$ satisfying the inequality $u(t) > 0$ for $t \ne \tau$; $\Lambda^1_{t_1}(t_1,t_2)$ is the set of all $l \in \Lambda(t_1,t_2)$ such that $l(u) < u(t_1)$ for an arbitrary nonnegative decreasing function $u \in C([t_1,t_2])$; and $\Lambda^1_{t_2}(t_1,t_2)$ is the set of all $l \in \Lambda(t_1,t_2)$ such that $l(u) < u(t_2)$ for an arbitrary nonnegative increasing function $u \in C([t_1,t_2])$.

A function $u \in C^1([a,b])$ is called a *solution* of Eq. (1.1) if ru' is absolutely continuous and the relation $(r(t)u'(t))' = p(t)u(t) + q(t)$ holds almost everywhere on $[a,b]$.

A solution u of Eq. (1.1) satisfying the boundary conditions (1.k), $k \in \{2,3,4\}$, is called a solution of problem (1.1), (1.k).

Problem (1.1), (1.k) is said to be *well-posed* if it is uniquely solvable for arbitrarily fixed $c_i \in \mathbb{R}$ $(i = 1, 2)$ and Lebesgue integrable function $q : [a,b] \to \mathbb{R}$ and if there exists a positive constant ϱ independent of the c_i $(i = 1, 2)$ and q such that the solution admits the estimate

$$\|u\|_{C^1} \le \varrho(|c_1| + |c_2| + \|\widetilde{q}\|_C),$$

where $\widetilde{q}(t) = \int_a^t q(s)\,ds$.

Along with the inhomogeneous equations (1.1), consider the corresponding homogeneous equation

$$(r(t)u')' = p(t)u \tag{1.1$_0$}$$

with the homogeneous boundary conditions

$$u(a) = l_1(u), \qquad u(b) = l_2(u), \tag{1.2$_0$}$$
$$u(a) = l_1(u), \qquad r(b)u'(b) = l_2(ru'), \tag{1.3$_0$}$$
$$r(a)u'(a) = l_1(ru'), \qquad r(b)u'(b) = l_2(ru'). \tag{1.4$_0$}$$

Theorems 1.1 and 1.2 in [19] imply the following assertion.

Proposition 1.1. *Let $k \in \{2,3,4\}$. Problem* (1.1), (1.k) *is well posed if and only if the corresponding homogeneous problem* (1.1), (1.k_0) *has only the trivial solution.*

We study problems (1.1), (1.k) ($k = 2,3,4$) for the cases in which the two functionals l_1 and l_2 satisfy one of the following conditions:

$$l_1 \in \Lambda^-(a,t_0), \qquad l_2 \in \Lambda^-(t_0,b), \tag{1.7}$$
$$l_1 \in \Lambda^-(a,t_0), \qquad l_2 \in \Lambda_b^1(t_0,b), \tag{1.8}$$
$$l_1 \in \Lambda_a^1(a,t_0), \qquad l_2 \in \Lambda^-(t_0,b), \tag{1.9}$$
$$l_1 \in \Lambda_a^+(a,t_0), \qquad l_2 \in \Lambda_b^+(t_0,b), \qquad l_1(1) = l_2(1) = 1, \tag{1.10}$$
$$l_1 \in \Lambda_a^+(a,t_0), \qquad l_1(1) = 1, \qquad l_2 \in \Lambda^-(t_0,b). \tag{1.11}$$

For problems (1.1), (1.k') ($k = 2,3,4$), the conditions corresponding to (1.7)–(1.11) have the form

$$l_{1k} \le 0, \qquad l_{2k} \le 0 \qquad (k = 1,\dots,m), \tag{1.7$'$}$$
$$l_{1k} \le 0 \qquad (k = 1,\dots,m), \qquad S_m(l_{21},\dots,l_{2m}) \le 1, \tag{1.8$'$}$$
$$S_m(l_{11},\dots,l_{1m}) \le 1, \qquad l_{2k} \le 0 \qquad (k = 1,\dots,m), \tag{1.9$'$}$$
$$l_{1k} \ge 0, \qquad l_{2k} \ge 0 \qquad (k = 1,\dots,m); \qquad \sum_{i=1}^m l_{1i} = \sum_{i=1}^m l_{2i} = 1, \tag{1.10$'$}$$
$$l_{1k} \ge 0, \qquad l_{2k} \le 0 \qquad (k = 1,\dots,m); \qquad \sum_{i=1}^m l_{1i} = 1. \tag{1.11$'$}$$

