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SU(6) CASIMIR INVARIANTS AND SU(2) ⊗ SU(3) SCALARS FOR A MIXED QUBIT-QUTRIT STATE

V. Gerdt,* D. Mladenov, Yu. Palii,† and A. Khvedelidze‡** UDC 517.986

In the present paper, a few steps are undertaken towards the description of the "qubit–qutrit" pair – a quantum bipartite system composed of two- and three-level subsystems. Calculations of the Molien functions and Poincaré series for the qubit-qubit and qubit-qutrit "local unitary invariants" are outlined and compared with the known results. The requirement of the positive semi-definiteness of the density operator is formulated explicitly as a set of inequalities in five Casimir invariants of the enveloping algebra $\mathfrak{su}(6)$. *Bibliography: 26 titles.*

1. Introduction

The present article discusses several computational aspects of pure quantum effects in composite systems valuable for the modern theory of quantum computing and quantum information [1, 2].

The cornerstone of these latest trends is an extraordinary quantum phenomenon – the "*entanglement*" of quantum states. Basically, by entanglement one means the occurrence of diverse nonlocal correlations in a composite multipartite quantum system, which have no classical analog. From the mathematical point of view, characteristics of entanglement can be understood within the classical theory of invariants (cf. [3, 4]). The central object in these studies is the ring of G-invariant polynomials in the elements of density matrices, with the group G consisting of the so-called *local unitary transformations* acting separately on each part of a multipartite composite system. The program of describing this ring for multipartite mixed states was outlined in [5], and during the last decade it has been intensively developed. Many interesting physical and pure mathematical results have been obtained. In particular, for the simplest bipartite system of two qubits, the structure of the corresponding ring has been clarified (see, e.g., [6–8]). However, comparatively little is known for multipartite states, as well as for bipartite mixed states composed of arbitrary d-level subsystems, i.e., for so-called qudits [9, 10]. The main reason is the great computational difficulty we are faced with. Indeed, even when dealing with a 3-level subsystem, a qutrit, a large number of independent elements of the density matrix leads to a wide variety of local polynomial invariants and makes the direct usage of known computer algebra packages noneffective.

Below, attempting to construct the polynomial ring of invariants for the qubit-qutrit pair, we obtained additional evidence of the complexity of the problem. The known results [23] and our calculation of the Molien functions and Poincaré series for the qubit-qutrit show that the number of local invariants increases significantly compared with the case of two qubits. Nevertheless, the obtained information is very useful for the analysis of the ring of $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ invariants. As a preliminary result, here we present a set of linearly independent $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ invariant polynomials up to the fourth order constructed via trace operation from noncommutative monomials in three elements of a special decomposition of the qubit-qutrit density matrix. Using the subset of $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ invariant polynomials consisting of the Casimir invariants of the enveloping algebra $\mathfrak{U}(\mathfrak{su}(6))$, the positive semi-definiteness of the density matrix of the qubit-qutrit pair is derived in the form of a system of algebraic inequalities.

2. The SU(n) Casimir invariants

Here we present basic facts about the unitary symmetry of quantum mechanics and its role in the description of composite multipartite states.

The density operator and SU(n)-invariants

According to conventional quantum theory, the complete information on a generic n-dimensional system is accumulated in the self-adjoint positive semi-definite density operator ϱ with unit trace, $\varrho \in \mathfrak{P}_+$. For a closed

*Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna, Russia, e-mail: gerdt@jinr.ru.

**Department of Physics, Sofia State University, Sofia, Bulgaria, e-mail: dimitar.mladenov@phys.uni-sofia.bg.

†Institute of Applied Physics, Chisinau, Moldova, e-mail: palii@jinr.ru.

‡Department of Theoretical Physics, A. Razmadze Mathematical Institute, Tbilisi, Georgia, e-mail: akhved@jinr.ru.

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quantum system, this description is highly redundant; the equivalence relation on $\mathfrak{P}_+$, due to the invariance of observables under the adjoint action of the group $\mathrm{SU}(n)$,

$$(\mathrm{Ad}\ g)\varrho = g\,\varrho\,g^{-1}, \quad g \in \mathrm{SU}(n), \tag{2.1}$$

guarantees that the physically relevant information about quantum states can be extracted from the orbit space $\mathfrak{P}_+ \,|\, \mathrm{SU}(n)$.[1] Relaxing for a moment the semi-definiteness condition, the density operator ϱ can be expressed via the Lie algebra $\mathfrak{su}(n)$ of $\mathrm{SU}(n)$ [11]:

$$\varrho = \frac{1}{n}\,\mathbb{I}_n + \tilde{\kappa}\,\imath\,\mathfrak{g}, \quad \mathfrak{g} \in \mathfrak{su}(n), \quad \imath^2 = -1, \tag{2.2}$$

with some normalization factor $\tilde{\kappa}$. Therefore, the density operator can be decomposed in terms of $n^2 - 1$ basis elements, e_i, of the Lie algebra $\mathfrak{su}(n)$,

$$\mathfrak{g} = \sum_{i=1}^{n^2-1} \xi_i\, e_i, \tag{2.3}$$

and any other operator $\mathcal{A}[\varrho]$ constructed from the density operator ρ admits a representation as a graded power series:

$$\mathcal{A}(\mathbf{e}) = A^{(0)}\,\mathrm{I} + A^{(1)}_i\, e_i + \frac{1}{2!}A^{(2)}_{ij}\, e_i e_j + \frac{1}{3!}A^{(3)}_{ijk}\, e_i e_j e_k + \dots\,. \tag{2.4}$$

According to the Poincaré–Birkhoff–Witt theorem [12], the ordered monomials

$$e_0 = 1, \quad e_{i_1 i_2 \cdots i_k} = e_{i_1} e_{i_2}\,\dots\, e_{i_k}, \quad e_{i_1} < e_{i_2} < \cdots < e_{i_k}, \tag{2.5}$$

form a linear basis of the universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(n))$ of $\mathfrak{su}(n)$. A direct corollary of this theorem is that the symmetrized monomials of degree d in (2.4) span a linear space $\mathfrak{U}^d(\mathfrak{su}(n))$ and the universal enveloping algebra

$$\mathfrak{U}(\mathfrak{su}(n)) = \bigoplus_{d=0}^{\infty} \mathfrak{U}^d(\mathfrak{su}(n))$$

is isomorphic as a linear space to the polynomial algebra in commutative real variables ξ_i , $i = 1, \dots, n^2 - 1$.

