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Generalized Calogero–Moser–Sutherland models from geodesic motion on $GL^+(n, \mathbb{R})$ group manifold

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Abstract

It is shown that geodesic motion on the $GL(n, \mathbb{R})$ group manifold endowed with the bi-invariant metric $ds^2 = \text{tr}(g^{-1} dg)^2$ corresponds to a generalization of the hyperbolic n -particle Calogero–Moser–Sutherland model. In particular, considering the motion on principal orbit stratum of the $SO(n, \mathbb{R})$ group action, we arrive at dynamics of a generalized n -particle Calogero–Moser–Sutherland system with two types of internal degrees of freedom obeying $SO(n, \mathbb{R}) \oplus SO(n, \mathbb{R})$ algebra. For the singular orbit strata of $SO(n, \mathbb{R})$ group action the geodesic motion corresponds to certain deformations of the Calogero–Moser–Sutherland model in a sense of description of particles with different masses. The mass ratios depend on the type of singular orbit stratum and are determined by its degeneracy. Using reduction due to discrete and continuous symmetries of the system a relation to IIA_n Euler–Calogero–Moser–Sutherland model is demonstrated. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Almost twenty years ago a possibility was discovered to maintain the integrability of Calogero–Moser–Sutherland models [1] (classification and description can be found in [2]) supposing that the particles moving on a line have additional internal degrees of freedom [3,4]. Later it was shown [5,6] that the generic elliptic Calogero–Moser–Sutherland type system, which consists of n particles on a line interacting with pair-

wise potential in the form of Weierstrass elliptic function $V(z) = \wp(z)$, admits the following generalization:

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \sum_{i \neq j}^n f_{ij} f_{ji} \wp(x_i - x_j). \quad (1)$$

Here apart from the canonical pairs (x_i, p_i) , describing the position and the momenta of particles and obeying nonvanishing Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \quad (2)$$

the “internal” degrees of freedom f_{ab} which satisfy the algebra

$$\{f_{ab}, f_{cd}\} = \delta_{bc} f_{ad} - \delta_{ad} f_{cb} \quad (3)$$

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are included. The recent comprehensive discussion of the integrability of generic spin Calogero–Moser–Sutherland systems can be found in the papers [7,8]. One of the most effective and transparent way to convince in the integrability of a Hamiltonian system is to find a known higher-dimensional exactly solvable model, whose dynamics on a certain invariant submanifold coincides with the dynamics of the given Hamiltonian system. This method is known as symplectic reduction method [9–11]. The Calogero–Sutherland–Moser systems with the so-called degenerate cases of the potential, when the Weierstrass elliptic function $\wp(z)$ reduces to $1/\sinh^2 z$, $1/\sin^2 z$ or to the rational function $1/z^2$, have a well-known interpretation as symplectic reductions of geodesic motions on symmetric spaces [12,13]. Furthermore, it was argued in [14] that symplectic reduction relates the elliptic Calogero–Moser–Sutherland system with certain integrable Hamiltonian system on the cotangent bundle to the central extension of two-dimensional Lie algebra of $SL(n, \mathbb{C})$ -valued currents on the some elliptic curve. New types of generalizations of the spin Calogero–Moser–Sutherland systems with nonstandard spin interactions have been constructed in [15] using discrete symmetries of the model.

In the present Letter we shall exploit the idea of symplectic reduction considering certain generalization of the Calogero–Sutherland–Moser model. Namely, we shall consider the integrable finite-dimensional model corresponding to the geodesic motion on the general linear matrix group with a positive determinant $GL^+(n, \mathbb{R})$,¹ endowed with the left- and right-invariant metric $ds^2 = \text{tr}(g^{-1} dg)^2$, where $g \in GL(n, \mathbb{R})$. In terms of this bi-invariant metric on $GL(n, \mathbb{R})$ group manifold the equations of motion for the corresponding dynamical system are encoded in the Lagrangian [9,10]

$$L_{GL} = \frac{1}{2} \text{tr}(g^{-1} \dot{g})^2, \tag{4}$$

where overdot denotes differentiation with respect to time. Bellow we shall represent the Hamiltonian corresponding to Lagrangian (4) in terms of a special parameterization, adapted to the action of $SO(n, \mathbb{R})$ symmetry group of the system. We shall demonstrate that

on the principal orbit stratum of $SO(n, \mathbb{R})$ group action the resulting Hamiltonian defines a new generalization of the Calogero–Sutherland–Moser model by introducing two internal variables “spin” and “isospin”. Furthermore, performing the Hamiltonian reduction owing to two types of symmetry: continuous and discrete, we show how to arrive at the conventional Hamiltonian of Euler–Calogero–Sutherland model

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{8} \sum_{i \neq j}^n \frac{l_{ij}^2}{\sinh^2(x_i - x_j)} \tag{5}$$

with internal variables $l_{ab} = -l_{ba}$, obeying the $SO(n, \mathbb{R})$ Poisson bracket algebra

$$\{l_{ab}, l_{cd}\} = \delta_{ac} l_{bd} - \delta_{ad} l_{bc} + \delta_{bd} l_{ac} - \delta_{bc} l_{ad}. \tag{6}$$

Another interesting systems arise when the dynamics takes place on the singular orbit strata of the $SO(n, \mathbb{R})$ group action. We found in this case new models representing a certain class of mass-deformed Calogero–Moser–Sutherland models. In particular, for the case of $GL(3, \mathbb{R})$ group, our analysis shows that the dynamics on singular orbit stratum with isotropy group $SO(2) \otimes Z_2$ corresponds to Calogero–Moser–Sutherland model, describing two particles, whose mass ratios is $m_1 : m_2 = 2 : 1$ (see Eq. (68)). The question of integrability of the mass-deformed Calogero–Moser–Sutherland models has been discussed in [16] and references therein.