In the theorems and corollaries given below, the function p is subjected to one of the following conditions:

$$\left(\frac{\pi^2}{\delta^3(b)} \int_a^b r^{\lambda-1}(t)\delta(t)(\delta(b)-\delta(t))[p(t)]_-^\lambda \, dt\right)^{1/\lambda} \le \frac{\pi^2}{\delta^2(b)}, \tag{1.12}$$

$$\left(\frac{\pi^2}{4\delta^2(b)} \int_a^b r^{\lambda-1}(t)\delta(t)[p(t)]_-^\lambda \, dt\right)^{1/\lambda} \le \frac{\pi^2}{4\delta^2(b)}, \tag{1.13}$$

$$\left(\frac{\pi^2}{4\delta^2(b)} \int_a^b r^{\lambda-1}(t)(\delta(b)-\delta(t))[p(t)]_-^\lambda \, dt\right)^{1/\lambda} \le \frac{\pi^2}{4\delta^2(b)}, \tag{1.14}$$

$$p(t) < 0 \quad \text{for almost all} \quad t \in (a,b), \qquad \left(\frac{\pi^2}{4\delta(b)} \int_a^b r^{\lambda-1}(t)|p(t)|^\lambda \, dt\right)^{1/\lambda} \le \frac{\pi^2}{\delta^2(b)}, \tag{1.15}$$

$$p(t) > 0 \quad \text{for almost all} \quad t \in (a,b), \tag{1.16}$$

where $\lambda \ge 1$ and

$$\delta(t) = \int_a^t \frac{ds}{r(s)}. \tag{1.17}$$

Theorem 1.1. *Problem* (1.1), (1.2) *is well posed if either conditions* (1.7) *and* (1.12), *or conditions* (1.8) *and* (1.13) *or conditions* (1.9) *and* (1.14), *or condition* (1.10) *and one of conditions* (1.15) *and* (1.16) *are satisfied.*

Corollary 1.1. *Problem* (1.1), (1.2′) *is well posed if either conditions* (1.7′) *and* (1.12), *or conditions* (1.8′) *and* (1.13), *or conditions* (1.9′) *and* (1.14), *or condition* (1.10′) *and one of conditions* (1.15) *and* (1.16) *are satisfied.*

Theorem 1.2. *Problem* (1.1), (1.3) *is well posed if either conditions* (1.7) *and* (1.13), *or conditions* (1.8) *and* (1.16), *or condition* (1.11) *and one of conditions* (1.15) *and* (1.16) *are satisfied.*

Corollary 1.2. *Problem* (1.1), (1.3′) *is well posed if either conditions* (1.7′) *and* (1.13), *or conditions* (1.8′) *and* (1.16), *or condition* (1.11′) *and one of conditions* (1.15) *and* (1.16) *are satisfied.*

By Corollary 1.1 (respectively, Corollary 1.2), if $l_{1k} = 0$ $(k = 1, \dots, m)$, $S_m(l_{21}, \dots, l_{2m}) \le 1$, and condition (1.13) [respectively, (1.16)] is satisfied, then problem (1.1), (1.2′) [respectively, problem (1.1), (1.3′)] is well posed. This generalizes the above-mentioned results obtained by Il'in and Moiseev [11, 12] and Dovletov [13]. For the case in which condition (1.10′) [respectively, condition (1.11′)] is satisfied, problem (1.1), (1.2′) [respectively, problem (1.1), (1.3′)] has not been studied yet.

Theorem 1.3. *Problem* (1.1), (1.4) *is well posed if either condition* (1.7) *and one of conditions* (1.15) *and* (1.16) *or one of conditions* (1.8) *and* (1.9) *and condition* (1.16) *are satisfied.*

Corollary 1.3. *Problem* (1.1), (1.4′) *is well posed if either condition* (1.7′) *and one of conditions* (1.15) *and* (1.16) *or one of conditions* (1.8′) *and* (1.9′) *and condition* (1.16) *are satisfied.*

Example 1.1. For an arbitrarily fixed $\varepsilon \in (0, 1)$ and for a continuous function $r : [a, b] \to (0, +\infty)$, take numbers $a_0 \in (a, b)$, $b_0 \in (a_0, b)$, and $\lambda \ge 1$ such that

$$\delta(b_0) - \delta(a_0) \ge \left(\frac{\pi}{2}\right)^{1/\lambda} (1 + \varepsilon)^{-1/2} \delta(b). \tag{1.18}$$

Then

$$\delta(a_0) < 2\delta(a_0) < 2\delta(b_0) - \delta(b) < \delta(b_0).$$

This, together with relation (1.17), implies the existence of numbers $a_1 \in (a_0, b_0)$ and $b_1 \in (a_0, b_0)$ such that

$$\delta(a_1) = 2\delta(a_0), \qquad \delta(b_1) = 2\delta(b_0) - \delta(b). \tag{1.19}$$

Set

$$p(t) = -\frac{\gamma}{r(t)} \left(\frac{\pi}{\delta(b_0) - \delta(a_0)}\right)^2, \qquad m = 1 \tag{1.20}$$

and consider problem (1.1), (1.2′), where γ is a positive constant and l_{11} and l_{21} are numbers satisfying one of the conditions

$$l_{11} = -1, \qquad l_{21} = -1, \tag{1.21}$$

$$l_{11} = 1, \qquad l_{21} = 1, \tag{1.22}$$

$$l_{11} = -1, \qquad l_{21} = 1, \tag{1.23}$$

$$l_{11} = 1, \qquad l_{21} = -1. \tag{1.24}$$

If $\gamma = 1$, then, by conditions (1.17), (1.18), and (1.20), we obtain the inequalities

$$\left(\frac{\pi^2}{\delta^3(b)} \int_a^b r^{\lambda-1}(t)\delta(t)(\delta(b) - \delta(t))[p(t)]_-^\lambda \, dt\right)^{1/\lambda} \le (1 + \varepsilon)\frac{\pi^2}{\delta^2(b)}, \tag{1.12$_\varepsilon$}$$

$$p(t) < 0 \quad \text{for almost all} \quad t \in (a, b), \quad \left(\frac{\pi^2}{4\delta(b)} \int_a^b r^{\lambda-1}(t)|p(t)|^\lambda \, dt\right)^{1/\lambda} \le (1 + \varepsilon)\frac{\pi^2}{\delta^2(b)}. \tag{1.15$_\varepsilon$}$$

