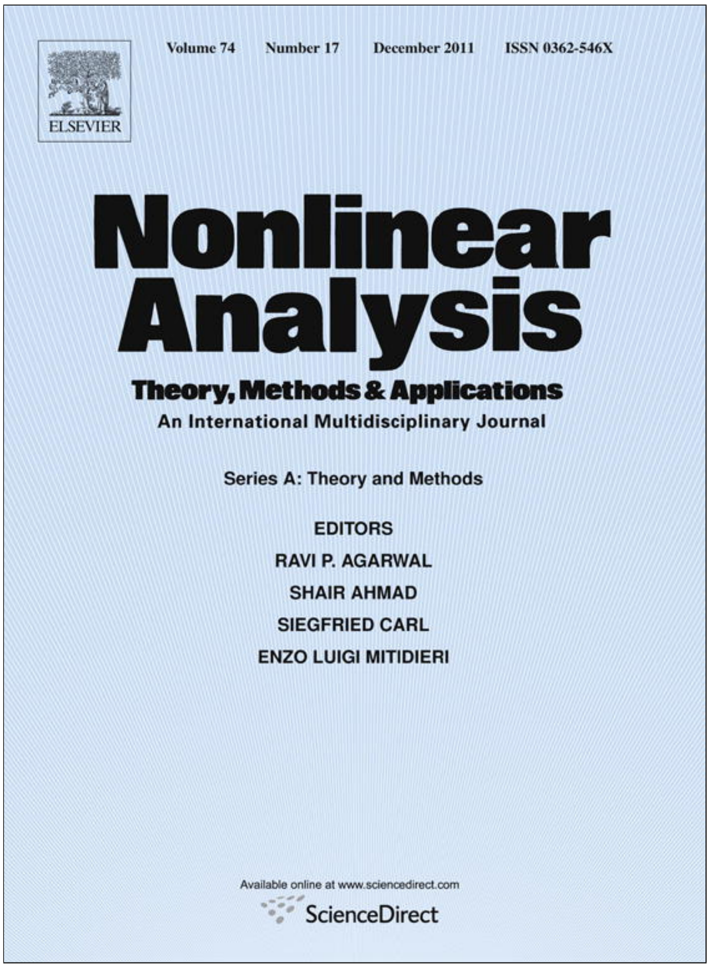

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Solvability conditions for non-local boundary value problems for two-dimensional half-linear differential systems

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ABSTRACT

In this paper, we consider two non-local boundary value problems for two-dimensional half-linear differential systems. We prove general Fredholm type theorems, which allow one to derive new efficient solvability criteria for the problems studied.

1. Statement of problem and formulation of main results

On the interval $[a, b]$, we consider the differential system

$$\frac{du_1}{dt} = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2 + q_1(t, u_1, u_2), \quad \frac{du_2}{dt} = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 + q_2(t, u_1, u_2) \tag{1.1}$$

subjected to one of the following boundary conditions,

$$\int_a^{a_0} u_1(s)\, d\alpha_1(s) = \gamma_1(u_1, u_2), \qquad \int_{b_0}^{b} u_1(s)\, d\alpha_2(s) = \gamma_2(u_1, u_2) \tag{1.2}$$

and

$$\int_a^{a_0} u_1(s)\, d\alpha_1(s) = \gamma_1(u_1, u_2), \qquad \int_{b_0}^{b} u_2(s)\, d\alpha_2(s) = \gamma_2(u_1, u_2). \tag{1.3}$$

In the case, where $\lambda_1 = \lambda_2 = 1$, problems (1.1), (1.2) and (1.1), (1.3) as well as their particular cases are studied in detail (see, e.g., [1–18] and the references therein). As for the case, where system (1.1) is half-linear, i.e., if

$$\lambda_1 > 0, \qquad \lambda_1 \neq 1, \qquad \lambda_1\lambda_2 = 1, \tag{1.4}$$

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as far as we know there is still a broad field for further investigation (Fredholm type results for a particular case of (1.1) can be found, e.g., in [19–21]; comparison theorems and their applications are obtained in [22–24]; for some results closely related to those given below see also [25,26]). In this paper, we try to fill this gap in a certain sense. For problems (1.1), (1.2) and (1.1), (1.3) we prove Fredholm type theorems (see Section 1.1), which allow one to derive new efficient solvability criteria in Sections 1.2 and 1.3.

The following notation is used throughout the paper: $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all natural and real numbers, respectively, $\mathbb{R}_+ = [0, +\infty[$. For any $x \in \mathbb{R}$, we put

$$[x]_+ = \frac{1}{2}(|x| + x), \qquad [x]_- = \frac{1}{2}(|x| - x).$$

$\mathcal{C}$ stands for the Banach space of continuous functions $u: [a, b] \to \mathbb{R}$ endowed with the norm

$$\|u\|_{\mathcal{C}} = \max\big\{|u(t)| : a \le t \le b\big\}.$$

Moreover, we denote

$$\ell(\lambda) := \lambda\left(\frac{1+\lambda}{\pi}\sin\frac{\pi}{1+\lambda}\right)^{-1-\lambda} \quad \text{for } \lambda > 0 \tag{1.5}$$

and

$$\eta(h,\lambda)(t) := \frac{\left(\int_a^t h(s)\,\mathrm{d}s\right)^{\lambda}\left(\int_t^b h(s)\,\mathrm{d}s\right)^{\lambda}}{\left(\int_a^t h(s)\,\mathrm{d}s\right)^{\lambda} + \left(\int_t^b h(s)\,\mathrm{d}s\right)^{\lambda}} \quad \text{for } a \le t \le b,\ \lambda > 0, \tag{1.6}$$

if $h: [a, b] \to \mathbb{R}_+$ is a Lebesgue integrable function which is not equal to zero on a set of positive measure.

In what follows we assume that $p_i: [a, b] \to \mathbb{R}$ $(i = 1, 2)$ are Lebesgue integrable functions and $q_i: [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are functions integrable in the first argument and continuous in the last two arguments. As for the boundary conditions, $a < a_0 \le b$, $a \le b_0 < b$, $\alpha_1: [a, a_0] \to \mathbb{R}$ and $\alpha_2: [b_0, b] \to \mathbb{R}$ are functions of bounded variation, and $\gamma_i: \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ $(i = 1, 2)$ are continuous functionals.

A pair (u_1, u_2) of functions $u_1, u_2: [a, b] \to \mathbb{R}$ is said to be a solution to system (1.1), if the functions u_1, u_2 are absolutely continuous and satisfy both equations in (1.1) almost everywhere on $[a, b]$. A solution (u_1, u_2) to system (1.1) verifying boundary conditions (1.2) (respectively, (1.3)) is called a solution to problem (1.1), (1.2) (respectively, (1.1), (1.3)).

For every $\varrho > 0$ and almost all $t \in [a, b]$, we put

$$q^*(t,\varrho) := \sum_{k=1}^{2}\max\Big\{|q_{3-k}(t, x_1, x_2)| : |x_k| \le \varrho^{\lambda_k},\ |x_{3-k}| \le \varrho\Big\} \tag{1.7}$$

and

$$\begin{gathered} \gamma_0^*(\varrho) := \sum_{k=1}^{2}\sup\Big\{|\gamma_k(u_1,u_2)| : \|u_1\|_{\mathcal{C}} \le \varrho,\ \|u_2\|_{\mathcal{C}} \le \varrho^{\lambda_2}\Big\}, \\ \gamma^*(\varrho) := \sum_{k=1}^{2}\sup\Big\{|\gamma_{3-k}(u_1,u_2)| : \|u_k\|_{\mathcal{C}} \le \varrho^{\lambda_k},\ \|u_{3-k}\|_{\mathcal{C}} \le \varrho\Big\}. \end{gathered} \tag{1.8}$$

Problems (1.1), (1.2) and (1.1), (1.3) will be investigated under the assumptions

$$\lim_{\varrho\to+\infty}\int_a^b \frac{q^*(s,\varrho)}{\varrho}\,\mathrm{d}s = 0, \qquad \lim_{\varrho\to+\infty}\frac{\gamma_0^*(\varrho)}{\varrho} = 0 \tag{1.9}$$

and

$$\lim_{\varrho\to+\infty}\int_a^b \frac{q^*(s,\varrho)}{\varrho}\,\mathrm{d}s = 0, \qquad \lim_{\varrho\to+\infty}\frac{\gamma^*(\varrho)}{\varrho} = 0, \tag{1.10}$$

respectively. For example, in view of (1.4), for the validity of relations (1.9) it is sufficient that the inequalities

$$|q_k(t, x_1, x_2)| \le r\Big(1 + |x_k|^{1-\varepsilon} + |x_{3-k}|^{\lambda_k-\varepsilon}\Big) \quad (k = 1, 2) \tag{1.11}$$

and

$$|\gamma_i(u_1, u_2)| \le r\Big(1 + \|u_1\|_{\mathcal{C}}^{1-\varepsilon} + \|u_2\|_{\mathcal{C}}^{\lambda_1-\varepsilon}\Big) \quad (i = 1, 2)$$

are satisfied on the sets $[a, b] \times \mathbb{R}^2$ and $\mathcal{C} \times \mathcal{C}$, respectively, where r is a positive constant and ε is a positive number small enough. As for the validity of relations (1.10), it sufficient to assume that, together with (1.11), the inequalities

$$|\gamma_i(u_1, u_2)| \le r\left(1 + \|u_i\|_{\mathcal{C}}^{1-\varepsilon} + \|u_{3-i}\|_{\mathcal{C}}^{\lambda_i-\varepsilon}\right) \quad (i = 1, 2)$$

hold.