Furthermore, according to the well-known Gelfand's theorem [13], the description of the center $\mathcal{Z}(\mathfrak{su}(n))$ of the enveloping algebra $\mathfrak{U}(\mathfrak{su}(n))$ reduces to the study of invariants in the commutative symmetrized algebra $S(\mathfrak{su}(n))$, which is isomorphic to the algebra of invariant polynomials over $\mathfrak{su}(n)$. The elements of the center $\mathcal{Z}(\mathfrak{su}(n))$ are in a one-to-one correspondence with the $\mathrm{SU}(n)$-invariant polynomials in $n^2 - 1$ real variables, coordinates in $\mathfrak{su}(n)$. More precisely, an element

$$\mathfrak{C}_r = \sum \frac{1}{r!}\, c_{i_1 \cdots i_r} \sum_{\sigma \in S_r} e_{i_{\sigma(1)}} e_{i_{\sigma(2)}}\, \dots\, e_{i_{\sigma(r)}} \tag{2.6}$$

of $\mathfrak{U}(\mathfrak{su}(n))$, where S_r is the group of permutations of $1, 2, \dots, r$, belongs to $\mathcal{Z}(\mathfrak{su}(n))$ if and only if $c_{i_1 \cdots i_r}$ are the coefficients of a polynomial

$$\phi(\xi_1, \xi_2, \dots, \xi_r) = \sum c_{i_1 \cdots i_r}\, \xi_{i_1} \xi_{i_2} \dots \xi_{i_r} \tag{2.7}$$

in variables $\xi_1, \xi_2, \dots, \xi_r$ that is invariant under the adjoint action

$$\phi(\xi_1, \xi_2, \dots, \xi_r) = \phi((\mathrm{Ad}\ g)^T \xi_1, (\mathrm{Ad}\ g)^T \xi_2, \dots (\mathrm{Ad}\ g)^T \xi_r), \tag{2.8}$$

with $(\mathrm{Ad}\ g)^T$, the matrix of the adjoint operator $\mathrm{Ad}\ g$, calculated in the basis $e_{i_1}, e_{i_2}\, \dots, e_{i_r}$.

Therefore, from the algebraic point of view, the study of the orbit space $\mathfrak{P}_+ \,|\, \mathrm{SU}(n)$, as well as of any characteristic of quantum-mechanical observables invariant under the unitary action (2.1), reduces to the computation of the center $\mathcal{Z}(\mathfrak{su}(n))$ of $\mathfrak{U}(\mathfrak{su}(n))$.

[1] The orbit space $\mathfrak{P}_+ \,|\, \mathrm{SU}(n)$ of $\mathrm{SU}(n)$ is defined as the set of all $\mathrm{SU}(n)$-orbits endowed with the quotient topology and differentiable structure, and the subset of all $\mathrm{SU}(n)$-orbits with the same orbit type forms a stratum of $\mathfrak{P}_+ \,|\, \mathrm{SU}(n)$.

The elements $\mathfrak{C}_r$ belonging to the center are called Casimir operators. The number of independent homogeneous Casimir generators for $\mathrm{SU}(n)$ is equal to $\mathrm{rank}\,\mathfrak{su}(n) = n-1$.

It is well known that a quadratic Casimir operator is unique up to a constant factor and is expressible in terms of the Cartan tensor:

$$C_{ij} = \mathrm{tr}((\mathrm{Ad}\ e_i)(\mathrm{Ad}\ e_j)). \tag{2.9}$$

Therefore, for the algebra $\mathfrak{su}(\mathrm{n})$ the quadratic Casimir operator reads as

$$\mathfrak{C}_2 = \sum e_i e_i. \tag{2.10}$$

The higher-dimensional Casimirs can be expressed in terms of the symmetric structure constants d_{ijk} of $\mathfrak{su}(n)$ [15]. Since in what follows, when dealing with the qubit-qutrit system, the Casimirs of SU(6) will be used,[2] the expressions for $\mathfrak{C}_i$ are given below:

$$\begin{aligned}
\mathfrak{C}_3 &= \sum d_{i_1 i_2 i_3}\, e_{i_1} e_{i_2} e_{i_3}\,,\\
\mathfrak{C}_4 &= \sum d_{j i_1 i_2} d_{j i_3 i_4}\, e_{i_1} e_{i_2} e_{i_3} e_{i_4}\,,\\
\mathfrak{C}_5 &= \sum d_{i i_1 i_2} d_{i j i_3} d_{j i_4 i_5}\, e_{i_1} e_{i_2} e_{i_3} e_{i_4} e_{i_5} e_{i_6}\,,\\
\mathfrak{C}_6 &= \sum d_{i i_1 i_2} d_{i j i_3} d_{j k i_4} d_{k i_5 i_6}\, e_{i_1} e_{i_2} e_{i_3} e_{i_4} e_{i_5} e_{i_6}\,.
\end{aligned}$$

Now, using these operators and decomposition (2.3) based on the isomorphism between the center $\mathcal{Z}(\mathfrak{su}(n))$ and the $\mathrm{SU}(n)$-invariant polynomials, the following scalars, hereafter referred as Casimir invariants, can be written:

$$\mathfrak{C}_2 = (n-1)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi}, \tag{2.11}$$

$$\mathfrak{C}_3 = (n-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\cdot\boldsymbol{\xi}, \tag{2.12}$$

$$\mathfrak{C}_4 = (n-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\cdot(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,), \tag{2.13}$$

$$\mathfrak{C}_5 = (n-1)\,\Big((\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\vee(\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)\Big)\cdot\boldsymbol{\xi}, \tag{2.14}$$

$$\mathfrak{C}_6 = (n-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi}\vee\boldsymbol{\xi}\,)^2, \tag{2.15}$$

$$\dots\dots,$$

where

$$(\boldsymbol{U}\vee\boldsymbol{V}\,)_a := \kappa\, d_{abc} U_b V_c,$$

with the normalization constant $\kappa := \sqrt{n(n-1)/2}$.

These scalars will be used for an explicit formulation of the positive semi-definiteness of the density matrices for an arbitrary n-level quantum system.