But if $\gamma = 1/4$, then

$$\left(\frac{\pi^2}{4\delta^2(b)}\int_a^b r^{\lambda-1}(t)\delta(t)[p(t)]_-^\lambda\,dt\right)^{1/\lambda} \le (1+\varepsilon)\frac{\pi^2}{4\delta^2(b)}, \tag{1.13_ε}$$

$$\left(\frac{\pi^2}{4\delta^2(b)}\int_a^b r^{\lambda-1}(t)(\delta(b)-\delta(t))[p(t)]_-^\lambda\,dt\right)^{1/\lambda} \le (1+\varepsilon)\frac{\pi^2}{4\delta^2(b)}. \tag{1.14_ε}$$

On the other hand, if $\gamma = 1$ and condition (1.21) [respectively, condition (1.22)] is satisfied, then, by virtue of relations (1.19) and (1.20), the homogeneous differential equation (1.1_0) with the homogeneous boundary conditions $(1.2'_0)$ has the nontrivial solution

$$u(t) = \sin\left(\frac{\pi(\delta(t)-\delta(a_0))}{\delta(b_0)-\delta(a_0)}\right) \qquad \left(u(t) = \cos\left(\frac{\pi(\delta(t)-\delta(a_0))}{\delta(b_0)-\delta(a_0)}\right)\right).$$

Consequently, problem (1.1), $(1.2')$ is ill posed, and this is caused by the fact that inequality (1.12_ε) [respectively, (1.15_ε)] holds instead of (1.12) [respectively, (1.15)].

But if $\gamma = 1/4$ and condition (1.23) [respectively, (1.24)] is satisfied, then the homogeneous problem (1.1_0), $(1.2'_0)$ has the nontrivial solution

$$u(t) = \sin\left(\frac{\pi(\delta(t)-\delta(a_0))}{2(\delta(b_0)-\delta(a_0))}\right) \qquad \left(u(t) = \cos\left(\frac{\pi(\delta(t)-\delta(a_0))}{2(\delta(b_0)-\delta(a_0))}\right)\right).$$

Consequently, in this case, problem (1.1), $(1.2')$ is ill posed again, and this is caused by the fact that inequality (1.13_ε) [respectively, (1.14_ε)] holds instead of (1.13) [respectively, (1.14)].

The constructed example shows that, for each $k \in \{12, 13, 14, 15\}$, condition $(1.k)$ in Theorem 1.1 and Corollary 1.1 is sharp and cannot be replaced by condition $(1.k_\varepsilon)$, however small $\varepsilon > 0$ is.

If we now consider problem (1.1), $(1.3')$ [respectively, problem (1.1), $(1.4')$] for the case in which condition (1.20) is satisfied and the numbers a_1, b_1, l_{11}, and l_{21} are chosen in the above-described way, then we find that conditions (1.13) and (1.15) in Theorem 1.2 and Corollary 1.2 [respectively, condition (1.15) in Theorem 1.3 and Corollary 1.3] cannot be replaced by conditions (1.13_ε) and (1.15_ε) [respectively, condition (1.15_ε)] for an arbitrarily small $\varepsilon > 0$.

Example 1.2. Let

$$p(t) = -\frac{2}{r(t)(\varrho + \delta(t)(\delta(b)-\delta(t)))}, \tag{1.25}$$

where $\varrho = 1 + \delta^2(b)$. Then condition (1.15) is satisfied. This, together with Corollary 1.1, implies that if the numbers l_{ik} $(i = 1, 2,\ k = 1, \dots, m)$ satisfy condition $(1.10')$, then problem (1.1), $(1.2')$ is well posed. Let us show that condition $(1.10')$ cannot be replaced by the condition

$$l_{1k} > 0, \quad l_{2k} > 0 \quad (k = 1, \dots, m), \quad 1 - \varepsilon < \sum_{j=1}^m l_{ij} < 1 \quad (i = 1, 2), \tag{1.26}$$

however small $\varepsilon > 0$ is. Indeed, for an arbitrarily fixed $\varepsilon \in (0, 1)$, take numbers a_k, b_k, l_{1k}, and l_{2k} $(k = 1, \dots, m)$ so as to satisfy condition (1.5) and the conditions

$$\begin{gathered}\delta(b)\delta(a_1) < \varepsilon, \quad \delta(b)(\delta(b)-\delta(b_1)) \le \varepsilon, \quad l_{1k} > 0, \quad l_{2k} > 0 \quad (k = 1, \dots, m), \\ \sum_{i=1}^m l_{1i}(\varrho + \delta(a_i)(\delta(b)-\delta(a_i))) = \varrho, \qquad \sum_{i=1}^m l_{2i}(\varrho + \delta(b_i)(\delta(b)-\delta(b_i))) = \varrho.\end{gathered} \tag{1.27}$$