1.1. Fredholm type theorems

For any $\mu \in [0, 1]$, we consider the half-linear differential system

$$\frac{du_1}{dt} = \mu p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2, \qquad \frac{du_2}{dt} = \mu p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 \tag{1.1$_\mu$}$$

together with the homogeneous boundary conditions

$$\int_a^{a_0} u_1(s)\, d\alpha_1(s) = 0, \qquad \int_{b_0}^{b} u_1(s)\, d\alpha_2(s) = 0 \tag{1.2$_0$}$$

and

$$\int_a^{a_0} u_1(s)\, d\alpha_1(s) = 0, \qquad \int_{b_0}^{b} u_2(s)\, d\alpha_2(s) = 0. \tag{1.3$_0$}$$

Theorem 1.1. *Let*

$$\lambda_1 > 1, \quad \lambda_1\lambda_2 = 1, \tag{1.12}$$

α_1, α_2 be non-decreasing functions satisfying the inequalities

$$\alpha_1(a_0) > \alpha_1(a), \qquad \alpha_2(b) > \alpha_2(b_0), \tag{1.13}$$

$$a_0 < b_0, \quad \int_{a_0}^{b_0} p_1(s)\, ds \ne 0, \tag{1.14}$$

and there exist $\sigma \in \{-1, 1\}$ such that

$$\sigma p_1(t) \ge 0 \quad \text{for a.e. } t \in [a, b]. \tag{1.15}$$

Moreover, let for every $\mu \in]0, 1]$ problem (1.1$_\mu$), (1.2$_0$) have only the trivial solution and conditions (1.9) hold. Then problem (1.1), (1.2) possesses at least one solution.

Remark 1.1. The assumption in Theorem 1.1 that problem (1.1$_\mu$), (1.2$_0$) has only the trivial solution for every $\mu \in]0, 1]$ cannot be weakened to $\mu \in]0, 1[$. Indeed, let $\lambda_1 > 0$, $\lambda_2 = 1/\lambda_1$, $p_1(t) \equiv -1$, $p_2(t) \equiv \left(\frac{\pi_p}{b-a}\right)^p$, $q_1(t, x_1, x_2) \equiv 0$, and $q_2(t, x_1, x_2) \equiv 1$, where

$$\pi_p = (p-1)^{1/p} \frac{2\pi}{p \sin \frac{\pi}{p}}, \quad p = 1 + \lambda_2. \tag{1.16}$$

Moreover, let $\gamma_k(v_1, v_2) \equiv 0 (k = 1, 2)$, $a < a_0 < b_0 < b$, and

$$\alpha_1(s) = \begin{cases} 0 & \text{for } s = a, \\ 1 & \text{for } s \in]a, a_0], \end{cases} \qquad \alpha_2(s) = \begin{cases} 0 & \text{for } s \in [b_0, b[, \\ 1 & \text{for } s = b. \end{cases} \tag{1.17}$$

Then problem (1.1), (1.2) has the form

$$\frac{du_1}{dt} = -|u_2|^{\lambda_1} \operatorname{sgn} u_2, \qquad \frac{du_2}{dt} = \left(\frac{\pi_p}{b-a}\right)^p |u_1|^{\lambda_2} \operatorname{sgn} u_1 + 1,$$
$$u_1(a) = 0, \qquad u_1(b) = 0.$$

It follows from [27, Section 1] that problem (1.1$_\mu$), (1.2$_0$) has only the trivial solution for every $\mu \in]0, 1[$. However, [27, Theorem 2.1(b)] yields that problem (1.1), (1.2) has no solution.

Theorem 1.2. *Let conditions (1.4) and (1.13) be satisfied. Moreover, let for every $\mu \in]0, 1]$ problem (1.1$_\mu$), (1.3$_0$) have only the trivial solution and conditions (1.10) hold. Then problem (1.1), (1.3) possesses at least one solution.*

Remark 1.2. The assumption in Theorem 1.2 that problem (1.1_μ), (1.3_0) has only the trivial solution for every $\mu \in]0,1]$ cannot be weakened to $\mu \in]0,1[$. Indeed, let $\lambda_1 > 0$, $\lambda_2 = 1/\lambda_1$, $p_1(t) \equiv -1$, $p_2(t) \equiv \left(\frac{\pi_p}{2(b-a)}\right)^p$, $q_1(t,x_1,x_2) \equiv 0$, and $q_2(t,x_1,x_2) \equiv 1$, where the numbers π_p and p are defined by formulas (1.16). Moreover, let $\gamma_k(v_1,v_2) \equiv 0 (k = 1,2)$, $a < a_0 < b_0 < b$, and the functions α_1, α_2 are given by relations (1.17). Then problem (1.1), (1.3) has the form

$$\frac{du_1}{dt} = -|u_2|^{\lambda_1} \operatorname{sgn} u_2, \qquad \frac{du_2}{dt} = \left(\frac{\pi_p}{2(b-a)}\right)^p |u_1|^{\lambda_2} \operatorname{sgn} u_1 + 1,$$
$$u_1(a) = 0, \qquad u_2(b) = 0.$$

It is not difficult to deduce from discussion presented in [27, Section 1] that problem (1.1_μ), (1.3_0) has only the trivial solution for every $\mu \in]0,1[$. However, as follows from [27, Theorem 2.1(b)], problem (1.1), (1.3) has no solution.

1.2. Solvability conditions for problem (1.1), (1.2)

In this section, we present new efficient conditions guaranteeing the solvability of problem (1.1), (1.2).

Theorem 1.3. *Let conditions (1.9), (1.12)–(1.14) be satisfied and the functions α_1, α_2 be non-decreasing. Moreover, let there exist numbers $\sigma \in \{-1,1\}$ and $p_0 \ge 0$ such that*

$$\sigma p_1(t) \ge 0, \qquad \sigma p_2(t) \ge -p_0|p_1(t)| \quad \text{for a.e. } t \in [a,b], \tag{1.18}$$

and

$$p_0\left(\int_a^b |p_1(s)|\, ds\right)^{1+\lambda_2} < 2^{1+\lambda_2}\ell(\lambda_2), \tag{1.19}$$

where the function ℓ is defined by relation (1.5). Then problem (1.1), (1.2) has at least one solution.

Remark 1.3. The example constructed in Remark 1.1 also shows that the strict inequality (1.19) in Theorem 1.3 cannot be replaced by the non-strict one.

Theorem 1.4. *Let conditions (1.9), (1.12)–(1.14) be satisfied and the functions α_1, α_2 be non-decreasing. Moreover, let there exist a number $\sigma \in \{-1,1\}$ such that along with (1.15) the inequality*

$$\int_a^b \eta(|p_1|,\lambda_2)(s)[\sigma p_2(s)]_-\, ds < 1 \tag{1.20}$$

holds, where the operator η is defined by relation (1.6). Then problem (1.1), (1.2) has at least one solution.

As an example of non-local boundary conditions (1.2) we consider the multi-point conditions

$$\sum_{k=1}^{m_1} \beta_{1k} u_1(a_k) = \gamma_1(u_1,u_2), \qquad \sum_{k=1}^{m_2} \beta_{2k} u_1(b_k) = \gamma_2(u_1,u_2), \tag{1.21}$$

where $a \le a_1 < \cdots < a_{m_1} \le a_0$, $b_0 \le b_1 < \cdots < b_{m_2} \le b$, and β_{ik} are positive numbers ($k = 1,\dots,m_i,\ i = 1,2$).

