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Entanglement of spins under a strong laser influence

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Abstract

The fate of entanglement of spins for two heavy constituents of a bound state moving in a strong laser field is analyzed within the semiclassical approach. The bound state motion as a whole is considered classically beyond the dipole approximation and taking into account the magnetic field effect by using the exact solution to the Newton equation. At the same time the evolution of constituent spins under the laser influence is studied quantum mechanically. The spin density matrix is determined as a solution to the von Neumann equation with the effective Hamiltonian, describing spin–laser interaction along the bound state classical trajectory. Based on the solution, the dynamics of concurrence of spins is calculated for the maximally entangled Werner states as well as for an initially uncorrelated state.

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1. Introduction

The entanglement of quantum states is a fundamental resource in quantum communication and computation [1]. In spite of the enormous progress in theoretical and experimental studies, the problem of effective control over entanglement and other quantum characteristics of multi-particle states as yet remains unsolved. Over the last two decades, special attention was devoted to models based on the usage of a coherent laser radiation. In most cases, for instance in the Cirac–Zoller model of quantum computation with cold trapped ions [2], the laser frequency and polarization play a role in the relevant control parameters. However, the recent impressive success in the construction of high-intensity lasers [3] offers an alternative control parameter, the *intensity of the coherent electromagnetic radiation*. Indeed, entering the regime of a high intensity, a variety of new physical phenomena have been discovered (see, e.g., [4] and references therein). Very interesting expositions on a strong laser influence on spins of single particles [5] as well on atoms [6] have been predicted. Evidently, it deserves attention to investigate whether the new effects due to the laser intensity can be used for the purposes of quantum engineering.

Moving toward this direction, it is worth emphasizing that the entanglement description in the presence of a strong laser

radiation is a highly complex problem owing to the following circumstances:

- (a) The dynamics of a non-relativistic charged particle driven by a low-intensity laser is fully determined by the electric component of electromagnetic field. The electric field dominance together with the dipole approximation provides a consistent solution to the equation of motion for the classical trajectory of a charged particle [7] in a way not affecting its spin dynamics [8]. This scheme works perfectly well for various applications of low-intensity lasers [9]. With a laser intensity large enough, the relativistic corrections to the electric charge motion become unavoidable. This requires one to abandon the electric dipole approximation and take into account the magnetic field influence [10–13] on classical trajectory as well as on the particle spin evolution.
- (b) Since in dealing with a high-intensity laser the relativistic description is inevitable, it is necessary to extend the conventional non-relativistic quantum formalism for the entanglement phenomena to the relativistic case. Today, in spite of the undertaken efforts, we are still far from having a complete picture of the relativistic entanglement.

In subsequent sections, taking into account these observations, we adopt the conventional semiclassical attitude to the dynamics of charged particles in an electromagnetic background. The semiclassical motion of a binary bound state driven by a high-intensity monochromatic and elliptically polarized laser radiation is modeled and studied in the context of dynamics of entanglement. We address the question of how the intensity of a laser can affect the correlations between constituent spins. The analysis of the entanglement in a strong laser background presented here was developed out of the recent studies [14, 15].

In the spirit of the Wentzel–Kramers–Brillouin (WKB) approximation, the binary composed system motion as a whole is studied classically, assuming that the back reaction of spin of the constituents on the particle dynamics is negligible (see this approximation in [16, 17]). Our analysis is based on the exact solution to the Hamilton–Jacobi equations for the trajectory of a charged particle interacting with the electromagnetic plane wave [14]. In doing so, the spin evolution will be treated quantum mechanically, as required by a spin nature, using the von Neumann equation for the spins density matrix with the leading relativistic corrections included. Furthermore, the spin–radiation interaction is encoded in the effective spatially homogeneous Hamiltonian which is determined solely by the bound state classical trajectory (see discussions on this approximation in [5]).

2. Semiclassical model for strong laser–bound state interaction

Consider a massive (M_B), electrically charged ($-q_B$) bound state composed of two charged ($e^{(n)}, e^{(p)}$), massive ($m^{(n)}, m^{(p)}$), spin-1/2 particles interacting with a laser radiation modeled by the monochromatic plane wave propagating along the z -axis:

$$A(t, \mathbf{x}) := a \left(\varepsilon \cos(\omega_L \xi), \sqrt{1 - \varepsilon^2} \sin(\omega_L \xi), 0 \right), \quad \xi = t - \frac{z}{c}. \quad (1)$$

In (1) the parameter $\varepsilon \in [0, 1]$ denotes the light polarization and ω_L is the wave frequency. The constant a determines the dimensionless laser field strength parameter [18, 19]

$$\eta^2 = \frac{q_B^2 a^2}{M_B^2 c^4},$$

setting the scale for the intensity of a laser–bound state interaction.