Then, on the one hand, the numbers l_{ik}, $(i = 1, 2;\ k = 1, \dots, m)$ satisfy inequalities (1.26), and on the other hand, it follows from relations (1.25) and (1.27) that Eq. (1.1_0) has the nontrivial solution

$$u(t) = \varrho + \delta(t)(\delta(b)-\delta(t))$$

satisfying the homogeneous boundary conditions

$$\sum_{k=1}^{m} l_{1k} u(a_k) = 0, \qquad \sum_{k=1}^{m} l_{2k} u(b_k) = 0.$$

Consequently, problem (1.1), (1.2′) is ill posed.

The constructed example shows that condition (1.10) in Theorem 1.1 and condition (1.10′) in Corollary 1.1 are in a sense sharp.

2. AUXILIARY ASSERTIONS

2.1. Lemmas on the Unique Solvability of Two-Point Boundary Value Problems

Consider the homogeneous differential equation (1.1_0) with two-point conditions of one of the following four forms:

$$u(a_0) = 0, \qquad u(b_0) = 0, \tag{2.1}$$

$$u(a_0) = 0, \qquad u'(b_0) = 0, \tag{2.2}$$

$$u'(a_0) = 0, \qquad u(b_0) = 0, \tag{2.3}$$

$$u'(a_0) = 0, \qquad u'(b_0) = 0, \tag{2.4}$$

where $a \le a_0 < b_0 \le b$. Just as above, we assume that $r : [a, b] \to (0, +\infty)$ is a continuous function, $p : [a, b] \to \mathbb{R}$ is an integrable function, and δ is the function given by (1.17).

Lemma 2.1. *If inequality* (1.12) *holds for some* $\lambda \ge 1$, *then problem* (1.1_0), (2.1) *has only the trivial solution.*

Proof. The transformation

$$x = \delta(t), \qquad w(x) = u(t) \tag{2.5}$$

reduces problem (1.1_0), (2.1) to the problem

$$w'' = p_0(x) w, \tag{2.6}$$

$$w(x_1) = 0, \qquad w(x_2) = 0, \tag{2.7}$$

where

$$p_0(\delta(t)) = r(t)p(t), \qquad x_1 = \delta(a_0), \qquad x_2 = \delta(b_0). \tag{2.8}$$

On the other hand, by Theorem 1.2 in [8], if

$$\int_{x_1}^{x_2} \frac{(x - x_1)(x_2 - x)}{x_2 - x_1} [p_0(x)]_-^{\lambda}\, dx \le \left(\frac{\pi}{x_2 - x_1}\right)^{2\lambda - 2}, \tag{2.9}$$

then problem (2.6), (2.7) and hence problem (1.1_0), (2.1) have only the trivial solution. Therefore, to prove the lemma, it suffices to show that inequality (1.12) implies inequality (2.9). Indeed, if, in addition to (1.12), we take into account relations (1.17) and (2.8), then we obtain the inequalities

$$\frac{(x - x_1)(x_2 - x)}{x_2 - x_1} \le \frac{x(\delta(b) - x)}{\delta(b)} \quad \text{if} \quad x_1 \le x \le x_2,$$

$$\int_{x_1}^{x_2} \frac{(x - x_1)(x_2 - x)}{x_2 - x_1} [p_0(x)]_-^{\lambda}\, dx \le \int_{0}^{\delta(b)} \frac{x(\delta(b) - x)}{\delta(b)} [p_0(x)]_-^{\lambda}\, dx$$

$$= \frac{1}{\delta(b)} \int_a^b r^{\lambda - 1}(t)\delta(t)(\delta(b) - \delta(t))[p(t)]_-^{\lambda}\, dt \le \left(\frac{\pi}{\delta(b)}\right)^{2\lambda - 2} \le \left(\frac{\pi}{x_2 - x_1}\right)^{2\lambda - 2}.$$

The proof of the lemma is complete.

Lemma 2.2. *If the inequality*

$$\int_{a_0}^{b_0} r^{\lambda-1}(t)(\delta(t)-\delta(a_0))[p(t)]_-^{\lambda}\,dt \le \left(\frac{\pi}{2(\delta(b_0)-\delta(a_0))}\right)^{2\lambda-2} \tag{2.10}$$

holds for some $\lambda \ge 1$, *then problem* (1.1_0), (2.2) *has only the trivial solution. If*

$$\int_{a_0}^{b_0} r^{\lambda-1}(t)(\delta(b_0)-\delta(t))[p(t)]_-^{\lambda}\,dt \le \left(\frac{\pi}{2(\delta(b_0)-\delta(a_0))}\right)^{2\lambda-2}, \tag{2.11}$$

then problem (1.1_0), (2.3) *has only the trivial solution.*

Proof. The transformation

$$x = \delta(t)-\delta(a_0), \qquad w(x) = u(t)$$

reduces problem (1.1_0), (2.2) to Eq. (2.6) with the boundary conditions

$$w(0) = 0, \qquad w'(x_0) = 0, \tag{2.12}$$

where

$$p_0(\delta(t)-\delta(a_0)) = r(t)p(t) \quad \text{if} \quad a_0 \le t \le b_0, \quad x_0 = \delta(b_0)-\delta(a_0). \tag{2.13}$$