Theorems 1.3 and 1.4 immediately yield

Corollary 1.1. *Let conditions (1.9), (1.12) and (1.14) be satisfied and there exist numbers $\sigma \in \{-1,1\}$ and $p_0 \ge 0$ (respectively, a number $\sigma \in \{-1,1\}$) such that inequalities (1.18) and (1.19) (respectively, (1.15) and (1.20)) hold. Then problem (1.1), (1.21) possesses at least one solution.*

1.3. Solvability conditions for problem (1.1), (1.3)

In this section we present new efficient conditions guaranteeing the solvability of problem (1.1), (1.3).

Put

$$\alpha_1(s) = \alpha_1(a_0) \quad \text{for } a_0 \le s \le b, \qquad \alpha_2(s) = \alpha_2(b_0) \quad \text{for } a \le s \le b_0, \tag{1.22}$$

and

$$\delta_i(s) = \max\big\{|\alpha_i(s) - \alpha_i(a)|, |\alpha_i(b) - \alpha_i(s)|\big\} \quad \text{for } a \le s \le b,\ i = 1,2. \tag{1.23}$$

Theorem 1.5. *Let conditions* (1.4) *and* (1.10) *be satisfied,*

$$\alpha_1(a_0) - \alpha_1(a) = 1, \qquad \alpha_2(b) - \alpha_2(b_0) = 1, \tag{1.24}$$

and

$$\int_a^b \delta_1(s)|p_1(s)|\,\mathrm{d}s\left(\int_a^b \delta_2(s)|p_2(s)|\,\mathrm{d}s\right)^{\lambda_1} < 1. \tag{1.25}$$

Then problem (1.1), (1.3) *has at least one solution.*

Theorem 1.6. *Let the functions* α_1, α_2 *be non-decreasing,*

$$a_0 \le b_0, \quad \alpha_1(a_0) > \alpha_1(a), \qquad \alpha_2(b) > \alpha_2(b_0), \tag{1.26}$$

and conditions (1.4) *and* (1.10) *hold. If, moreover, for each* $\sigma \in \{-1, 1\}$ *one of the inequalities*

$$\int_a^b [\sigma p_1(s)]_+\left(\int_s^b [\sigma p_2(\xi)]_-\,\mathrm{d}\xi\right)^{\lambda_1} \mathrm{d}s < 1 \tag{1.27}$$

and

$$\int_a^b [\sigma p_2(s)]_-\left(\int_a^s [\sigma p_1(\xi)]_+\,\mathrm{d}\xi\right)^{\lambda_2} \mathrm{d}s < 1 \tag{1.28}$$

is fulfilled, then problem (1.1), (1.3) *possesses at least one solution.*

Theorem 1.7. *Let the functions* α_1, α_2 *be non-decreasing and conditions* (1.4), (1.10) *and* (1.26) *hold. Moreover, let there exist numbers* $\sigma \in \{-1, 1\}$ *and* $p_0 \ge 0$ *such that inequalities* (1.18) *are satisfied and*

$$p_0\left(\int_a^b |p_1(s)|\,\mathrm{d}s\right)^{1+\lambda_2} < \ell(\lambda_2), \tag{1.29}$$

where the function ℓ *is defined by formula* (1.5). *Then problem* (1.1), (1.3) *has at least one solution.*

Remark 1.4. The example constructed in Remark 1.2 also shows that the strict inequality (1.29) in Theorem 1.7 cannot be replaced by the non-strict one.

At last we consider the case, where boundary conditions (1.3) have the form

$$\sum_{k=1}^{m_i} \beta_{ik} u_i(t_{ik}) = \gamma_i(u_1, u_2) \quad (i = 1, 2) \tag{1.30}$$

in which $t_{ik} \in [a, b]$ and $\beta_{ik} \in \mathbb{R}$ $(k = 1, \dots, m_i,\ i = 1, 2)$. The following statements follow immediately from Theorems 1.5–1.7.

Corollary 1.2. *Let*

$$\sum_{k=1}^{m_i} \beta_{ik} = 1 \quad \textit{for } i = 1, 2$$

and

$$\delta_0 \int_a^b |p_1(s)|\,\mathrm{d}s\left(\int_a^b |p_2(s)|\,\mathrm{d}s\right)^{\lambda_1} < 1,$$

where

$$\delta_0 = \left(\sum_{k=1}^{m_1} |\beta_{1k}|\right)\left(\sum_{k=1}^{m_2} |\beta_{2k}|\right)^{\lambda_1}.$$

If, moreover, conditions (1.4) *and* (1.10) *be satisfied, then problem* (1.1), (1.30) *possesses at least one solution.*

Corollary 1.3. *Let $a \le t_{1j} \le t_{2k} \le b$ $(j = 1, \dots, m_1, k = 1, \dots, m_2)$, $\beta_{ik} > 0$ $(k = 1, \dots, m_i,\ i = 1, 2)$, and conditions (1.4) and (1.10) be satisfied. Moreover, let either for each $\sigma \in \{-1, 1\}$ one of inequalities (1.27) and (1.28) be fulfilled, or there exist numbers $\sigma \in \{-1, 1\}$ and $p_0 \ge 0$ such that inequalities (1.18) and (1.29) hold. Then problem (1.1), (1.30) has at least one solution.*

2. Auxiliary statements

In this section we establish auxiliary statements that will be used in the proofs of the main results. For the sake of clarity we divide lemmas into the following five subsections.

2.1. Lemmas on properties of solutions to a certain first-order differential inequality

Let $h\colon [a, b] \to \mathbb{R}_+$ be a Lebesgue integrable functions, which is not equal to zero on a set of positive measure, $u_2\colon [a, b] \to \mathbb{R}$ be an essentially bounded measurable function, and λ_1 be a positive parameter.

Consider the differential inequality

$$|u_1'(t)| \le h(t)|u_2(t)|^{\lambda_1}. \tag{2.1}$$

A function $u_1\colon [a, b] \to \mathbb{R}$ is said to be a solution to inequality (2.1), if it is absolutely continuous and satisfies inequality (2.1) almost everywhere on $[a, b]$.

Lemma 2.1. *Let $t_0 \in [a, b]$ and u_1 be a solution to differential inequality (2.1) satisfying the condition*

$$u_1(t_0) = 0. \tag{2.2}$$

Then

$$|u_1(t)|^{1+\lambda_2} \le \left|\int_{t_0}^{t} h(s)\,ds\right|^{\lambda_2} \left|\int_{t_0}^{t} h(s)|u_2(s)|^{1+\lambda_1}\,ds\right| \quad \text{for } t \in [a, b] \tag{2.3}$$

and

$$\ell(\lambda_2)\int_a^b h(s)|u_1(s)|^{1+\lambda_2}\,ds \le \left(\int_a^b h(s)\,ds\right)^{1+\lambda_2} \int_a^b h(s)|u_2(s)|^{1+\lambda_1}\,ds, \tag{2.4}$$

where $\lambda_2 = 1/\lambda_1$ and the function ℓ is defined by formula (1.5).

To prove this lemma we need the following result that belongs to A. Levin.

Lemma 2.2 (*A. Levin, [28]*)**.** [1] *Let $\lambda > 0$, $c > 0$, $x_0 \in [0, c]$, and $u\colon [0, c] \to \mathbb{R}$ be an absolutely continuous function such that*

$$u(x_0) = 0, \qquad \int_0^c |u'(x)|^{1+\lambda}\,dx < +\infty. \tag{2.5}$$

Then

$$\ell(\lambda)\int_0^c |u(x)|^{1+\lambda}\,dx \le c^{1+\lambda}\int_0^c |u'(x)|^{1+\lambda}\,dx, \tag{2.6}$$

where the function ℓ is defined by relation (1.5).

Proof of Lemma 2.1. In view of condition (2.2), it follows from inequality (2.1) that

$$|u_1(t)|^{1+\lambda_2} \le \left|\int_{t_0}^{t} h(s)|u_2(s)|^{\lambda_1}\,ds\right|^{1+\lambda_2} \qquad \text{for } t \in [a, b]. \tag{2.7}$$

On the other hand, by using the Hölder inequality, we obtain

$$\begin{aligned}\left|\int_{t_0}^{t} h(s)|u_2(s)|^{\lambda_1}\,ds\right| &= \left|\int_{t_0}^{t} h^{\frac{\lambda_2}{1+\lambda_2}}(s)\Big(h(s)|u_2(s)|^{1+\lambda_1}\Big)^{\frac{1}{1+\lambda_2}}\,ds\right| \\ &\le \left|\int_{t_0}^{t} h(s)\,ds\right|^{\frac{\lambda_2}{1+\lambda_2}} \left|\int_{t_0}^{t} h(s)|u_2(s)|^{1+\lambda_1}\,ds\right|^{\frac{1}{1+\lambda_2}}\end{aligned}$$

for $t \in [a, b]$ which, together with (2.7), results in desired estimate (2.3).