Positivity of density operators

To the best of our knowledge, the first analysis of the consequences of the constraints on the density operator due to its positive semi-definiteness was carried out in the 1960s when studying the production and decay of resonant states in strong interaction processes [16–18]. Nowadays, quantum computing and quantum information reveal the new role of these constraints, and recently they have been derived once again [19, 20].[3]

To formulate the semi-definiteness, we choose the Bloch representation of the density operator (2.2) (see [11]),

$$\varrho = \frac{1}{n}\,(\,\mathbb{I}_n + \omega\,)\,, \quad \omega = \kappa\,\boldsymbol{\xi}\cdot\boldsymbol{\lambda}, \tag{2.16}$$

characterized by an (n^2-1)-dimensional Bloch vector $\boldsymbol{\xi}\in\mathbb{R}^{n^2-1}$ contracted with Hermitian basis elements $\lambda_i\,,\ i=1,\dots,n^2-1$, of the Lie algebra $\mathfrak{su}(n)$. According to [17],[4] a necessary and sufficient condition for a Hermitian matrix to be positive is that the coefficients S_k of its characteristic equation

$$|\mathbb{I}x-\varrho| = x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^n S_n = 0 \tag{2.17}$$

[2]The tensorial $\mathfrak{su}(2)\otimes\mathfrak{su}(3)$ product type basis for $\mathfrak{su}(6)$ is given in Appendix A. There we also present formulas for the symmetric structure constants d_{ijk}, as well as for the antysymmetric structure constants f_{ijk} for $\mathfrak{su}(n)$.

[3]In our recent publication [8], the positivity conditions for density operators were analyzed in the context of consequences for an integrity basis of the ring of $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ polynomial invariants, as well as for entanglement characteristics of mixed qubit states [21].

[4]Note that P. Minnaert attributed the same result to D. N. Williams.

are nonnegative:

$$\varrho \geq 0 \quad \Leftrightarrow \quad S_k \geq 0, \quad k = 1, \ldots, n. \tag{2.18}$$

It is convenient to rewrite these inequalities in terms of the normalized coefficients $\bar{S}_k := S_k / \max\{ S_k \}$. Observing that the maximum values of S_k correspond to the most degenerate roots $x_1 = x_2 = \ldots = x_n = 1/n$ of the characteristic equation (2.17), one can express them in terms of binomial coefficients:

$$\max\{ S_k \} = \frac{1}{n^k} \binom{n}{n-k},$$

and hence

$$0 \leq \bar{S}_k \leq 1, \quad k = 2, \ldots, n. \tag{2.19}$$

Now we are ready to rewrite the constraints (2.19) in terms of the Casimir invariants (2.11)–(2.15). This can be done since each of the three sets, $\mathfrak{C}_k$, S_k, and $t_k = \operatorname{tr}(\varrho^k)$, $k = 2, \ldots, n$, forms a basis of algebraically independent invariants of SU(n) (see, e.g., [14]). Expressions for the coefficients S_k in terms of t_m are well known; they are given by determinants:

$$S_k = \frac{1}{k!} \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 2 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{k-1} & t_{k-2} & t_{k-3} & t_{k-4} & \ldots & k-1 \\ t_k & t_{k-1} & t_{k-2} & t_{k-3} & \ldots & t_1 \end{vmatrix}.$$

Further, the t_m can be written as polynomials in the Casimir invariants. Using the expressions for the traces of symmetrized products of basis elements of a Lie algebra (see Appendix A; cf. also [20]), we have

$$\operatorname{tr}(\omega^2) = n\mathfrak{C}_2, \tag{2.20}$$

$$\operatorname{tr}(\omega^3) = n\mathfrak{C}_3, \tag{2.21}$$

$$\operatorname{tr}(\omega^4) = n\,(\mathfrak{C}_2^2 + \mathfrak{C}_4), \tag{2.22}$$

$$\operatorname{tr}(\omega^5) = n\,(2\,\mathfrak{C}_2\mathfrak{C}_3 + \mathfrak{C}_5), \tag{2.23}$$

$$\operatorname{tr}(\omega^6) = n\,(\mathfrak{C}_2^3 + 2\,\mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_3^2 + \mathfrak{C}_6). \tag{2.24}$$

Finally, imposing the following normalization for the Casimir invariants,

$$C_k = \frac{(k-1)!}{(n-1)(n-2)\ldots(n-k+1)}\,\mathfrak{C}_k, \tag{2.25}$$

we arrive at a system of inequalities in the Casimir invariants of $\mathfrak{su}(6)$ that determines the positive semi-definiteness of the density matrix of the qubit-qutrit pair:

$$0 \leq C_2 \leq 1, \tag{2.26}$$

$$0 \leq 3\,C_2 - C_3 \leq 1, \tag{2.27}$$

$$0 \leq 6\,C_2 - 5\,C_2^2 - 4\,C_3 + C_4 \leq 1, \tag{2.28}$$

$$0 \leq (1-5\,C_2)^2 - 30\,C_2C_3 + 10\,C_3 - 5\,C_4 + C_5 \leq 1 \tag{2.29}$$

$$0 \leq (1-5C_2)^3 - 180C_2C_3 + 125C_2C_4 + 20C_3(1+5C_3) - 15C_4 + 6C_5 - C_6 \leq 1. \tag{2.30}$$

To discuss the role of the positive semi-definiteness in the entanglement problem, we need to write the obtained system in terms of local $\mathrm{SU}(2) \otimes \mathrm{SU}(3)$ invariants.

The local invariance of composite states

When a quantum system is obtained by combining r subsystems with $n_1, n_2, \dots, n_r$ levels each, nonlocal properties of the composite system are in a correspondence with a certain decomposition of the unitary operations (2.1).

In order to discuss this decomposition, consider the subgroup

$$\mathrm{SU}(n_1) \otimes \mathrm{SU}(n_2) \otimes \cdots \otimes \mathrm{SU}(n_r) \tag{3.1}$$

of the unitary group formed by the *local unitary transformations* acting independently on the density matrix of each subsystem:

$$\varrho^{(n_i)} \quad \to \quad \varrho^{(n_i)\prime} = g\varrho^{(n_i)}g^{-1}, \quad g \in \mathrm{SU}(n_i), \quad i = 1, 2, \dots, r. \tag{3.2}$$

Two states of the composite system connected by local unitary transformations (3.1) have the same nonlocal properties. The latter can be changed only by the remaining unitary actions

$$\frac{\mathrm{SU}(n)}{\mathrm{SU}(n_1) \otimes \mathrm{SU}(n_2) \otimes \cdots \otimes \mathrm{SU}(n_r)}, \quad n = n_1 n_2 \cdots n_r, \tag{3.3}$$

generating the class of nonlocal transformations.