On the other hand, the transformation

$$x = \delta(b_0)-\delta(t), \qquad w(x) = u(t)$$

reduces problem (1.1_0), (2.3) to problem (2.6), (2.12), where

$$p_0(\delta(b_0)-\delta(t)) = r(t)p(t) \quad \text{if} \quad a_0 \le t \le b_0, \quad x_0 = \delta(b_0)-\delta(a_0). \tag{2.14}$$

It follows from Theorem 1.4 in [8] that if the function p_0 satisfies the inequality

$$\int_0^{x_0} x[p_0(x)]_-^{\lambda}\,dx \le \left(\frac{\pi}{2x_0}\right)^{2\lambda-2}, \tag{2.15}$$

then problem (2.6), (2.12) has only the trivial solution. Consequently, to prove the lemma, it suffices to show that conditions (2.10) and (2.13) [respectively, (2.11) and (2.14)] ensure inequality (2.15). Indeed, if conditions (2.10) and (2.13) are satisfied, then we have

$$\int_0^{x_0} x[p_0(x)]_-^{\lambda}\,dx = \int_{a_0}^{b_0} r^{\lambda-1}(t)(\delta(t)-\delta(a_0))[p(t)]_-^{\lambda}\,dt \le \left(\frac{\pi}{2x_0}\right)^{2\lambda-2}.$$

If conditions (2.11) and (2.14) hold, then

$$\int_0^{x_0} x[p_0(x)]_-^{\lambda}\,dx = \int_{a_0}^{b_0} r^{\lambda-1}(t)(\delta(b_0)-\delta(t))[p(t)]_-^{\lambda}\,dt \le \left(\frac{\pi}{2x_0}\right)^{2\lambda-2}.$$

The proof of the lemma is complete.

Lemma 2.3. *Suppose that there exists a* $\sigma \in \{-1, 1\}$ *such that*

$$\sigma p(t) > 0 \quad \textit{for almost all} \quad t \in (a_0, b_0). \tag{2.16}$$

Then an arbitrary solution u *of problem* (1.1_0), (2.4) *has at least one zero in the interval* (a_0, b_0).

Proof. Assume that the lemma is false. Then there exists a solution u of problem (1.1_0), (2.4) such that

$$u(t) > 0 \quad \text{if} \quad a_0 < t < b_0. \tag{2.17}$$

If we integrate the identity

$$\sigma(r(t)u'(t))' = \sigma p(t)u(t)$$

from a_0 to b_0, then, by condition (2.4), we obtain

$$\sigma \int_{a_0}^{b_0} p(t)u(t)\,dt = 0.$$

But this relation contradicts conditions (2.16) and (2.17). The resulting contradiction proves the lemma.

Lemma 2.4. *If one of conditions* (1.15) *and* (1.16) *is satisfied, then problem* (1.1_0), (2.4) *has only the trivial solution.*

Proof. First, note that condition (1.15) [respectively, condition (1.16)] implies inequalities (2.16) with $\sigma = -1$ (respectively, $\sigma = 1$).

Now assume that the lemma is false; i.e., one of conditions (1.15) and (1.16) is satisfied, but nevertheless, problem (1.1_0), (2.4) has a nontrivial solution u. Then, by Lemma 2.3 and inequality (2.16), there exists a $c \in (a_0, b_0)$ such that

$$u(c) = 0. \tag{2.18}$$

On the other hand, it follows from Lemma 2.2 and relations (2.4) and (2.18) that

$$\int_{a_0}^{c} r^{\lambda-1}(t)(\delta(c) - \delta(t))[p(t)]_-^{\lambda}\,dt > \left(\frac{\pi}{2}\right)^{2\lambda-2} (\delta(c) - \delta(a_0))^{2-2\lambda},$$

$$\int_{c}^{b_0} r^{\lambda-1}(t)(\delta(t) - \delta(c))[p(t)]_-^{\lambda}\,dt > \left(\frac{\pi}{2}\right)^{2\lambda-2} (\delta(b_0) - \delta(c))^{2-2\lambda}.$$

Therefore,

$$\int_{a_0}^{c} r^{\lambda-1}(t)[p(t)]_-^{\lambda}\,dt > \left(\frac{\pi}{2}\right)^{2\lambda-2} (\delta(c) - \delta(a_0))^{1-2\lambda},$$

$$\int_{c}^{b_0} r^{\lambda-1}(t)[p(t)]_-^{\lambda}\,dt > \left(\frac{\pi}{2}\right)^{2\lambda-2} (\delta(b_0) - \delta(c))^{1-2\lambda}.$$