[1] See also [29, Theorem 256].

It remains to show the validity of inequality (2.4). Let $\varepsilon > 0$ be arbitrary but fixed. We put

$$x = \int_a^t \big(\varepsilon + h(s)\big)\,ds, \qquad u(x) = u_1(t) \quad \text{for } t \in [a, b], \tag{2.8}$$

and

$$x_0 = \int_a^{t_0} \big(\varepsilon + h(s)\big)\,ds, \qquad c = \int_a^b \big(\varepsilon + h(s)\big)\,ds.$$

Then the function $u\colon [0, c] \to \mathbb{R}$ is absolutely continuous. Moreover, by virtue of assumptions (1.4), (2.1) and (2.2), the relation

$$|u'(x)|^{1+\lambda_2} = \left|\frac{u_1'(t)}{\varepsilon + h(t)}\right|^{1+\lambda_2} \le \left(\frac{h(t)}{\varepsilon + h(t)}|u_2(t)|^{\lambda_1}\right)^{1+\lambda_2} \le |u_2(t)|^{1+\lambda_1}$$

holds for a.e. $x \in [0, c]$, $u(x_0) = 0$, and

$$\int_0^c |u'(x)|^{1+\lambda_2}\,dx \le \int_a^b \big(\varepsilon + h(s)\big)|u_2(s)|^{1+\lambda_1}\,ds < +\infty. \tag{2.9}$$

Consequently, condition (2.5) with $\lambda = \lambda_2$ is satisfied and thus Lemma 2.2 yields that relation (2.6) holds. Hence, in view of (2.1), (2.8) and (2.9), it follows from (2.6) that

$$\ell(\lambda_2)\int_a^b \big(\varepsilon + h(s)\big)|u_1(s)|^{1+\lambda_2}\,ds \le \left(\int_a^b \big(\varepsilon + h(s)\big)\,ds\right)^{1+\lambda_2} \int_a^b \big(\varepsilon + h(s)\big)|u_2(s)|^{1+\lambda_1}\,ds.$$

Letting $\varepsilon \to 0$ in the last inequality gives desired estimate (2.4). □

Lemma 2.3. *Let $a \le t_1 < t_2 \le b$ and u_1 be a solution to differential inequality (2.1) such that*

$$u_1(t_1) = u_1(t_2) = 0. \tag{2.10}$$

Then

$$|u_1(t)|^{1+\lambda_2} \le \eta(h, \lambda_2)(t)\int_{t_1}^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1, t_2] \tag{2.11}$$

and

$$2^{1+\lambda_2}\ell(\lambda_2)\int_{t_1}^{t_2} h(s)|u_1(s)|^{1+\lambda_2}\,ds \le \left(\int_{t_1}^{t_2} h(s)\,ds\right)^{1+\lambda_2}\int_{t_1}^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds, \tag{2.12}$$

where $\lambda_2 = 1/\lambda_1$, the function ℓ and the operator η are defined by formulas (1.5) and (1.6), respectively.

Proof. In view of equalities (2.10), it follows from Lemma 2.1 that

$$\begin{aligned}|u_1(t)|^{1+\lambda_2} &\le \left(\int_{t_1}^t h(s)\,ds\right)^{\lambda_2}\int_{t_1}^t h(s)|u_2(s)|^{1+\lambda_1}\,ds\\ &\le \left(\int_a^t h(s)\,ds\right)^{\lambda_2}\int_{t_1}^t h(s)|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1, t_2]\end{aligned}$$

and

$$\begin{aligned}|u_1(t)|^{1+\lambda_2} &\le \left(\int_t^{t_2} h(s)\,ds\right)^{\lambda_2}\int_t^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds\\ &\le \left(\int_t^b h(s)\,ds\right)^{\lambda_2}\int_t^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1, t_2].\end{aligned}$$

Consequently, we have

$$\left(\int_t^b h(s)\,ds\right)^{\lambda_2}|u_1(t)|^{1+\lambda_2} \le \left(\int_a^t h(s)\,ds\right)^{\lambda_2}\left(\int_t^b h(s)\,ds\right)^{\lambda_2}\int_{t_1}^t h(s)|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1, t_2]$$

and

$$\left(\int_a^t h(s)\,ds\right)^{\lambda_2} |u_1(t)|^{1+\lambda_2} \le \left(\int_a^t h(s)\,ds\right)^{\lambda_2} \left(\int_t^b h(s)\,ds\right)^{\lambda_2} \int_t^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1, t_2].$$

Summing the last two inequalities results in

$$\left[\left(\int_a^t h(s)\,ds\right)^{\lambda_2} + \left(\int_t^b h(s)\,ds\right)^{\lambda_2}\right] |u_1(t)|^{1+\lambda_2}$$
$$\le \left(\int_a^t h(s)\,ds\right)^{\lambda_2} \left(\int_t^b h(s)\,ds\right)^{\lambda_2} \int_{t_1}^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1, t_2]$$

which, in view of notation (1.6), guarantees the validity of estimate (2.11).

It remains to show that inequality (2.12) also holds. Indeed, let $t_0 \in [t_1, t_2]$ be such that

$$\int_{t_1}^{t_0} h(s)\,ds = \int_{t_0}^{t_2} h(s)\,ds = \frac{1}{2} \int_{t_1}^{t_2} h(s)\,ds.$$

Then, by virtue of equalities (2.10), it follows from Lemma 2.1 that

$$\ell(\lambda_2) \int_{t_1}^{t_0} h(s)|u_1(s)|^{1+\lambda_2}\,ds \le \left(\int_{t_1}^{t_0} h(s)\,ds\right)^{1+\lambda_2} \int_{t_1}^{t_0} h(s)|u_2(s)|^{1+\lambda_1}\,ds$$
$$= \frac{1}{2^{1+\lambda_2}} \left(\int_{t_1}^{t_2} h(s)\,ds\right)^{1+\lambda_2} \int_{t_1}^{t_0} h(s)|u_2(s)|^{1+\lambda_1}\,ds$$

and

$$\ell(\lambda_2) \int_{t_0}^{t_2} h(s)|u_1(s)|^{1+\lambda_2}\,ds \le \left(\int_{t_0}^{t_2} h(s)\,ds\right)^{1+\lambda_2} \int_{t_0}^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds$$
$$= \frac{1}{2^{1+\lambda_2}} \left(\int_{t_1}^{t_2} h(s)\,ds\right)^{1+\lambda_2} \int_{t_0}^{t_2} h(s)|u_2(s)|^{1+\lambda_1}\,ds,$$

whose summing we obtain desired estimate (2.12). □

2.2. Lemmas on properties of solutions to system (1.1_μ)

Throughout this section we assume that $\mu \in]0, 1]$ and that condition (1.4) holds.

Lemma 2.4. *Let $t_0 \in [a, b]$. Then system (1.1_μ) has only the trivial solution satisfying the initial conditions*

$$u_i(t_0) = 0 \quad (i = 1, 2). \tag{2.13}$$

Proof. Let (u_1, u_2) be a solution to problem (1.1_μ), (2.13). Put

$$u(t) := \max\left\{|u_1(s)| : 0 \le (s - t_0)\,\mathrm{sgn}(t - t_0) \le |t - t_0|\right\} \quad \text{for } t \in [a, b]$$

and

$$p(t) := |p_1(t)| \left|\int_{t_0}^t |p_2(s)|\,ds\right|^{\lambda_1} \quad \text{for a.e. } t \in [a, b].$$

Then, by virtue of conditions (1.4) and (2.13), we get from (1.1_μ) the relation

$$u(t) \le \left|\int_{t_0}^t |p_1(s)| \left|\int_{t_0}^s |p_2(\xi)|\, |u_1(\xi)|^{\lambda_2}\,d\xi\right|^{\lambda_1} ds\right|$$
$$\le \left|\int_{t_0}^t p(s)u(s)\,ds\right| \quad \text{for } t \in [a, b].$$

By using the Gronwall–Bellman lemma we get from the last inequalities that $u(t) \equiv 0$. Consequently, we have $u_1(t) \equiv 0$ and $u_2(t) \equiv \mu \int_{t_0}^{t} p_2(s)|u_1(s)|^{\lambda_2} \, ds \equiv 0$. □

Lemma 2.5. *Let $a \le t_1 < t_2 \le b$ and (u_1, u_2) be a solution to system (1.1$_\mu$) such that*

$$u_1(t_2)u_2(t_2) = u_1(t_1)u_2(t_1). \tag{2.14}$$

Then

$$\int_{t_1}^{t_2} p_1(s)|u_2(s)|^{1+\lambda_1} \, ds = -\int_{t_1}^{t_2} p_2(s)|u_1(s)|^{1+\lambda_2} \, ds. \tag{2.15}$$