The semiclassical picture is mathematically formulated as follows. Let us divide all configuration variables into three parts: the center-mass coordinate

$$\mathbf{R} = (m^{(n)} \mathbf{r}_n + m^{(p)} \mathbf{r}_p) / M_B,$$

the relative coordinate between constituents $\mathbf{r} = \mathbf{r}_n - \mathbf{r}_p$ and their spin variables. Correspondingly, the Hilbert space \mathcal{H} is decomposed as

$$\mathcal{H} = \mathcal{H}_{CM} \otimes \mathcal{H}_{RM} \otimes \mathcal{H}_{SPIN}.$$

The dynamics on \mathcal{H} is assumed to be driven by the following interactions:

- The electric charge of the bound state has a point-like distribution, peaked at the position of its center of mass \mathbf{R} with a ‘point-like’ part of the laser–charge interaction V_{CL} described by the conventional radiation scattering of the electric charge $-q_B$, moving with the velocity $\mathbf{v}_R = d\mathbf{R}/dt$

$$V_{CL} := \frac{q_B}{c} \mathbf{v}_R \cdot \mathbf{A}(t, \mathbf{R}).$$

- The degrees of freedom of the constituents evolve in time and interact with each other (V_B) as well as with a laser radiation (V_{SL})

$$V_B = V_0(r) + V_{SS}(r), \quad V_{SS}(r) := V_S(r) \mathbf{S} \otimes \mathbf{S},$$

where $V_0(r)$ and $V_S(r)$ are scalar functions of the relative distance $r = |\mathbf{r}_n - \mathbf{r}_p|$ between constituents, and the Pauli matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are used to describe the spin of constituents, $\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\sigma}$. The spin–laser coupling V_{SL} is determined by the magnetic moments of constituents in the relativistically modified Larmor form

$$V_{SL} := -\boldsymbol{\Omega}^{(n)}(t, \mathbf{r}_n) \cdot \mathbf{S}^{(n)} - \boldsymbol{\Omega}^{(p)}(t, \mathbf{r}_p) \cdot \mathbf{S}^{(p)},$$

where the vector $\boldsymbol{\Omega}^{(i)}$ reads

$$\boldsymbol{\Omega}^{(i)} := \frac{e^{(i)} g^{(i)}}{2 m^{(i)} c} \left(\mathbf{B} - \frac{1}{c} [\mathbf{v}^{(i)} \times \mathbf{E}] \right) + \frac{1}{2 c^2} [\mathbf{v}^{(i)} \times \mathbf{a}^{(i)}]. \quad (2)$$

\mathbf{E} and \mathbf{B} in (2) are, respectively, the electric and magnetic components of a laser field evaluated along the trajectory of the i th particle (with the gyromagnetic ratio $g^{(i)}$) moving with the velocity $\mathbf{v}^{(i)}$ and acceleration $\mathbf{a}^{(i)}$ seen in the laboratory frame. The term in parentheses is the magnetic field in the instantaneous rest frame of a charged particle (Galilei boosted), while the last contribution in (2) is the leading relativistic Thomas precession correction [20] due to the non-vanishing curvature of the particle trajectory, see, e.g., [7].

Gathering all the above together, the evolution of a bound state traveling in the laser background is governed by the total Hamiltonian

$$H = H_0 + V_{SS} + V_{CL} + V_{SL},$$

where H_0 is the Hamiltonian of free spinless constituents.

3. The terse summary of computation

3.1. Evolution of center of mass motion of bound state

In the leading semiclassical approximation the contribution to the phase of wave function that comes from the laser–spin interaction term V_{SL} is negligibly small. This term will come into play later on, when we turn to a study of dynamics of spin degrees. Therefore the density matrix of our system admits the *charge and spin decomposition*

$$\rho = \sum_{\alpha=\pm} c_\alpha |\psi_\alpha\rangle \otimes \varrho_\alpha, \quad (3)$$