Now we are in a position to discuss the structure of the ring of polynomial local invariants, i.e., polynomials in the elements of the density matrices that are scalars under the adjoint local unitary transformations. It is well known that for any reductive linear algebraic group G (in particular, a Lie group) and for any finite-dimensional G-module V, the ring $\mathcal{R}^{\mathrm{G}}$ has the Cohen–Macaulay property [22] and possesses a Hironaka decomposition

$$\mathcal{R}^G = \bigoplus_{a=0}^{r} J_a \, \mathbb{C}[K_1, K_2, \dots, K_n], \tag{3.4}$$

where K_b, $b = 1, 2, \dots, n$, are primary algebraically independent polynomials and J_a, $a = 0, 1, 2, \dots, r$, with $J_0 = 1$, are secondary linearly independent invariants, respectively. According to this, the corresponding Molien function $M_G(q)$ for $\mathcal{R}^G$ (see [7]) can be expressed as follows:

$$M_G(q) = \frac{\sum\limits_{a=0}^{r} q^{\deg J_a}}{\prod\limits_{b=1}^{n} (1 - q^{\deg K_b})}. \tag{3.5}$$

In this form, it provides us with information about the number of algebraically independent polynomials, as well as linearly independent invariants.

The Molien function for $\mathbb{C}[\mathfrak{P}_+^{(2\otimes 2)}]$ and $\mathbb{C}[\mathfrak{P}_+^{(2\otimes 3)}]$

Let us start with a remark concerning the adjoint action (2.1). Consider the case of nondegenerate density matrices. In this case, using the natural identification of the linear space spanned by the Hermitian $n \times n$ matrices with $\mathbb{R}^{n^2-1}$,

$$\varrho \to \rho_{ij},$$

one can, instead of the adjoint action (2.1), consider the linear representation on $\mathbb{R}^{n^2-1}$:

$$V_A' = L_{AB} V_B, \qquad L_{AB} \in \mathrm{SU}(n) \otimes \overline{\mathrm{SU}(n)},$$

where the overline means complex conjugation.

After this identification, in order to get some insight into the structure of the ring of polynomial invariants of a linear action of a Lie group G on a linear space V, we may compute the Molien function

$$M(\mathbb{C}[V]^G, q) = \int\limits_G \frac{d\mu(g)}{\det(\mathbb{I} - q\pi(g))}, \quad |q| < 1, \tag{3.6}$$

where $d\mu(g)$ is the Haar measure for the Lie group G and $\pi(g)$ is the corresponding representation on V. We start with the system of two qubits.

Two qubits. In this case, the local unitary group is

$$G = \mathrm{SU}(2) \otimes \mathrm{SU}(2). \tag{3.7}$$

As is well known, for any reductive linear group, the integration in (3.6) reduces to integration over the maximal compact subgroup K of G (see [4]). In the present case, this results in integration over the maximal torus

$$\pi(g) = \mathrm{diag}\,(1, 1, z, z^{-1}) \otimes \mathrm{diag}\,(1, 1, w, w^{-1}), \tag{3.8}$$

where z, w are coordinates on the one-dimensional tori. Therefore, the computations reduce to the following two-dimensional integral:

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(2)}(q) = \frac{1}{(2\pi i)^2} \int\limits_{|z|=1} \int\limits_{|w|=1} \frac{d\,\mu}{\Psi(z, w, q)}, \tag{3.9}$$

where

$$\begin{aligned} d\,\mu &= (1-z)^2(1-w)^2 \frac{dz}{z^2}\frac{dw}{w^2}, \\ \det(\mathbb{I} - q\pi(g)) &= (1-q)\,\Psi(z, w, q) \\ \Psi(z, w, q) &= (1-q)^3(1-qz)^2(1-qw)^2(1-qz^{-1})^2(1-qw^{-1})^2 \\ &\quad \times (1-qzw)(1-qz^{-1}w)(1-qzw^{-1})(1-qz^{-1}w^{-1}). \end{aligned} \tag{3.10}$$

After the integration we obtain the Molien function (see [7])

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(2)}(q) = \frac{1 + q^4 + q^5 + 3q^6 + 2q^7 + 2q^8 + 3q^9 + q^{10} + q^{11} + q^{15}}{(1-q^2)^3(1-q^3)^2(1-q^4)^3(1-q^6)}, \tag{3.11}$$

which is palindromic:

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(2)}(1/q) = -q^{15} M_{\mathrm{SU}(2)\otimes\mathrm{SU}(2)}(q),$$

in accordance with the fact that

$$\dim\ \mathrm{SU}(4) = 15.$$

Qubit–qutrit. Now the local unitary group is $G := \mathrm{SU}(2) \otimes \mathrm{SU}(3)$, and owing to the symmetries of the integrand (3.6), the nontrivial contribution to the integral comes from the diagonal components of the $\pi(g)$-representation of the form

$$\pi(g)_{\mathrm{diag}} = \mathrm{diag}\,(1, 1, x, x^{-1}) \otimes \mathrm{diag}\,(1, 1, 1, y, z, yz, y^{-1}, z^{-1}, (yz)^{-1}), \tag{3.12}$$

where x, y, and z are coordinates on the one-dimensional tori. Therefore, the computation of the Molien function (3.6) reduces to the evaluation of a multiple contour integral over the unit circles in the complex planes:[5]