Proof. By direct calculation we get

$$\begin{aligned}\mu \int_{t_1}^{t_2} p_1(s)|u_2(s)|^{1+\lambda_1} \, ds &= \int_{t_1}^{t_2} u_1'(s)u_2(s) \, ds \\ &= u_1(t_2)u_2(t_2) - u_1(t_1)u_2(t_1) - \int_{t_1}^{t_2} u_1(s)u_2'(s) \, ds \\ &= -\mu \int_{t_1}^{t_2} p_2(s)|u_1(s)|^{1+\lambda_2} \, ds. \quad \square\end{aligned}$$

Lemma 2.6. *Let $a \le t_1 < t_2 \le b$ and (u_1, u_2) be a nontrivial solution to system (1.1$_\mu$) such that*

$$u_i(t_i) = 0 \quad (i = 1, 2). \tag{2.16}$$

Then

$$\int_{t_1}^{t_2} |p_1(s)| \, |u_2(s)|^{1+\lambda_1} \, ds > 0 \quad \text{and} \quad \int_{t_1}^{t_2} |p_2(s)| \, |u_1(s)|^{1+\lambda_2} \, ds > 0. \tag{2.17}$$

Proof. Assume that, on the contrary, at least one of inequalities (2.17) is violated. Then, by virtue the integral representations

$$u_1(t) = \mu \int_{t_1}^{t} p_1(s)|u_2(s)|^{\lambda_1} \operatorname{sgn} u_2(s) \, ds \quad \text{for } t \in [t_1, t_2]$$

and

$$u_2(t) = -\mu \int_{t}^{t_2} p_2(s)|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s) \, ds \quad \text{for } t \in [t_1, t_2],$$

it is clear that $u_1(t) = 0$ and $u_2(t) = 0$ for $t \in [t_1, t_2]$. Consequently, Lemma 2.4 guarantees that $u_1(t) \equiv 0$ and $u_2(t) \equiv 0$ on $[a, b]$, which contradicts the assumption of the lemma. □

2.3. Lemma on the unique solvability of problem (1.1$_\mu$), (1.2$_0$)

Lemma 2.7. *Let conditions (1.4), (1.13) and (1.14) be satisfied and the functions α_1, α_2 be non-decreasing. Moreover, let there exist numbers $\sigma \in \{-1, 1\}$ and $p_0 \ge 0$ (respectively, a number $\sigma \in \{-1, 1\}$) such that inequalities (1.18) and (1.19) (respectively, (1.15) and (1.20)) hold. Then, for every $\mu \in]0, 1]$, problem (1.1$_\mu$), (1.2$_0$) has only the trivial solution.*

Proof. Assume that, on the contrary, (u_1, u_2) is a nontrivial solution to problem (1.1$_\mu$), (1.2$_0$) with some $\mu \in]0, 1]$. Since the functions α_1, α_2 are non-decreasing and satisfy inequalities (1.13) and (1.14), there exist $t_1 \in [a, a_0]$ and $t_2 \in [b_0, b]$, $t_1 < t_2$, such that equalities (2.10) are fulfilled. The integration of the first equation in (1.1$_\mu$) from t_1 to t_2 results in

$$\int_{t_1}^{t_2} p_1(s)|u_2(s)|^{\lambda_1} \operatorname{sgn} u_2(s) \, ds = 0$$

which, together with assumptions (1.14) and (1.15), guarantees that there is a point $t_0 \in [t_1, t_2]$ such that

$$u_2(t_0) = 0. \tag{2.18}$$

Clearly, $t_0 > t_1$ because, in the contrary case, we obtain a contradiction with the assertion of Lemma 2.4. Therefore, in view of (2.10) and (2.18), Lemma 2.6 yields

$$\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds \geq \int_{t_1}^{t_0} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds > 0. \tag{2.19}$$

On the other hand, it follows from Lemma 2.3 with $h(t) \equiv |p_1(t)|$ that

$$2^{1+\lambda_2}\ell(\lambda_2)\int_{t_1}^{t_2} |p_1(s)|\,|u_1(s)|^{1+\lambda_2}\,ds \leq \left(\int_{t_1}^{t_2} |p_1(s)|\,ds\right)^{1+\lambda_2}\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds \tag{2.20}$$

and

$$|u_1(t)|^{1+\lambda_2} \leq \eta(|p_1|,\lambda_2)(t)\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds \quad \text{for } t \in [t_1,t_2], \tag{2.21}$$

where the function ℓ and the operator η are defined by formulas (1.5) and (1.6), respectively. By using inequalities (1.18) and (2.20) (respectively, (1.15) and (2.21)) and Lemma 2.5 we get

$$\begin{aligned}
\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds &= \int_{t_1}^{t_2} \big(-\sigma p_2(s)\big)|u_1(s)|^{1+\lambda_2}\,ds \\
&\leq p_0\int_{t_1}^{t_2} |p_1(s)|\,|u_1(s)|^{1+\lambda_2}\,ds \\
&\leq \frac{p_0}{2^{1+\lambda_2}\ell(\lambda_2)}\left(\int_a^b |p_1(s)|\,ds\right)^{1+\lambda_2}\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds
\end{aligned}$$

$$\begin{aligned}
\Bigg(\text{respectively, } \int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds &= \int_{t_1}^{t_2} \big(-\sigma p_2(s)\big)|u_1(s)|^{1+\lambda_2}\,ds \\
&\leq \int_{t_1}^{t_2} [\sigma p_2(s)]_-|u_1(s)|^{1+\lambda_2}\,ds \\
&\leq \int_a^b \eta(|p_1|,\lambda_2)(s)[\sigma p_2(s)]_-\,ds\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds\Bigg)
\end{aligned}$$

which, in view of (2.19), contradicts assumption (1.19) (respectively, (1.20)). □

2.4. Lemmas on the unique solvability of problem (1.1$_\mu$), (1.3$_0$)

Lemma 2.8. *Let conditions (1.4), (1.24) and (1.25) be satisfied, where the functions δ_1, δ_2 are defined by formulas (1.22) and (1.23). Then, for every $\mu \in]0,1]$, problem (1.1$_\mu$), (1.3$_0$) has only the trivial solution.*

Proof. Let (u_1,u_2) be a solution to problem (1.1$_\mu$), (1.3$_0$) with some $\mu \in]0,1]$. Then, in view of (1.22) and (1.24), equalities

$$\int_a^b u_i(s)\,d\alpha_i(s) = 0, \qquad \alpha_i(b)-\alpha_i(a) = 1 \quad (i=1,2)$$

are satisfied. Therefore, u_1 and u_2 admit the integral representations

$$\begin{aligned}
u_1(t) &= \mu\int_a^b g_1(t,s)p_1(s)|u_2(s)|^{\lambda_1}\operatorname{sgn} u_2(s)\,ds \quad \text{for } t \in [a,b], \\
u_2(t) &= \mu\int_a^b g_2(t,s)p_2(s)|u_1(s)|^{\lambda_2}\operatorname{sgn} u_1(s)\,ds \quad \text{for } t \in [a,b],
\end{aligned} \tag{2.22}$$

where

$$g_i(t,s) = \begin{cases} \alpha_i(s)-\alpha_i(a) & \text{for } s \leq t, \\ \alpha_i(s)-\alpha_i(b) & \text{for } s > t \end{cases} \quad (i=1,2).$$

Moreover, in view of (1.23), we have

$$|g_i(t,s)| \leq \delta_i(s) \quad \text{for } a \leq s,\ t \leq b,\ i=1,2. \tag{2.23}$$

Put

$$\varrho_i := \|u_i\|_C \quad \text{for } i = 1, 2.$$

Then, by virtue of (1.4) and (2.23), it follows from equalities (2.22) that

$$\varrho_1 \le \varrho_2^{\lambda_1} \int_a^b \delta_1(s)|p_1(s)|\,ds, \qquad \varrho_2 \le \varrho_1^{\lambda_2} \int_a^b \delta_2(s)|p_2(s)|\,ds,$$

whence we get

$$\varrho_1 \le \varrho_1 \int_a^b \delta_1(s)|p_1(s)|\,ds \left(\int_a^b \delta_2(s)|p_2(s)|\,ds \right)^{\lambda_1}.$$

Consequently, in view of inequality (1.25), we obtain $\varrho_1 = 0$ and $\varrho_2 = 0$, i.e., $u_i(t) \equiv 0$ $(i = 1, 2)$. □

Lemma 2.9. *Let the functions α_1, α_2 be non-decreasing and conditions (1.4) and (1.26) hold. If, moreover, for each $\sigma \in \{-1, 1\}$ one of inequalities (1.27) and (1.28) is satisfied then, for every $\mu \in]0, 1]$, problem (1.1$_\mu$), (1.3$_0$) has only the trivial solution.*

Proof. Assume that, on the contrary, (u_1, u_2) is a nontrivial solution to problem (1.1$_\mu$), (1.3$_0$) with some $\mu \in]0, 1]$. Since the functions α_1, α_2 are non-decreasing and satisfy inequalities (1.26), there exist $t_1 \in [a, a_0]$ and $t_2 \in [b_0, b]$ such that

$$u_i(t_i) = 0 \quad (i = 1, 2). \tag{2.24}$$

Clearly, Lemma 2.4 yields

$$t_1 < t_2, \quad u_2(t_1) \ne 0, \qquad u_1(t_2) \ne 0.$$

Therefore, we can assume without loss of generality that

$$u_1(t) > 0 \quad \text{for } t_1 < t \le t_2, \qquad \sigma u_2(t) > 0 \quad \text{for } t_1 \le t < t_2, \tag{2.25}$$

where $\sigma \in \{-1, 1\}$.