If we add these two inequalities, then we obtain the estimate

$$\int_{a_0}^{b_0} r^{\lambda-1}(t)[p(t)]_-^{\lambda}\,dt > \left(\frac{\pi}{2}\right)^{2\lambda-2} \varrho, \tag{2.19}$$

where

$$\varrho = (\delta(c) - \delta(a_0))^{1-2\lambda} + (\delta(b_0) - \delta(c))^{1-2\lambda}.$$

By the Radon theorem (see [20, Th. 65] or [5, Lemma 2.1]), we have

$$\varrho \geq 2^{2\lambda}(\delta(b_0) - \delta(a_0))^{1-2\lambda} \geq 2^{2\lambda}\delta^{1-2\lambda}(b).$$

Therefore, it follows from (2.19) that

$$\int\limits_a^b r^{\lambda-1}(t)[p(t)]_-^\lambda \, dt \geq \int\limits_{a_0}^{b_0} r^{\lambda-1}(t)[p(t)]_-^\lambda \, dt > \frac{4}{\delta(b)}\left(\frac{\pi}{\delta(b)}\right)^{2\lambda-2}.$$

But this inequality contradicts both condition (1.15) and condition (1.16). The resulting contradiction completes the proof of the lemma.

2.2. Lemmas on Functionals Belonging to the Sets $\Lambda_\tau^+(t_1,t_2)$ and $\Lambda_{t_i}^1(t_1,t_2)$ $(i=1,2)$

Throughout this section, we assume that $-\infty < t_1 < t_2 < +\infty$ and $\tau \in [t_1,t_2]$.

Lemma 2.5. *If*

$$l \in \Lambda_\tau^+(t_1,t_2), \tag{2.20}$$

then for each function $u \in C([t_1,t_2])$, there exists a $\tau_0 \in [t_1,t_2]$ such that $\tau_0 \neq \tau$ and

$$l(u) = u(\tau_0)l(1).$$

Proof. It follows from condition (2.20) that $l(1) > 0$. Set

$$u_0 = l(u)/l(1). \tag{2.21}$$

We should show that $u(\tau_0) = u_0$ for some $\tau_0 \in [t_1,t_2]\backslash\{\tau\}$. Assume the contrary: there does not exist a τ_0 with this property. Then, without loss of generality, one can assume that

$$u(t) > u_0 \quad \text{for} \quad t \in [t_1,t_2]\backslash\{\tau\}.$$

This, together with condition (2.20), implies that $l(u-u_0) > 0$. But this contradicts relation (2.21), and the proof of the lemma is complete.

Lemma 2.6. *Let m be a positive integer, let $x_i \in \mathbb{R}$ $(i=1,\ldots,m)$, and let*

$$0 \leq y_1 < \cdots < y_m. \tag{2.22}$$

Then

$$\sum_{i=1}^m x_i y_i \leq S_m(x_1,\ldots,x_m)y_m. \tag{2.23}$$

Proof. Since y_1 is nonnegative, we have

$$x_1 y_1 \leq [x_1]_+ y_1 = S_1(x_1)y_1.$$

Now assume that the inequality

$$\sum_{i=1}^k x_i y_i \leq S_k(x_1,\ldots,x_k)y_k$$

holds for some $k \in \{1, \dots, m-1\}$. Then, by virtue of conditions (1.6) and (2.22), we obtain the relations

$$\sum_{i=1}^{k+1} x_i y_i = \sum_{i=1}^{k} x_i y_i + x_{k+1} y_{k+1} \le S_k(x_1, \dots, x_k) y_k + x_{k+1} y_{k+1}$$
$$\le (S_k(x_1, \dots, x_k) + x_{k+1}) y_{k+1} \le [S_k(x_1, \dots, x_k) + x_{k+1}]_+ y_{k+1} = S_{k+1}(x_1, \dots, x_{k+1}) y_{k+1}.$$

This, by induction, implies inequality (2.23). The proof of the lemma is complete.

Lemma 2.7. *Let* $l : C([t_1, t_2]) \to \mathbb{R}$ *be the functional given by the relation*

$$l(u) = \sum_{i=1}^{m} x_i u(\tau_i), \tag{2.24}$$

where

$$t_1 < \tau_1 < \dots < \tau_m < t_2 \qquad (t_1 < \tau_m < \dots < \tau_1 < t_2), \tag{2.25}$$
$$S_m(x_1, \dots, x_m) \le 1. \tag{2.26}$$

Then

$$l \in \Lambda^1_{t_2}(t_1, t_2) \qquad (l \in \Lambda^1_{t_1}(t_1, t_2)). \tag{2.27}$$

Proof. Let $u : [t_1, t_2] \to [0, +\infty)$ be an arbitrary continuous increasing (respectively, decreasing) function. Then, by condition (2.25), the numbers $y_i = u(\tau_i)$ $(i = 1, \dots, m)$ satisfy inequalities (2.22). This, together with Lemma 2.6, implies the estimate (2.23). By virtue of this estimate and inequalities (2.25) and (2.26), it follows from the representation (2.24) that $l(u) \le u(\tau_m)$ and

$$l(u) < u(t_2) \qquad [\text{respectively, } l(u) < u(t_1)].$$

Consequently, the functional l satisfies condition (2.27). The proof of the lemma is complete.