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(q) = \frac{1}{(2\pi i)^3} \int\limits_{|x|=1} \int\limits_{|y|=1} \int_{|z|=1} f(x, y, z, q)\, dx\, dy\, dz, \tag{3.13}$$

where

$$f(x, y, z, q) = \frac{1}{xyz} \frac{(1-x^{-1})(1-y^{-1})(1-z^{-1})(1-(yz)^{-1})}{\Psi(x, y, z, q)}, \tag{3.14}$$

$$\det(\mathbb{I} - q\pi(g)) = (1-q)\Psi(x, y, z, q), \tag{3.15}$$

[5]The multiple integral (3.13) was calculated by repeated application of Cauchy's residue theorem. Since f has poles of rather high orders, computer calculations of the residues were performed using the built-in command `Residue` of the Mathematica packet, which implements the standard limit formula for high-order poles (see `http://mathworld.wolfram.com/ComplexResidue.html`).

and

$$\Psi(x,y,z,q) = (1-q)^5(1-qy)^2(1-qz)^2(1-qyz)^2(1-\frac{q}{y})^2(1-\frac{q}{z})^2(1-\frac{q}{yz})^2$$
$$\times(1-qx)^3(1-qxy)(1-qxz)(1-qxyz)(1-\frac{qx}{y})(1-\frac{qx}{z})(1-\frac{qx}{yz})$$
$$\times(1-\frac{q}{x})^3(1-\frac{qy}{x})(1-\frac{qz}{x})(1-\frac{qyz}{x})(1-\frac{q}{xy})(1-\frac{q}{xz})(1-\frac{q}{xyz}).$$

As a result, the Molien function can be represented in a rational form (cf. [23]):

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(q) = \frac{N}{D},$$

where

$$\begin{aligned} N = {} & 1 + 4\,q^4 + 9\,q^5 + 38\,q^6 + 69\,q^7 + 173\,q^8 + 347\,q^9 + 733\,q^{10} + 1403\,q^{11} \\ & + 2796\,q^{12} + 5091\,q^{13} + 9286\,q^{14} + 16058\,q^{15} + 27208\,q^{16} + 44250\,q^{17} \\ & + 70537\,q^{18} + 108430\,q^{19} + 163158\,q^{20} + 238264\,q^{21} + 339974\,q^{22} \\ & + 472130\,q^{23} + 641187\,q^{24} + 848615\,q^{25} + 1098643\,q^{26} + 1388741\,q^{27} \\ & + 1717327\,q^{28} + 2075836\,q^{29} + 2456389\,q^{30} + 2843020\,q^{31} + 3222408\,q^{32} \\ & + 3575226\,q^{33} + 3884797\,q^{34} + 4133599\,q^{35} + 4308636\,q^{36} + 4398377\,q^{37} \\ & + 4398377\,q^{38} + \ldots + 38\,q^{69} + 9\,q^{70} + 4\,q^{71} + q^{75}, \\ D = {} & (1-q^2)^3(1-q^3)^4(1-q^4)^5(1-q^5)^4(1-q^6)^5(1-q^7)^2(1-q^8). \end{aligned} \tag{3.16}$$

This Molien function is palindromic:

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(1/q) = q^{35} M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(q),$$

since

$$\dim\ \mathrm{SU}(6) = 35.$$

This form of the Molien function serves as a source of information on the polynomial ring of $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ invariants. In particular, one may endeavor to identify the structure of algebraically independent local unitary scalars. According to (3.16), there are 24 independent scalars, in agreement with a simple count of $\dim[\mathrm{SU}(6)/\,\mathrm{SU}(2)\otimes\mathrm{SU}(3)] = 35 - 11 = 24$. The set of these 24 polynomial invariants can be composed of three invariants of degree 2, four of degree 3, five of degree 4, four of degree 5, five of degree 6, two of degree 7, and one of degree 8.

Note that the Poincaré series of $M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(q)$,

$$M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(q) = \sum_{d=0}^{\infty} \dim\left(\mathcal{P}_d^{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}\right) q^d, \tag{3.17}$$

determines the number of homogeneous polynomial invariants of degree d. According to the calculations of (3.13), a few terms of the Taylor expansion over q are

$$\begin{aligned} M_{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}(q) = {} & 1+3\,q^2+4\,q^3+15\,q^4+25\,q^5+90\,q^6+170\,q^7+489\,q^8 \\ & + 1059\,q^9 + 2600\,q^{10} + 5641\,q^{11} + 12872\,q^{12} + 27099\,q^{13} \\ & + 57990\,q^{14} + 118254\,q^{15} + 240187\,q^{16} + O\left(q^{17}\right). \end{aligned} \tag{3.18}$$

Now, having in mind the structure of the Molien function (3.16), we attempt to construct the local $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ unitary invariants.

Constructing $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ **invariants**

Let us introduce a decomposition of density matrices well adapted to the case of a composite qubit-qutrit system. The space $\mathfrak{su}(6)$ in (2.2) for $n=6$ admits a decomposition into the direct sum of three real spaces:

$$\mathfrak{su}(6)=\bigoplus_{a=1}^{3} V_a=\mathfrak{su}(2)\otimes \mathrm{I}_3+\mathrm{I}_2\otimes\mathfrak{su}(2)+\mathfrak{su}(2)\otimes\mathfrak{su}(3). \tag{3.19}$$

Using the Pauli matrices σ_i as a basis for $\mathfrak{su}(2)$ and the Gell–Mann matrices λ_a as a basis for $\mathfrak{su}(3)$ (see the Appendix), the density matrix (2.16) for the qubit-qutrit system can be written (see [9, 10]) as

$$\varrho=\frac{1}{6}\left[\mathrm{I}_6+\omega\right],\quad \omega=\alpha+\beta+\gamma, \tag{3.20}$$

where

$$\alpha:=\sum_{i=1}^{3} a_i\,\sigma_i\otimes \mathrm{I}_3,\quad \beta:=\sum_{a=1}^{8} b_i\ \mathrm{I}_2\otimes\lambda_a,\quad \gamma:=\sum_{i=1}^{3}\sum_{a=1}^{8} c_{ia}\,\sigma_i\otimes\lambda_a. \tag{3.21}$$

Among the 35=3+8+24 real parameters (a_i, b_a, c_{ia}), the first two sets, a_i and b_a, correspond to the Bloch vectors of an individual qubit and qutrit, respectively; the evaluation of the partial trace yields the reduced matrices for the subsystems:

$$\varrho^{(A)}:=\mathrm{tr}_B(\varrho)=\frac{1}{2}(\mathrm{I}_2+\vec{a}\cdot\vec{\sigma}),\quad \varrho^{(B)}:=\mathrm{tr}_A(\varrho)=\frac{1}{3}(\mathrm{I}_3+\vec{b}\cdot\vec{\lambda}),$$

while the variables c_{ia} are the entries of the so-called correlation matrix $c_{ia}=\|C\||_{ia}$.