By using relations (2.24) and (2.25), from (1.1$_\mu$) we get the inequalities

$$\begin{aligned} 0 < u_1(t) &\le \int_{t_1}^t [\sigma p_1(s)]_+ |u_2(s)|^{\lambda_1}\,ds \quad \text{for } t_1 < t \le t_2, \\ 0 < \sigma u_2(t) &\le \int_t^{t_2} [\sigma p_2(s)]_- |u_1(s)|^{\lambda_2}\,ds \quad \text{for } t_1 \le t < t_2. \end{aligned} \tag{2.26}$$

Let

$$\varrho_i := \max\big\{|u_i(t)| : t \in [t_1, t_2]\big\} \quad \text{for } i = 1, 2.$$

Clearly, $\varrho_1 > 0$ and $\varrho_2 > 0$.

If inequality (1.27) holds then, in view of relations (1.4), it follows from inequalities (2.26) the contradiction

$$\varrho_1 \le \int_{t_1}^{t_2} [\sigma p_1(s)]_+ \left(\int_s^{t_2} [\sigma p_2(\xi)]_- |u_1(\xi)|^{\lambda_2}\,d\xi \right)^{\lambda_1} ds \le \varrho_1 \int_{t_1}^{t_2} [\sigma p_1(s)]_+ \left(\int_s^{t_2} [\sigma p_2(\xi)]_-\,d\xi \right)^{\lambda_1} ds < \varrho_1.$$

If inequality (1.28) is satisfied then, by virtue of relations (1.4), from inequalities (2.26) we get

$$\varrho_2 \le \int_{t_1}^{t_2} [\sigma p_2(s)]_- \left(\int_{t_1}^{s} [\sigma p_1(\xi)]_+ |u_2(\xi)|^{\lambda_1}\,d\xi \right)^{\lambda_2} ds \le \varrho_2 \int_{t_1}^{t_2} [\sigma p_2(s)]_- \left(\int_{t_1}^{s} [\sigma p_1(\xi)]_+\,d\xi \right)^{\lambda_2} ds < \varrho_2,$$

which is a contradiction. The contradictions obtained prove the lemma. □

Lemma 2.10. *Let the functions α_1, α_2 be non-decreasing and conditions (1.4) and (1.26) hold. Moreover, let there exist numbers $\sigma \in \{-1, 1\}$ and $p_0 \ge 0$ such that inequalities (1.18) and (1.29) are satisfied, where the function ℓ is defined by formula (1.5). Then, for every $\mu \in]0, 1]$, problem (1.1$_\mu$), (1.3$_0$) has only the trivial solution.*

Proof. Assume that, on the contrary, (u_1, u_2) is a nontrivial solution to problem (1.1$_\mu$), (1.3$_0$) with some $\mu \in]0, 1]$. Since the functions α_1, α_2 are non-decreasing and satisfy inequalities (1.26), there exist $t_1 \in [a, a_0]$ and $t_2 \in [b_0, b]$ such that equalities (2.24) hold. Clearly, $t_1 < t_2$ because, in the contrary case, we obtain a contradiction to the assertion of Lemma 2.4. Therefore, Lemma 2.6 yields

$$\int_{t_1}^{t_2} |p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds > 0. \tag{2.27}$$

On the other hand, it follows from Lemmas 2.5 and 2.1 with $h(t) \equiv |p_1(t)|$ that equality (2.15) holds and

$$\ell(\lambda_2)\int_{t_1}^{t_2}|p_1(s)|\,|u_1(s)|^{1+\lambda_2}\,ds \le \left(\int_{t_1}^{t_2}|p_1(s)|\,ds\right)^{1+\lambda_2}\int_{t_1}^{t_2}|p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds, \tag{2.28}$$

where the function ℓ is defined by formula (1.5). By using relations (1.18), (2.15) and (2.28) we get

$$\begin{aligned}\int_{t_1}^{t_2}|p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds &= \int_{t_1}^{t_2}\big(-\sigma p_2(s)\big)|u_1(s)|^{1+\lambda_2}\,ds \le p_0\int_{t_1}^{t_2}|p_1(s)|\,|u_1(s)|^{1+\lambda_2}\,ds\\ &\le \frac{p_0}{\ell(\lambda_2)}\left(\int_a^b|p_1(s)|\,ds\right)^{1+\lambda_2}\int_{t_1}^{t_2}|p_1(s)|\,|u_2(s)|^{1+\lambda_1}\,ds,\end{aligned}$$

which, in view of (2.27), contradicts assumption (1.29). The contradiction obtained proves the lemma. □

2.5. *Lemmas on the solvability of problems (1.1), (1.2) and (1.1), (1.3)*

Along with problems (1.1), (1.2) and (1.1), (1.3) we consider the problems

$$\begin{aligned}\frac{du_1}{dt} &= (1-\delta)\sigma u_2 + \delta\Big[p_1(t)|u_2|^{\lambda_1}\,\mathrm{sgn}\,u_2 + q_1\big(t,u_1,u_2\big)\Big],\\ \frac{du_2}{dt} &= \delta\Big[p_2(t)|u_1|^{\lambda_2}\,\mathrm{sgn}\,u_1 + q_2\big(t,u_1,u_2\big)\Big],\end{aligned} \tag{2.29}$$

$$\int_a^{a_0}u_1(s)\,d\alpha_1(s) = \delta\gamma_1(u_1,u_2), \qquad \int_{b_0}^b u_1(s)\,d\alpha_2(s) = \delta\gamma_2(u_1,u_2) \tag{2.30}$$

with $\sigma \in \mathbb{R}$ and

$$\begin{aligned}\frac{du_1}{dt} &= \delta\Big[p_1(t)|u_2|^{\lambda_1}\,\mathrm{sgn}\,u_2 + q_1\big(t,u_1,u_2\big)\Big],\\ \frac{du_2}{dt} &= \delta\Big[p_2(t)|u_1|^{\lambda_2}\,\mathrm{sgn}\,u_1 + q_2\big(t,u_1,u_2\big)\Big],\end{aligned} \tag{2.31}$$

$$\int_a^{a_0}u_1(s)\,d\alpha_1(s) = \delta\gamma_1(u_1,u_2), \qquad \int_{b_0}^b u_2(s)\,d\alpha_2(s) = \delta\gamma_2(u_1,u_2) \tag{2.32}$$

depending on a parameter $\delta \in]0,1[$.

Lemma 2.11. *Let $a_0 < b_0$, the functions α_1, α_2 be non-decreasing and satisfy inequalities (1.13). Moreover, let there exist numbers $\sigma \in \{-1,1\}$ and $\varrho > 0$ such that, for any $\delta \in]0,1[$, every solution (u_1,u_2) to problem (2.29), (2.30) admits the estimate*

$$\|u_1\|_{\mathcal{C}} + \|u_2\|_{\mathcal{C}} \le \varrho. \tag{2.33}$$

Then problem (1.1), (1.2) has at least one solution.

Proof. According to [30, Corollary 2], in order to prove the lemma it is sufficient to show that, for every $\sigma \in \{-1,1\}$, the system

$$\frac{du_1}{dt} = \sigma u_2, \qquad \frac{du_2}{dt} = 0 \tag{2.34}$$

has only the trivial solution satisfying boundary conditions (1.2_0).