where two states $|\psi_{\pm}\rangle$ are linearly independent WKB solutions to the Schrödinger equation with the Hamiltonian $H_0 + V_{SS} + V_{CL}$ and ϱ_{α} is the density matrix of constituent spins. This Hamiltonian admits separation of the relative and absolute motion, and for the last, one can use *the exact solution* [14] to the analogous Hamilton–Jacobi problem for a point-like charged particle. According to [14], the Hamilton–Jacobi generating function for a spinless particle traveling in an arbitrary plane wave background of the form $A_{\mu} := (0, \mathbf{A}_{\perp}(\xi), 0)$ reads

$$\mathcal{F}(\xi, \mathbf{\Pi}) = -c(mc - \Pi_z)\xi + c \times \int_0^{\xi} du \sqrt{(mc - \Pi_z)^2 + W(u, \mathbf{\Pi}_{\perp})}, \quad (4)$$

where

$$W(\xi, \mathbf{\Pi}_{\perp}) := -\frac{e^2}{c^2} \mathbf{A}_{\perp}^2 + 2\frac{e}{c} \mathbf{A}_{\perp} \cdot \mathbf{\Pi}_{\perp}.$$

The constants Π_z and Π_{\perp} are determined from the initial value of the particle velocity. With the aid of (4) the standard calculations give the leading semiclassical wave function

$$\langle \mathbf{x}, t | \psi_{\pm} \rangle = \frac{1}{\sqrt{|\partial\mathcal{F}/\partial\xi|}} e^{\frac{i}{\hbar} \mathbf{\Pi}_{\perp} \cdot \mathbf{x}_{\perp}} e^{\frac{i}{\hbar} \varepsilon t} \exp\left\{\frac{i}{\hbar} \mathcal{F}(\xi, \mathbf{\Pi}_{\perp})\right\}.$$

To make formulae more compact, let us impose the initial condition on the classical trajectory $\mathbf{R}(t=0) = 0$ and fix the frame, where the time-average value of the component of particle velocity orthogonal to the electromagnetic wave propagation direction vanishes, $\langle \langle v_{\perp} \rangle \rangle = 0$. From the generating function (4), it follows that the bound state center of mass moves along the trajectory:

$$R_x(t) = -\frac{c}{\omega_L} \sqrt{\frac{\varepsilon^2}{1-2\varepsilon^2}} \arcsin[\mu \operatorname{sn}(u, \mu)], \quad (5)$$

$$R_y(t) = \frac{c}{\omega_L} \sqrt{\frac{1-\varepsilon^2}{1-2\varepsilon^2}} \ln \left[\frac{\mu \operatorname{cn}(u, \mu) + \operatorname{dn}(u, \mu)}{1+\mu} \right], \quad (6)$$

$$R_z(t) = ct - \frac{c}{\omega_L} \operatorname{am}(u, \mu). \quad (7)$$

The trajectory (5)–(7) is expressed in terms of the Jacobian elliptic functions $\operatorname{sn}(u, \mu)$, $\operatorname{cn}(u, \mu)$, $\operatorname{dn}(u, \mu)$ and the amplitude function $\operatorname{am}(u, \mu)$ [21]. The argument $u := \omega'_L t$ of these functions is the laboratory frame time t scaled by the laser frequency $\omega'_L = \gamma_z \omega_L$ non-relativistically Doppler shifted by $\gamma_z = 1 - v_z(0)/c$. The modulus μ is determined by the laser and the particle characteristics

$$\gamma_z^2 \mu^2 = (1 - 2\varepsilon^2) \eta^2.$$

In (5)–(7) the modulus belongs to the fundamental domain $0 < \mu^2 < 1$. The solution outside this interval can be reconstructed from it by using the modular properties of the Jacobian functions. For the corresponding details, see [14].

3.2. The evolution of spin degrees

The semiclassical calculations imply that the spin density matrix ϱ in the decomposition (3) satisfies the *spin evolution equation* written in the form of the von Neumann equation

$$\dot{\varrho}(t) = -\frac{i}{\hbar} [H_S(t), \varrho(t)]. \quad (8)$$

The effective spin Hamiltonian H_S is defined as the projection of the Hamiltonian $V_{SS} + V_{SL}$ to the classical trajectory of the constituent particles:

$$H_S(t) = V_{SS} + V_{SL} \Big|_{\text{Particles classical trajectory}}. \quad (9)$$