Before suggesting a set of local $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ scalars, candidates for the elements of an integrity basis, let us make a few explanatory remarks. Consider the homogeneous polynomials in variables (a,b,c) of degrees s,t,q, respectively, constructed as follows.

By analogy with the generators (2.5) of the universal enveloping algebra, we introduce a general noncommutative monomial of total degree d,

$$\mathcal{M}_{i_1\ldots i_d}:=X_{i_1}\cdot X_{i_2}\cdot\ldots\cdot X_{i_d}, \tag{3.22}$$

in three matrix variables $X_{i_k}\in\{\alpha,\beta,\gamma\}, k=1,\ldots,d$. The trace operation on the monomial (3.22) determines a map

$$\mathrm{tr}\ :\ \mathcal{M}\to\mathcal{P},\quad \mathrm{tr}\left(\mathcal{M}_{i_1\ldots i_d}\right)\in\mathcal{P}_{stq}(a_i,b_a,c_{ia}), \tag{3.23}$$

where $\mathcal{P}_{stq}(a_i,b_a,c_{ia})$ is a polynomial in the variables (a_i,b_a,c_{ia}) of total degree $d=s+t+q$, where s,t, and q are the sums of the degrees of the variables a_i, b_a, and $c_{i,a}$, respectively.

Now it is easy to verify that the image of the trace map is a set of $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ invariants. Indeed, a generic term of the polynomial (3.23) consists of the convolution of monomials in (a_i,b_a,c_{ia}) with the traces of tensorial products in the monomials (3.22):

$$\mathrm{tr}\left(\sigma_1\sigma_2\cdots\sigma_p\otimes\lambda_1\lambda_2\cdots\lambda_r\right)=\mathrm{tr}\left(\sigma_1\sigma_2\cdots\sigma_p\right)\mathrm{tr}\left(\lambda_1\lambda_2\cdots\lambda_r\right),$$

where $p=s+q$ and $r=t+q$. Since under a transformation of the form $k_1\otimes k_2$, where $k_1\in\mathrm{SU}(2)$ and $k_2\in\mathrm{SU}(3)$, the basis elements are transformed independently in the adjoint way,

$$\sigma\to k_1\sigma k_1^{-1},\quad \lambda\to k_2\lambda k_2^{-1},$$

the polynomials $\mathrm{tr}\,(\mathcal{M})$ are invariant under the action of $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$.

Therefore, the polynomials $\mathcal{P}_{stq}(a_i,b_a,c_{ia})$ are a store for constructing an integrity basis for $\mathbb{C}[\mathfrak{P}]^{\mathrm{SU}(2)\otimes\mathrm{SU}(3)}$. Now, in contrast to the case of $\mathrm{SU}(n)$, where the Casimir invariants are built with the help of the symmetric structure constants only, the invariants are expressed in terms of the antisymmetric structure constants of product algebras as well. For example,

$$\mathrm{tr}\,\gamma^3=c_{ia}c_{jb}c_{kc}\,\mathrm{tr}\left(\sigma_i\sigma_j\sigma_k\otimes\lambda_a\lambda_b\lambda_c\right)=c_{ia}c_{jb}c_{kc}\,\mathrm{tr}\left(\sigma_i\sigma_j\sigma_k\right)\mathrm{tr}\left(\lambda_a\lambda_b\lambda_c\right).$$

This quantity, being invariant under the action of $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$, is expressible in terms of the totally antisymmetric tensor ϵ_{ijk} (the structure constants of $\mathfrak{su}(2)$) and f_{abc} (the structure constants of $\mathfrak{su}(3)$):

$$\mathrm{tr}\,\gamma^3 = -4\ \mathrm{tr}\,\varepsilon_{ijk} f_{abc}\, c_{ia} c_{jb} c_{kc}.$$

When choosing a basis for the local invariants, several types of algebraic dependences between polynomials in $\mathcal{P}_{stq}(a_i\,, b_a, c_{ia})$ must be taken into account. It is worth considering two illustrative examples. Applying the Hamilton–Cayley theorem to the elements $\alpha,\beta,$ and γ, regarded as Hermitian 6×6 matrices, one can determine algebraic identities for polynomials of the form $\mathrm{tr}(\gamma^n)$, $n>7$. A less obvious example of relations between polynomials arises from identities between the structure constants of the algebra.[6] Let us consider two invariants, both of the 4th order in the variables C, but the first one constructed using the invariant symmetric structure constants d, while the second one, using the antisymmetric structure constants f:

$$\mathfrak{I}^{004}(dd) = d_{abc}\, d_{cpq}\,(C^TC)_{ab}(C^TC)_{pq}, \tag{3.24}$$

$$\mathfrak{I}^{004}(ff) = f_{apc} f_{cbq}(C^TC)_{ab}(C^TC)_{pq}. \tag{3.25}$$

With the aid of identities (A.6) and (A.7) (see the Appendix) for the structure constants of $\mathfrak{su}(3)$, one can check that

$$\mathfrak{I}^{004}(dd) = \frac{2}{3}\,\mathfrak{I}^{004}(ff) - \frac{1}{3}\left[\left(\mathrm{tr}(C^TC)\right)^2 - 2\,\mathrm{tr}(C^TCC^TC)\right]. \tag{3.26}$$

According to the Poincaré series (3.18), there are 15 homogeneous scalars of order 4, while there are $81=3^4$ monomials in three noncommutative variables. But since the elements α and β commute, this number decreases. Taking into account this commutativity, as well as the invariance of the trace operation under cyclic permutations of products, one can find 18 valuable monomials:

$$\begin{gathered}\alpha^4,\ \beta^4,\ \gamma^4,\ \alpha^3\beta,\ \alpha\beta^3,\ \alpha^3\gamma,\ \alpha\gamma^3,\ \beta^3\gamma,\ \beta\gamma^3,\\ \alpha^2\beta^2,\ \alpha^2\gamma^2,\ \alpha\gamma\alpha\gamma,\ \beta^2\gamma^2,\ \beta\gamma\beta\gamma,\\ \alpha^2\beta\gamma,\ \alpha\beta^2\gamma,\ \alpha\beta\gamma^2,\ \alpha\gamma\beta\gamma.\end{gathered} \tag{3.27}$$