Indeed, let (u_1,u_2) be a solution to problem (2.34), (1.2_0) with some $\sigma \in \{-1,1\}$. Since the functions α_1, α_2 are non-decreasing and satisfy inequalities (1.13), there exist $t_1 \in [a,a_0]$ and $t_2 \in [b_0,b]$ such that $t_1 < t_2$ and equalities (2.10) are fulfilled. The integration of the first equation in (2.34) from t_1 to t_2 results in

$$\sigma\int_{t_1}^{t_2}u_2(s)\,ds = 0,$$

which guarantees that there is a point $t_0 \in [t_1,t_2]$ such that $u_2(t_0) = 0$. Consequently, (2.34) yields $u_2(t) \equiv 0$ and $u_1(t) \equiv 0$ as well. □

Lemma 2.12. *Let inequalities (1.13) hold and there exist a number $\varrho > 0$ such that, for any $\delta \in]0,1[$, every solution (u_1,u_2) to problem (2.31), (2.32) admits estimate (2.33). Then problem (1.1), (1.3) has at least one solution.*

Proof. The validity of the lemma follows immediately from the above-mentioned [30, Corollary 2] because it is clear that, in view of inequalities (1.13), the system

$$\frac{du_1}{dt} = 0, \qquad \frac{du_2}{dt} = 0$$

has only the trivial solution satisfying boundary conditions (1.3_0). □

3. Proofs of main results

Proof of Theorem 1.1. Assume that, on the contrary, there is no solution to problem (1.1), (1.2). Then, according to Lemma 2.11, there exist sequences $(u_{1n})_{n=1}^{+\infty}$, $(u_{2n})_{n=1}^{+\infty}$ of functions absolutely continuous on $[a, b]$ and a sequence $(\delta_n)_{n=1}^{+\infty}$ of numbers from the interval $]0, 1[$ such that the relations

$$\int_a^{a_0} u_{1n}(s)\,d\alpha_1(s) = \delta_n\gamma_1(u_{1n}, u_{2n}), \qquad \int_{b_0}^{b} u_{1n}(s)\,d\alpha_2(s) = \delta_n\gamma_2(u_{1n}, u_{2n}),$$

$$u'_{1n}(t) = (1-\delta_n)\sigma u_{2n}(t) + \delta_n p_1(t)|u_{2n}(t)|^{\lambda_1}\operatorname{sgn} u_{2n}(t) + \delta_n q_1\big(t, u_{1n}(t), u_{2n}(t)\big) \quad \text{for a.e. } t \in [a, b],$$

$$u'_{2n}(t) = \delta_n p_2(t)|u_{1n}(t)|^{\lambda_2}\operatorname{sgn} u_{1n}(t) + \delta_n q_2\big(t, u_{1n}(t), u_{2n}(t)\big) \quad \text{for a.e. } t \in [a, b],$$

and

$$\|u_{1n}\|_{\mathcal{C}} + \|u_{2n}\|_{\mathcal{C}}^{\lambda_1} \geq n \tag{3.1}$$

are satisfied for every $n \in \mathbb{N}$. Put

$$\varrho_n := \|u_{1n}\|_{\mathcal{C}} + \|u_{2n}\|_{\mathcal{C}}^{\lambda_1} \quad \text{for } n \in \mathbb{N} \tag{3.2}$$

and

$$z_{1n}(t) := \frac{u_{1n}(t)}{\varrho_n}, \qquad z_{2n}(t) := \frac{u_{2n}(t)}{\varrho_n^{\lambda_2}} \quad \text{for } t \in [a, b],\ n \in \mathbb{N}. \tag{3.3}$$

Then, for any $n \in \mathbb{N}$, we have

$$\|z_{1n}\|_{\mathcal{C}} + \|z_{2n}\|_{\mathcal{C}}^{\lambda_1} = 1, \tag{3.4}$$

$$\int_a^{a_0} z_{1n}(s)\,d\alpha_1(s) = \frac{\delta_n}{\varrho_n}\gamma_1(u_{1n}, u_{2n}), \qquad \int_{b_0}^{b} z_{1n}(s)\,d\alpha_2(s) = \frac{\delta_n}{\varrho_n}\gamma_2(u_{1n}, u_{2n}), \tag{3.5}$$

$$z'_{1n}(t) = (1-\delta_n)\sigma\frac{z_{2n}(t)}{\varrho_n^{1-\lambda_2}} + \delta_n p_1(t)|z_{2n}(t)|^{\lambda_1}\operatorname{sgn} z_{2n}(t) + \frac{\delta_n}{\varrho_n} q_1\big(t, u_{1n}(t), u_{2n}(t)\big) \quad \text{for a.e. } t \in [a, b], \tag{3.6}$$

and

$$z'_{2n}(t) = \delta_n p_2(t)|z_{1n}(t)|^{\lambda_2}\operatorname{sgn} z_{1n}(t) + \frac{\delta_n}{\varrho_n^{\lambda_2}} q_2\big(t, u_{1n}(t), u_{2n}(t)\big) \quad \text{for a.e. } t \in [a, b]. \tag{3.7}$$

By using relations (1.7), (1.12) and (3.2), we obtain

$$\frac{\delta_n}{\varrho_n}\left|q_1\big(t, u_{1n}(t), u_{2n}(t)\big)\right| \leq \frac{\delta_n}{\varrho_n} q^*(t, \varrho_n) \quad \text{for a.e. } t \in [a, b],\ n \in \mathbb{N} \tag{3.8}$$

and

$$\frac{\delta_n}{\varrho_n^{\lambda_2}}\left|q_2\big(t, u_{1n}(t), u_{2n}(t)\big)\right| \leq \frac{\delta_n}{\varrho_n^{\lambda_2}} q^*\big(t, \varrho_n^{\lambda_2}\big) \quad \text{for a.e. } t \in [a, b],\ n \in \mathbb{N}. \tag{3.9}$$

Therefore, in view of (1.12), (3.1), (3.2), (3.4), (3.8) and (3.9), the equalities (3.6) and (3.7) yield

$$|z_{1n}(t) - z_{1n}(s)| \leq \int_s^t (1 + |p_1(\xi)|)\,d\xi + \int_s^t \frac{q^*(\xi, \varrho_n)}{\varrho_n}\,d\xi \quad \text{for } a \leq s \leq t \leq b,\ n \in \mathbb{N} \tag{3.10}$$

and

$$|z_{2n}(t) - z_{2n}(s)| \leq \int_s^t |p_2(\xi)|\,d\xi + \int_s^t \frac{q^*\big(\xi, \varrho_n^{\lambda_2}\big)}{\varrho_n^{\lambda_2}}\,d\xi \quad \text{for } a \leq s \leq t \leq b,\ n \in \mathbb{N}. \tag{3.11}$$

Since we suppose that $\int_a^b \frac{q^*(s,\varrho)}{\varrho}\,ds \to 0$ as $\varrho \to +\infty$, it follows from (3.1), (3.2), and [31, Corollary IV.8.11] that, for any $\varepsilon > 0$, there exists $\omega > 0$ such that

$$\int_E \frac{q^*(s,\varrho_n)}{\varrho_n}\,ds < \varepsilon, \qquad \int_E \frac{q^*\big(s,\varrho_n^{\lambda_2}\big)}{\varrho_n^{\lambda_2}}\,ds < \varepsilon$$

for every $E \subseteq [a,b]$, mes $E < \omega$, and all $n \in \mathbb{N}$. Consequently, relations (3.4), (3.10) and (3.11) guarantee that the sequences $(z_{1n})_{n=1}^{+\infty}$ and $(z_{2n})_{n=1}^{+\infty}$ are uniformly bounded and equicontinuous. We can thus assume without loss of generality that there exist $z_1, z_2 \in \mathcal{C}$ and $\mu \in [0,1]$ such that

$$\lim_{n\to+\infty} \delta_n = \mu \tag{3.12}$$

and

$$\lim_{n\to+\infty} \|z_{1n} - z_1\|_{\mathcal{C}} = 0, \qquad \lim_{n\to+\infty} \|z_{2n} - z_2\|_{\mathcal{C}} = 0. \tag{3.13}$$

The integration of (3.6) and (3.7) from a to t implies

$$\begin{aligned} z_{1n}(t) &= z_{1n}(a) + \frac{(1-\delta_n)\sigma}{\varrho_n^{1-\lambda_2}} \int_a^t z_{2n}(s)\,ds + \delta_n \int_a^t p_1(s)|z_{2n}(s)|^{\lambda_1}\operatorname{sgn} z_{2n}(s)\,ds \\ &\quad + \frac{\delta_n}{\varrho_n}\int_a^t q_1\big(s,u_{1n}(s),u_{2n}(s)\big)\,ds \quad \text{for } t\in[a,b],\ n\in\mathbb{N} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} z_{2n}(t) &= z_{2n}(a) + \delta_n \int_a^t p_2(s)|z_{1n}(s)|^{\lambda_2}\operatorname{sgn} z_{1n}(s)\,ds \\ &\quad + \frac{\delta_n}{\varrho_n^{\lambda_2}}\int_a^t q_2\big(s,u_{1n}(s),u_{2n}(s)\big)\,ds \quad \text{for } t\in[a,b],\ n\in\mathbb{N}. \end{aligned} \tag{3.15}$$