To evaluate (9), we follow the spirit of the Born–Oppenheimer approximation [22]. Namely, we ‘freeze’ the relative motion of constituents inside the bound state, i.e. approximate their relative trajectory by the mean value $\langle r(t) \rangle = 0$ and neglect all contributions of order v_r/c , where v_r is the relative velocity of constituents. A straightforward evaluation of the effective laser–spin Hamiltonian (9) gives

$$H_S = -\mathfrak{B}^{(n)}(t) \cdot \mathbf{S} \otimes I - I \otimes \mathbf{S} \cdot \mathfrak{B}^{(p)}(t) + H_I, \quad (10)$$

where $\mathfrak{B}^{(i)}(t)$, $i = (n, p)$, for $\tilde{g}^{(i)} = (e^{(i)}/m^{(i)}) (M_B/q_B) g^{(i)}$ read

$$\begin{aligned} \mathfrak{B}_x^{(i)}(t) &= \eta \frac{\omega'_L}{2} \sqrt{1-\varepsilon^2} [(\tilde{g}^{(i)} + 1) \operatorname{dn}(u, \mu) - \gamma_z] \operatorname{cn}(u, \mu), \\ \mathfrak{B}_y^{(i)}(t) &= \eta \frac{\omega'_L}{2} \varepsilon [(\tilde{g}^{(i)} + 1) \operatorname{dn}(u, \mu) - \gamma_z (1 - \mu^2)] \operatorname{sn}(u, \mu), \\ \mathfrak{B}_z^{(i)}(t) &= -\eta^2 \frac{\omega'_L}{2} \varepsilon \sqrt{1-\varepsilon^2} [\tilde{g}^{(i)} - \gamma_z \operatorname{dn}(u, \mu)]. \end{aligned}$$

Similarly, the spin–spin interaction term H_I in (10) originates from V_{SS} under the same static approximation for the spatial relative degrees of freedom:

$$\hbar H_I = g \mathbf{S} \otimes \mathbf{S}. \quad (11)$$

The constant g in (11) is determined by the spin–spin potential evaluated at the mean value of the relative distance between the constituents, $g := \hbar V_S(0)$.

4. Dynamics of entanglement

Now one can analyze the dynamics of entanglement under the environment coupling [23] realized in our model by a background laser radiation. We postpone for a future analysis a generic case and consider the dynamics of density matrices of a special type only.

4.1. Werner states.

Consider first a family of entangled mixed states, the so-called Werner states [24], characterized by a single real parameter p that measures the overlap of a given Werner state with the maximally entangled pure Bell state

$$\varrho_W := \frac{1}{4} (\mathbf{I} + p \boldsymbol{\sigma} \otimes \boldsymbol{\sigma}). \quad (12)$$

For $\frac{1}{3} < p \leq 1$ the density matrix (12) describes the mixed entangled state.

To find the fate of the entanglement of the initial Werner state, we use the expression for the evolution operator $U(t) = X(t)W(t)$ given in the appendix. Since the entanglement properties are invariant under the local unitary transformation of the form $W(t) = U_{(n)}(t) \otimes U_{(p)}(t)$, only the action of the operator $X(t)$ may affect the entanglement. With this observation one can easily evaluate the leading, in a laser intensity, change of the density matrix

$$\delta_t \varrho_W = \frac{i}{\hbar} [V_I, \varrho_W].$$

This gives the expression

$$\delta_t \varrho_W = -\frac{1}{2} g \eta p \Delta \sin(\omega_L t) \left[\cos(4gt) \sigma_{[30]} + \frac{3}{4} \sin(4gt) \sigma_{[12]} \right],$$

where

$$\Delta = \tilde{g}^{(n)} - \tilde{g}^{(p)},$$

$$\sigma_{[\mu\nu]} := \sigma_\mu \otimes \sigma_\nu - \sigma_\nu \otimes \sigma_\mu (\mu, \nu = 0, 1, 2, 3),$$

and σ_0 is the unit 2×2 matrix. As a result, one can obtain that in the leading order in a laser intensity the concurrence is stable under the influence of a laser background:

$$C(\varrho_W) = \max \left(0, \frac{3p-1}{2} \right).$$

4.2. Initially uncorrelated spins

The same strategy can be applied to the initially uncorrelated spin state

$$\varrho_0 = \frac{1}{4} \left(I + \alpha \frac{1}{2} (\sigma_{03} + \sigma_{30}) + \beta \frac{1}{2} (\sigma_{03} - \sigma_{30}) \right).$$

Our calculations show that the concurrence, in leading order in η , is

$$C(\varrho_0) = \max \left(0, 4\eta |\beta g \Delta Q(t) - \sqrt{1-\alpha^2}| \right), \quad (13)$$

where

$$Q(t) = \frac{1}{\omega_L + 4g} \sin^2(\omega_L/2 + 2g)t$$

$$+ \frac{1}{\omega_L - 4g} \sin^2(\omega_L/2 - 2g)t.$$

This example demonstrates the possibility of formation of the entanglement solely due to the effects of a laser field intensity. Note that, remaining within the dipole approximation and ignoring the intensity influence on the spin dynamics, we will obtain the vanishing concurrence rather than the above derived result (13).