Taking the traces of these monomials, one can check that five of them form the kernel of the trace map,

$$\mathrm{tr}(\alpha^3\beta) = \mathrm{tr}(\alpha\beta^3) = \mathrm{tr}(\alpha^3\gamma) = \mathrm{tr}(\beta^3\gamma) = \mathrm{tr}(\alpha^2\beta\gamma) = 0,$$

and the images of the last two monomials in (3.27) coincide up to sign:

$$\mathrm{tr}\,(\alpha\beta\gamma^2) = -\,\mathrm{tr}\,(\alpha\gamma\beta\gamma).$$

Therefore, the set of twelve traces

$$\mathrm{tr}\,(\alpha^4),\quad \mathrm{tr}\,(\beta^4),\quad \mathrm{tr}\,(\alpha^2\beta^2),\quad \mathrm{tr}\,(\alpha^2\gamma^2), \tag{3.28}$$

$$\mathrm{tr}\,(\gamma^4),\quad \mathrm{tr}\,(\alpha\gamma^3),\quad \mathrm{tr}\,(\beta\gamma^3),\quad \mathrm{tr}\,(\alpha\gamma\alpha\gamma), \tag{3.29}$$

$$\mathrm{tr}\,(\beta^2\gamma^2),\quad \mathrm{tr}\,(\beta\gamma\beta\gamma),\quad \mathrm{tr}\,(\alpha\beta^2\gamma),\quad \mathrm{tr}\,(\alpha\beta\gamma^2), \tag{3.30}$$

plus the three fourth-order polynomials constructed as products of second-order polynomials $\mathrm{tr}\,(\alpha^2)\,\mathrm{tr}\,(\beta^2)$, $\mathrm{tr}\,(\alpha^2)\,\mathrm{tr}\,(\gamma^2)$, $\mathrm{tr}\,(\beta^2)\,\mathrm{tr}\,(\gamma^2)$, are 15 homogeneous invariant polynomials, in accordance with the Poincaré series (3.18).

How difficult is to extract independent scalars from this list? It is easy to verify that all traces in (3.28) can be expressed in terms of second-order polynomials; e.g., $\mathrm{tr}\,(\alpha^2\beta^2) = \frac{1}{6}\,\mathrm{tr}\,(\alpha^2)\,\mathrm{tr}\,(\beta^2)$. As concerns the remaining monomials, one can see that some of them have the same multidegree. Namely, the "trace" polynomials

(1) $\mathrm{tr}\,(\alpha^2\gamma^2) = \frac{1}{6}\,\mathrm{tr}\,(\alpha^2)\,\mathrm{tr}\,(\gamma^2)$ and $\mathrm{tr}\,(\alpha\gamma\alpha\gamma)$,

(2) $\mathrm{tr}\,(\beta^2)\,\mathrm{tr}\,(\gamma^2)$, $\mathrm{tr}\,(\beta^2\gamma^2)$, and $\mathrm{tr}\,(\beta\gamma\beta\gamma)$

[6] For a detailed analysis of relations of this type, we refer to [24].

belong to the spaces $\mathcal{P}_{202}$ and $\mathcal{P}_{022}$, respectively. Being linearly independent monomials, they obey the following relations:

$$\begin{aligned}
\operatorname{tr}(\alpha^2\gamma^2) + \operatorname{tr}(\alpha\gamma\alpha\gamma) &= 8\, a_{i_1} a_{i_2} c_{i_1 j_1} c_{i_2 j_1}\,, \\
\operatorname{tr}(\beta^2\gamma^2) - \frac{1}{6}\operatorname{tr}(\beta^2)\operatorname{tr}(\gamma^2) &= 4\, d_{j_1 j_2 k}\, d_{k j_3 j_4}\, b_{j_1} b_{j_2} c_{i_1 j_3} c_{i_1 j_4}\,, \\
\operatorname{tr}(\beta^2\gamma^2) + \operatorname{tr}(\beta\gamma\beta\gamma) &= 8(\tfrac{2}{3} b_{j_1} b_{j_2} c_{i_1 j_1} c_{i_1 j_2} + d_{j_1 j_2 k}\, d_{k j_3 j_4}\, b_{j_1} b_{j_3} c_{i_1 j_2} c_{i_1 j_4}),
\end{aligned}$$

where the summation over all indices is assumed. This fact leaves open the question of how to build the elements of the integrity basis with a certain multidegree with the aid of the invariant "trace" polynomials.

To sum up our analysis, we present the following list of linearly independent $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ scalars that are not products of low-order ones:[7]

- degree 2, three invariants:

$$\operatorname{tr}(\alpha^2),\ \operatorname{tr}(\beta^2),\ \operatorname{tr}(\gamma^2);$$

- degree 3, four invariants:

$$\operatorname{tr}(\beta^3),\ \operatorname{tr}(\gamma^3),\ \operatorname{tr}(\alpha\beta\gamma),\ \operatorname{tr}(\beta\gamma^2);$$

- degree 4, eight invariants:

$$\begin{aligned}
&\operatorname{tr}(\gamma^4), \quad \operatorname{tr}(\alpha\gamma^3), \quad \operatorname{tr}(\beta\gamma^3), \quad \operatorname{tr}(\alpha\gamma\alpha\gamma), \\
&\operatorname{tr}(\beta^2\gamma^2), \quad \operatorname{tr}(\beta\gamma\beta\gamma), \quad \operatorname{tr}(\alpha\beta^2\gamma), \quad \operatorname{tr}(\alpha\beta\gamma^2).
\end{aligned}$$