Observe that, in view of (3.4), the relation

$$\frac{1-\delta_n}{\varrho_n^{1-\lambda_2}}\left|\int_a^t z_{2n}(s)\,ds\right| \le \frac{b-a}{\varrho_n^{1-\lambda_2}} \quad \text{for } t\in[a,b],\ n\in\mathbb{N}$$

holds. Therefore, by virtue of (1.9), (1.12), (3.1), (3.2), (3.8), (3.9), (3.12) and (3.13), we get from equalities (3.14) and (3.15) that

$$z_1(t) = z_1(a) + \mu\int_a^t p_1(s)|z_2(s)|^{\lambda_1}\operatorname{sgn} z_2(s)\,ds \quad \text{for } t\in[a,b]$$

and

$$z_2(t) = z_2(a) + \mu\int_a^t p_2(s)|z_1(s)|^{\lambda_2}\operatorname{sgn} z_1(s)\,ds \quad \text{for } t\in[a,b].$$

Consequently, the functions z_1 and z_2 are absolutely continuous and (z_1, z_2) is a solution to system (1.1_μ).

On the other hand, by using relations (1.8), (1.12) and (3.2), we obtain

$$\frac{\delta_n}{\varrho_n}\big|\gamma_k\big(u_{1n},u_{2n}\big)\big| \le \frac{\delta_n}{\varrho_n}\gamma_0^*(\varrho_n) \quad \text{for } n\in\mathbb{N},\ k=1,2 \tag{3.16}$$

and thus, in view of (1.9), (3.1), (3.2) and (3.13), it follows from (3.4) and (3.5) that

$$\|z_1\|_{\mathcal{C}} + \|z_2\|_{\mathcal{C}}^{\lambda_1} = 1 \tag{3.17}$$

and

$$\int_a^{a_0} z_1(s)\,d\alpha_1(s) = 0, \qquad \int_{b_0}^b z_1(s)\,d\alpha_2(s) = 0. \tag{3.18}$$

Thus we have shown that (z_1, z_2) is a nontrivial solution to problem (1.1_μ), (1.2_0). On the other hand, according to one of the conditions of the theorem, problem (1.1_μ), (1.2_0) has only the trivial solution for every $\mu\in]0,1]$. Therefore it is clear that $\mu = 0$.

Now, for any $n \in \mathbb{N}$, we choose $a_n \in [a, a_0]$ and $b_n \in [b_0, b]$ such that

$$|z_{1n}(a_n)| = \min\{|z_{1n}(t)| : t \in [a, a_0]\},\\ |z_{1n}(b_n)| = \min\{|z_{1n}(t)| : t \in [b_0, b]\} \tag{3.19}$$

and we find $c_n \in [a_n, b_n]$ with the property

$$|z_{2n}(c_n)| = \min\{|z_{2n}(t)| : t \in [a_n, b_n]\}. \tag{3.20}$$

Clearly, we can assume without loss of generality that

$$\lim_{n\to+\infty} c_n = c_0, \tag{3.21}$$

where $c_0 \in [a, b]$.

Since the functions α_1, α_2 are non-decreasing and satisfy inequalities (1.13), by virtue of relations (3.16), it follows from equalities (3.5) that

$$|z_{1n}(a_n)| \le \frac{\delta_n}{\alpha_1(a_0) - \alpha_1(a)} \frac{\gamma_0^*(\varrho_n)}{\varrho_n} \quad \text{for } n \in \mathbb{N} \tag{3.22}$$

and

$$|z_{1n}(b_n)| \le \frac{\delta_n}{\alpha_2(b) - \alpha_2(b_0)} \frac{\gamma_0^*(\varrho_n)}{\varrho_n} \quad \text{for } n \in \mathbb{N}. \tag{3.23}$$

The integration of equality (3.6) from a_n to b_n implies

$$\sigma z_{1n}(b_n) - \sigma z_{1n}(a_n) = \frac{1-\delta_n}{\varrho_n^{1-\lambda_2}} \int_{a_n}^{b_n} z_{2n}(s)\,\mathrm{d}s + \delta_n \int_{a_n}^{b_n} \sigma p_1(s)|z_{2n}(s)|^{\lambda_1} \operatorname{sgn} z_{2n}(s)\,\mathrm{d}s \\ + \frac{\delta_n \sigma}{\varrho_n} \int_{a_n}^{b_n} q_1\big(s, u_{1n}(s), u_{2n}(s)\big)\,\mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Therefore, by using (1.15), (3.8), (3.20), (3.22) and (3.23), we get

$$|z_{2n}(c_n)|^{\lambda_1} \int_{a_0}^{b_0} |p_1(s)|\,\mathrm{d}s \le |z_{2n}(c_n)|^{\lambda_1} \int_{a_n}^{b_n} \sigma p_1(s)\,\mathrm{d}s \\ \le \left(\frac{1}{\alpha_1(a_0) - \alpha_1(a)} + \frac{1}{\alpha_2(b) - \alpha_2(b_0)}\right) \frac{\gamma_0^*(\varrho_n)}{\varrho_n} + \frac{1}{\varrho_n} \int_a^b q^*(s, \varrho_n)\,\mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Consequently, by virtue of (1.9), (1.14), (3.1), (3.2), (3.13) and (3.21), letting $n \to +\infty$ in the last inequality gives

$$z_2(c_0) = 0. \tag{3.24}$$

On the other hand, since the function z_1 satisfies (3.18) and the function α_1 is non-decreasing with the property (1.13), there exists $t_0 \in [a, a_0]$ such that

$$z_1(t_0) = 0. \tag{3.25}$$

As we have proved above, (z_1, z_2) is a solution to the system

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}z_2}{\mathrm{d}t} = 0$$

and thus, in view of (3.24) and (3.25), we obtain $z_1(t) \equiv 0$ and $z_2(t) \equiv 0$, which contradicts equality (3.17). □

Proof of Theorem 1.2. Assume that, on the contrary, there is no solution to problem (1.1), (1.3). Then, according to Lemma 2.12, there exist sequences $(u_{1n})_{n=1}^{+\infty}$, $(u_{2n})_{n=1}^{+\infty}$ of functions absolutely continuous on $[a, b]$ and a sequence $(\delta_n)_{n=1}^{+\infty}$ of numbers from the interval $]0, 1[$ such that the relations

$$\int_a^{a_0} u_{1n}(s)\,\mathrm{d}\alpha_1(s) = \delta_n \gamma_1(u_{1n}, u_{2n}), \qquad \int_{b_0}^{b} u_{2n}(s)\,\mathrm{d}\alpha_2(s) = \delta_n \gamma_2(u_{1n}, u_{2n}),$$

$$u_{1n}'(t) = \delta_n p_1(t)|u_{2n}(t)|^{\lambda_1} \operatorname{sgn} u_{2n}(t) + \delta_n q_1\big(t, u_{1n}(t), u_{2n}(t)\big) \quad \text{for a.e. } t \in [a, b],\\ u_{2n}'(t) = \delta_n p_2(t)|u_{1n}(t)|^{\lambda_2} \operatorname{sgn} u_{1n}(t) + \delta_n q_2\big(t, u_{1n}(t), u_{2n}(t)\big) \quad \text{for a.e. } t \in [a, b],$$

and

$$\|u_{1n}\|_{\mathcal{C}} + \|u_{2n}\|_{\mathcal{C}}^{\lambda_1} \ge n$$

are satisfied for every $n \in \mathbb{N}$. Define numbers ϱ_n ($n \in \mathbb{N}$) by formula (3.2) and functions z_{1n}, z_{2n} ($n \in \mathbb{N}$) by equalities (3.3). Following similar steps as in the proof of Theorem 1.1 we construct a nontrivial solution to problem (1.1_μ), (1.3_0), where $\mu \in [0, 1]$. Consequently, according to one of the assumptions of the theorem, we have $\mu = 0$, which is a contradiction because, in view of inequalities (1.13), it clear that problem (1.1_0), (1.3_0) has only the trivial solution. □

Proof of Theorem 1.3–1.7. Theorems 1.3 and 1.4 follow from Theorem 1.1 and Lemma 2.7. Theorem 1.5 follows from Theorem 1.2 and Lemma 2.8. Theorem 1.6 (respectively, Theorem 1.7) follow from Theorem 1.2 and Lemma 2.9 (respectively, Lemma 2.10). □

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