3. PROOF OF THE MAIN RESULTS

3.1. Proof of Theorem 1.1

By Proposition 1.1, to prove the theorem, it suffices to show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution.

First, consider the case in which conditions (1.7) and (1.12) are satisfied. By (1.7), for an arbitrary solution u of problem (1.1_0), (1.2_0), there exist $a_0 \in [a, t_0)$ and $b_0 \in (t_0, b]$ such that relations (2.1) hold. However, by Lemma 2.1 and inequality (1.12), problem (1.1_0), (2.1) has only the trivial solution. Consequently, $u(t) \equiv 0$.

Now let us prove that problem (1.1_0), (1.2_0) also has only the trivial solution for the case in which conditions (1.8) and (1.13) are satisfied. Assume the contrary: this problem has a nontrivial solution u. Conditions (1.2_0) and (1.8) ensure the existence of an $a_0 \in [a, t_0)$ such that $u(a_0) = 0$. Without loss of generality, one can assume that $u'(a_0) = 1$. On the other hand, by inequality (1.13), inequality (2.10) holds for each $b_0 \in (a_0, b]$. This, together with Lemma 2.1, implies that

$$u(t) > 0, \quad u'(t) > 0 \quad \text{for} \quad a_0 < t \le b.$$

If, in addition, we use the condition $l_2 \in \Lambda^1_b(t_0, b)$, then it becomes clear that $l_2(u) < u(b)$. But this inequality contradicts the second relation in (1.2_0). The resulting contradiction shows that, in the considered case, problem (1.1_0), (1.2_0) has only the trivial solution. In a similar way, one can show that problem (1.1_0), (1.2_0) also has only the trivial solution for the case in which conditions (1.9) and (1.14) are satisfied.

To complete the proof of the theorem, it remains to consider the case in which, in addition to (1.10), one of condition (1.15) and (1.16) is satisfied. In this case, by Lemma 2.5, for an arbitrary solution u of problem (1.1_0), (1.2_0), there exist numbers $\tau_1 \in (a, t_0]$ and $\tau_2 \in [t_0, b)$ such that

$$l_1(u) = u(\tau_1), \qquad l_2(u) = u(\tau_2);$$

consequently,

$$u(a) = u(\tau_1), \quad u(b) = u(\tau_2).$$

This, together with Rolle's theorem, implies that the function u satisfies relation (2.4) for some $a_0 \in (a, t_0)$ and $b_0 \in (t_0, b)$. However, problem (1.1_0), (2.4) has only the trivial solution by Lemma 2.4. Consequently, $u(t) \equiv 0$. The proof of the theorem is complete.

3.2. Proof of Theorem 1.2

Let u be an arbitrary solution of problem (1.1_0), (1.3_0). By Proposition 1.1, to prove Theorem 1.2, it suffices to show that $u(t) \equiv 0$.

First, consider the case in which conditions (1.7) and (1.13) are satisfied. By (1.3_0) and (1.7), there exist numbers $a_0 \in [a, t_0)$ and $b_0 \in (t_0, b]$ such that the function u satisfies relation (2.2). On the other hand, inequality (1.13) implies (2.10). If we now use Lemma 2.2, then we find that $u(t) \equiv 0$.

Now let us proceed to the case in which conditions (1.8) and (1.16) are satisfied. Suppose that $u(t) \not\equiv 0$ in this case. It follows from the conditions $u(a) = l_1(u)$ and $l_1 \in \Lambda^-(a, t_0)$ that there exists an $a_0 \in [a, t_0)$ such that $u(a_0) = 0$. Without loss of generality, one can assume that $r(a_0)u'(a_0) = 1$. Then, by condition (1.16), we have

$$r(t)u'(t) > 1 \quad \text{for} \quad t_0 \le t \le b, \quad (r(t)u'(t))' > 0 \quad \text{for almost all} \quad t \in (t_0, b).$$

This, together with the condition $l_2 \in \Lambda_b^1(t_0, b)$, implies that $l_2(ru') < r(b)u'(b)$. But this inequality contradicts the second relation in (1.3_0). From the resulting contradiction, we have $u(t) \equiv 0$.

In conclusion, consider the case in which, in addition to (1.11), one of conditions (1.15) and (1.16) is satisfied. By Lemma 2.5 and conditions (1.3_0) and (1.11), there exist $\tau_0 \in (a, t_0]$ and $b_0 \in (t_0, b]$ such that

$$u(a) = u(\tau_0), \qquad u'(b_0) = 0.$$

This, together with Rolle's theorem, implies that the function u satisfies relation (2.4) for some $a_0 \in (a, \tau_0)$. However, problem (1.1_0), (2.4) has only the trivial solution by Lemma 2.4. Consequently, $u(t) \equiv 0$. The proof of the theorem is complete.

Theorem 1.3 can be proved by analogy with Theorem 1.2.