Decomposition of the Casimir invariants

The expansion of the Casimir invariants up to the 4th order (2.11)–(2.13) in terms of the $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ "trace" scalars suggested above reads as follows:

$$\begin{aligned}
6\,\mathfrak{C}_2 &= \operatorname{tr}(\alpha^2) + \operatorname{tr}(\beta^2) + \operatorname{tr}(\gamma^2), \\
6\,\mathfrak{C}_3 &= \operatorname{tr}(\beta^3) + \operatorname{tr}(\gamma^3) + 3\ \operatorname{tr}(\beta\gamma^2) + 6\ \operatorname{tr}(\alpha\beta\gamma), \\
6\,\mathfrak{C}_4 &= \frac{1}{3}\Big[\operatorname{tr}(\alpha^2)\Big(2\ \operatorname{tr}(\beta^2)+\operatorname{tr}(\gamma^2)\Big)+\tfrac{1}{4}\ \operatorname{tr}(\beta^2)^2-\tfrac{1}{2}\ \operatorname{tr}(\gamma^2)^2-\operatorname{tr}(\beta^2)\operatorname{tr}(\gamma^2)\Big] \\
&\quad + 4\Big[\operatorname{tr}(\alpha\gamma^3) + \operatorname{tr}(\beta\gamma^3) + \operatorname{tr}(\beta^2\gamma^2) + \operatorname{tr}(\alpha\beta\gamma^2) + 3\operatorname{tr}(\alpha\beta^2\gamma)\Big] \\
&\quad + 2\Big[\operatorname{tr}(\alpha\gamma\alpha\gamma) + \operatorname{tr}(\beta\gamma\beta\gamma)\Big] + \operatorname{tr}(\gamma^4).
\end{aligned}$$

We conclude with a final remark on the applicability of the obtained results to the problem of classification of mixed quantum states. Using inequalities (2.26)–(2.30) and the results from [21], the well-known Peres–Horodecki criterion for the separability of qubit-qutrit mixed states can be reformulated as a set of inequalities in $\mathrm{SU}(2)\otimes\mathrm{SU}(3)$ scalars.

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Appendix: Formulas for $\mathfrak{su}(6)$

The tensorial basis

For the algebra $\mathfrak{su}(6)$, we use the basis $\{\tau_A\}_{A=1,\dots,35}$ constructed from tensor products of the Pauli matrices $\sigma_i \in \mathfrak{su}(2)$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

[7]Note that invariants of the second and third order were suggested in [23].

and eight Gell–Mann matrices $\{\lambda_a\}_{a=1,\dots,8}$ forming a basis of $\mathfrak{su}(3)$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The elements τ_A are enumerated as

$$\begin{aligned} &\tau_i = \frac{1}{\sqrt{3}}\,\sigma_i \otimes \mathbb{I}_3, \quad \tau_{3+a} = \frac{1}{\sqrt{2}}\,\mathbb{I}_2 \otimes \lambda_a, \\ &\tau_{11+a} = \frac{1}{\sqrt{2}}\,\sigma_1 \otimes \lambda_a, \ \tau_{19+a} = \frac{1}{\sqrt{2}}\,\sigma_2 \otimes \lambda_a, \ \tau_{27+a} = \frac{1}{\sqrt{2}}\,\sigma_3 \otimes \lambda_a. \end{aligned} \tag{A.2}$$

The algebraic structures

The product of basis elements reads as follows:

$$\tau_A \tau_B = \frac{2}{n}\delta_{AB}\mathbb{I} + (d_{ABC} + \imath\, f_{ABC})\tau_C.$$

The structure constants d_{ABC} and f_{ABC} can be determined via the equations

$$d_{ABC} = \frac{1}{4}\,\mathrm{Tr}(\{\tau_A, \tau_B\}\tau_C), \quad f_{ABC} = -\frac{\imath}{4}\,\mathrm{Tr}([\tau_A, \tau_B]\tau_C),$$

where, apart from the Lie algebra product [,], the "anticommutator" of elements, i.e., $\{\tau_A, \tau_B\} = \tau_A\tau_B + \tau_B\tau_A$, has been used.

Identities for the structure constants

For SU(n), the structure constants obey the following identities:

$$f_{abc}f_{cpq} + f_{bpc}f_{caq} + f_{pac}f_{cbq} = 0, \tag{A.3}$$

$$d_{abc}f_{cpq} + d_{bpc}f_{caq} + d_{pac}f_{cbq} = 0, \tag{A.4}$$

$$f_{abc}f_{cpq} = d_{apc}d_{cbq} - d_{aqc}d_{cbp} + \frac{2}{n}\,(\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp}), \tag{A.5}$$

$$f_{abc}f_{cpq} + f_{aqc}f_{cpb} = 2d_{apc}d_{cbq} - d_{abc}d_{cpq} - d_{aqc}d_{cbp} + \frac{2}{n}\,(2\delta_{ap}\delta_{bq} - \delta_{ab}\delta_{pq} - \delta_{aq}\delta_{bp}). \tag{A.6}$$

The SU(3) symmetric constants satisfy (see [25, 26]) the important identities

$$d_{abc}d_{cpq} + d_{bpc}d_{caq} + d_{pac}d_{cbq} = \frac{1}{3}(\delta_{ab}\delta_{pq} + \delta_{ap}\delta_{bq} + \delta_{aq}\delta_{bp}). \tag{A.7}$$

The traces

The traces of symmetrized products of $\mathfrak{su}(n)$ basis elements are

$$\begin{aligned} \mathrm{tr}\,(\tau_{\{a}\tau_{b\}}) &= 2\,\delta_{ab}, \\ \mathrm{tr}\,(\tau_{\{a}\tau_b\tau_{c\}}) &= 2\,d_{abc}, \\ \mathrm{tr}\,(\tau_{\{a}\tau_b\tau_c\tau_{d\}}) &= \frac{2^2}{n}\,\delta_{ab}\delta_{cd} + 2\,d_{abe}d_{ecd}, \\ \mathrm{tr}\,(\tau_{\{a}\tau_b\tau_c\tau_d\tau_{e\}}) &= \frac{2^2}{n}\,(d_{abc}\delta_{de} + \delta_{ab}d_{cde}) + 2\,d_{abf}d_{fcg}d_{gde}, \\ \mathrm{tr}\,(\tau_{\{a}\tau_b\tau_c\tau_d\tau_e\tau_{f\}}) &= \frac{2^3}{n^2}\,\delta_{ab}\delta_{cd}\delta_{ef} + \frac{2^2}{n}\,(d_{abg}d_{gcd}\delta_{ef} + \delta_{ab}d_{cdg}d_{gef}) \\ &\quad + \frac{2^2}{n}\,d_{abc}d_{def} + 2\,d_{abg}d_{gch}d_{hdv}d_{vef}. \end{aligned}$$

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