5. Concluding remarks

In the present paper aimed to understand the dynamics of entanglement under a strong field influence, we formulated

a model for the bound state composed of two heavy charged spin-1/2 particles traveling in a laser field. The relative motion of constituents was treated within the Born–Oppenheimer method [22] and the semiclassical approximation has been used to find the evolution operator. Our studies show the following:

- The entanglement between constituents with different gyromagnetic ratios evolves in a manner strongly dependent on the intensity of a laser beam as well as on the coupling between spins.
- There are cases when the entanglement properties reveal stability in leading order. This happens particularly for the Werner states of the spin density matrix.
- There is a possibility to attain the entanglement manipulating with the laser intensity. In particular, in our model uncorrelated initial state evolving in laser field acquires non-trivial concurrence merely due to the strong laser intensity.

The last observation may have an impact on very interesting and promising studies of the nonlinear effects in spin–laser interactions. It may open an alternative way for the manipulation with entanglement of spins using a laser intensity as a control parameter.

It is worth noting that the methods suggested in this paper are adequate and well adapted to description of processes in the transition from the non-relativistic to relativistic regime only. For the completeness of the studies, the fully relativistic treatment is necessary. Today, its elaboration remains an open and intriguing research area.

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Appendix

Consider (8) for spins evolving in the linearly polarized plane wave ($\varepsilon = 0$). The spin density matrix ϱ , obeying a certain initial condition at $t = 0$, is given by the unitary evolution operator U

$$\varrho(t) = U(t)\varrho(0)U^\dagger(t).$$

It is convenient to pass to the ‘interaction picture’ $U(t) = W(t)X(t)$, i.e. factor out from U the operator $W(t) := U^{(n)}(t) \otimes U^{(p)}(t)$ that describes the dynamics of spins, (n) and (p), in a laser background, but not interacting with each other while precessing. The unknown operator $X(t)$ is subject to the equation

$$\dot{X}(t) = -\frac{i}{\hbar} H'_I(t) X(t) \quad (A.1)$$

with the Hamiltonian (11) written in the ‘interaction picture’

$$H_I'(t) := W^+ H_I W = g\hbar (\cos \vartheta_- \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} + \sin \vartheta_- \sigma_{[12]}), \quad (\text{A.2})$$

where

$$\vartheta_-(t) := \vartheta^{(n)} - \vartheta^{(p)} = \frac{1}{2} \eta \Delta \operatorname{sn}(u, \mu).$$

To obtain (A.2) the expression $U^{(i)}(t) = \exp\left(\frac{i}{2} \vartheta^{(i)}(t) \sigma_1\right)$ with

$$2\vartheta^{(i)}(t) = \eta (\tilde{g}^{(i)} + 1) \operatorname{sn}(u, \mu) - \arcsin(\mu \operatorname{sn}(u, \mu))$$

has been used. Note that $\vartheta^{(i)}(t)$ depends nonlinearly on the laser intensity η and only for small intensities $\eta \ll 1$ reduces to the well-known expression for the angle characterizing the non-relativistic precession

$$2\vartheta_{\text{NR}} = \eta \tilde{g}^{(n)} \sin(\omega_L t).$$

The solution to (A.1) is expressed in terms of the time-ordered exponent

$$X(t) = e^{ig\psi(t)\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} T \left(\exp \frac{i}{\hbar} \int_0^t V_I(\tau) d\tau \right).$$

Here

$$\psi(t) = \int_0^t dt \cos \vartheta_-(t)$$

and

$$V_I(t) := g\hbar \sin \vartheta_- \left[\cos(4g\psi(t)) \sigma_{[12]} + \frac{3}{4} \sin(4g\psi(t)) \sigma_{[30]} \right].$$

For the first factor in $X(t)$ one can use the remarkable Eulerian representation

$$e^{i\psi \boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} = \frac{1}{2} e^{i\psi} + \frac{1}{2} e^{-i\psi} \left[\cos 2\psi + i \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \sin 2\psi \right].$$

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