3.3. Proof of Corollaries 1.1–1.3

Set $t_0 = a_1$ and

$$l_1(v) = \sum_{i=1}^{m} l_{1i} v(a_i), \qquad l_2(v) = \sum_{i=1}^{m} l_{2i} v(b_i).$$

Then for each $k \in \{2, 3, 4\}$, the boundary conditions $(1.k')$ acquire the form $(1.k)$. If, in addition to the definitions of the sets $\Lambda^-(a, t_0)$, $\Lambda^-(t_0, b)$, $\Lambda_a^+(a, t_0)$, and $\Lambda_b^+(t_0, b)$, we use inequalities (1.5) and Lemma 2.7, then we find that, for each $k \in \{7, 8, 9, 10, 11\}$, condition $(1.k')$ implies condition $(1.k)$. Therefore, Corollaries 1.1–1.3 follow from Theorems 1.1–1.3, respectively.

REFERENCES

1. De la Vallée Poussin, C., Sur l'equation differentielle lineaire du second ordre. Détermination d'une integrale par deux valeurs assignées. Extension aux équations d'ordre n, *J. Math. Pures Appl.*, 1929, vol. 297, no. 2, pp. 131–151.

2. Epheser, H., Über die Existenz der Lösungen von Randwertaufgaben mit gewöhnlichen nichtlinearen Differentialgleichungen zweiter Ordnung, *Math. Z.*, 1955, vol. 61, no. 4, pp. 435–454.
3. Hartman, P., *Ordinary Differential Equations*, SIAM, Philadelphia, 2002.
4. Kiguradze, I.T. and Shekhter, B.L., Singular Boundary Value Problems for Second-Order Ordinary Differential Equations, *Itogi Nauki Tekhn. Ser. Sovrem. Probl. Mat.*, 1987, vol. 30, pp. 105–201.
5. Kiguradze, I., Some Optimal Conditions for the Solvability of Two-Point Singular Boundary Value Problems, *Funct. Differ. Equ.*, 2003, vol. 10, no. 1–2, pp. 259–281.
6. Agarwal, R.P. and Kiguradze, I., Two-Point Boundary Value Problems for Higher-Order Linear Differential Equations with Strong Singularities, *Boundary Value Problems*, 2006, pp. 1–32.
7. Kiguradze, I., The Neumann Problem for the Second Order Nonlinear Ordinary Differential Equations at Resonance, *Funct. Differ. Equ.*, 2009, vol. 16, no. 2, pp. 353–371.
8. Kiguradze, T., On Solvability and Unique Solvability of Two-Point Singular Boundary Value Problems, *Nonlinear Anal.*, 2009, vol. 71, pp. 789–798.
9. Partsvania, N., On Two-Point Boundary Value Problems for Two-Dimensional Linear Differential Systems with Singular Coefficients, *Mem. Differential Equations Math. Phys.*, 2010, vol. 51, pp. 155–162.
10. Kiguradze, I. and Lomtatidze, A., On Certain Boundary Value Problems for Second-Order Linear Ordinary Differential Equations with Singularities, *J. Math. Anal. Appl.*, 1984, vol. 101, no. 2, pp. 325–347.
11. Il'in, V.A. and Moiseev, E.I., A Nonlocal Boundary Value Problem of the First Kind for the Sturm–Liouville Operator in Differential and Difference Interpretations, *Differ. Uravn.*, 1987, vol. 23, no. 7, pp. 1198–1207.
12. Il'in, V.A. and Moiseev, E.I., A Nonlocal Boundary Value Problem of the Second Kind for the Sturm–Liouville Operator, *Differ. Uravn.*, 1987, vol. 23, no. 8, pp. 1422–1431.
13. Dovletov, D.M., A Nonlocal Boundary Value Problem of the First Kind in a Differential and a Difference Treatment, *Differ. Uravn.*, 1989, vol. 25, no. 8, pp. 1297–1307.
14. Lomtatidze, A., On a Nonlocal Boundary Value Problem for Second Order Linear Ordinary Differential Equations, *J. Math. Anal. Appl.*, 1995, vol. 193, pp. 889–908.
15. Lomtatidze, A.G., A Nonlocal Boundary Value Problem for Second-Order Linear Ordinary Differential Equations, *Differ. Uravn.*, 1995, vol. 31, no. 3, pp. 446–455.
16. Kiguradze, T., On Some Nonlocal Boundary Value Problems for Linear Singular Differential Equations of Higher Order, *Mem. Differential Equations Math. Phys.*, 2009, vol. 47, pp. 169–174.
17. Kiguradze, T.I., On Conditions for the Well-Posedness of Linear Singular Boundary Value Problems, *Differ. Uravn.*, 2010, vol. 46, no. 2, pp. 183–190.
18. Kiguradze, I.T., On Boundary Value Problems for Linear Differential Systems with Singularities, *Differ. Uravn.*, 2003, vol. 39, no. 2, pp. 198–209.
19. Kiguradze, I.T., Boundary Value Problems for Systems of Ordinary Differential Equations, *Itogi Nauki Tekhn. Ser. Sovrem. Probl. Mat.*, 1987, vol. 30, pp. 3–103.
20. Hardy, G.H., Littlewood, J.E., and Pólya, G., *Inequalities*, Cambridge, 1934.