2. Geodesic motion on the principal orbit stratum

2.1. Symmetries and dynamics

If we choose the elements of the matrix $g \in GL(n, \mathbb{R})$ as n^2 Lagrangian coordinates, the Euler–Lagrange equations obtained from Lagrangian (4) can be represented in the form of current conservation

$$\frac{d}{dt}(g^{-1} \dot{g}) = 0. \tag{7}$$

This form allows to find the general solution, i.e., the geodesics of the bi-invariant metric are given by

$$g(t) = g(0) \exp(tJ), \tag{8}$$

where $g(0)$ and J are two arbitrary constant matrices. The special choice of these matrices corresponds to the

¹ Hereafter we shall omit the upper index + to simplify the expressions.

particular solutions describing the motion on a certain invariant submanifold.

To show that (7) are equations of motion of many-particle system representing a certain generalization of the Calogero–Moser–Sutherland model, it is useful to pass to the Hamiltonian form of the geodesic motion on $GL(n, \mathbb{R})$ group. Performing Legendre transformation of Lagrangian (4)

$$\pi_{ab}^T = \frac{\partial L_{GL}}{\partial \dot{g}_{ab}} = (g^{-1} \dot{g} g^{-1})_{ab} \quad (9)$$

we arrive at the canonical Hamiltonian

$$H_{GL} = \frac{1}{2} \text{tr}(\pi^T g)^2 \quad (10)$$

generating the Hamilton equations of motion

$$\dot{g} = \{g, H_{GL}\} = g \pi^T g, \quad (11)$$

$$\dot{\pi} = \{\pi, H_{GL}\} = -\pi g^T \pi. \quad (12)$$

The nonvanishing Poisson brackets between the fundamental phase space variables (g_{ab}, π_{ab}) are

$$\{g_{ab}, \pi_{cd}\} = \delta_{ac} \delta_{bd}. \quad (13)$$

From now on the purpose of the present Letter will be to rewrite this Hamiltonian in terms of coordinates, adapted to the symmetry possessing the system. At first we would like to analyze the following symmetry action of the $SO(n, \mathbb{R})$ group on $GL(n, \mathbb{R})$:

$$g \mapsto g' = Rg, \quad (14)$$

with time-independent orthogonal matrix R . In order to consider the configuration space as manifold with orbit and slice structure with respect to this action, it is convenient to use the polar decomposition [17] for an arbitrary element of the $GL(n, \mathbb{R})$ group. For the sake of technical simplicity we investigate in details the $GL(3, \mathbb{R})$ group hereinafter, i.e.,

$$g = OS, \quad (15)$$

where S is a positive definite 3×3 symmetric matrix, and $O(\phi_1, \phi_2, \phi_3) = e^{\phi_1 J_3} e^{\phi_2 J_1} e^{\phi_3 J_3}$ is an orthogonal matrix with $SO(3, \mathbb{R})$ generators in adjoint representation $(J_a)_{ij} = \varepsilon_{iaj}$. Since the matrix g represents an element of $GL(3, \mathbb{R})$ group, we can treat the polar decomposition (15) as a uniquely invertible transformation from the configuration variables g to a new set of Lagrangian variables: six coordinates S_{ij} and three

coordinates ϕ_i . In terms of these new variables Lagrangian (4) can be rewritten as

$$L_{GL} = \frac{1}{2} \text{tr}(\Theta_L + \dot{S} S^{-1})^2, \quad (16)$$

where $\Theta_L := O^{-1} \dot{O}$ is a left-invariant 1-form on the $SO(3, \mathbb{R})$ group. To find the corresponding Hamiltonian we note that the polar decomposition (15) induces the point canonical transformation from variables (g_{ab}, π_{ab}) to new canonical pairs (S_{ab}, P_{ab}) and (ϕ_a, P_a) obeying the nonvanishing Poisson bracket relations

$$\{S_{ab}, P_{cd}\} = \frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}), \quad (17)$$

$$\{\phi_a, P_b\} = \delta_{ab}. \quad (18)$$

The expression of the old π_{ab} as a function of the new coordinates is

$$\pi = O(P - k_a J_a), \quad (19)$$

where

$$k_a = \gamma_{ab}^{-1} (\eta_b^L - \varepsilon_{bmn} (SP)_{mn}), \quad (20)$$

$\gamma_{ik} = S_{ik} - \delta_{ik} \text{tr} S$ and η_a^L are three left-invariant vector fields on $SO(3, \mathbb{R})$ group:

$$\eta_1^L = \frac{\sin \phi_3}{\sin \phi_2} P_1 + \cos \phi_3 P_2 - \cot \phi_2 \sin \phi_3 P_3, \quad (21)$$

$$\eta_2^L = \frac{\cos \phi_3}{\sin \phi_2} P_1 - \sin \phi_3 P_2 - \cot \phi_2 \cos \phi_3 P_3, \quad (22)$$

$$\eta_3^L = P_3. \quad (23)$$

Hence, in terms of the new variables, the canonical Hamiltonian (10) takes the form

$$H_{GL} = \frac{1}{2} \text{tr}(PS)^2 + \frac{1}{2} \text{tr}(J_a S J_b S) k_a k_b, \quad (24)$$

where the canonical variables (S_{ab}, P_{ab}) are invariant under transformation (14), while the angular variables (ϕ_a, P_a) undergo changes generating by the right-invariant Killing vector fields η_a^R :

$$\begin{aligned} \eta_1^R &= -\sin \phi_1 \cot \phi_2 P_1 + \cos \phi_1 P_2 \\ &\quad + \frac{\sin \phi_1}{\sin \phi_2} P_3, \end{aligned} \quad (25)$$

$$\begin{aligned} \eta_2^R &= \cos \phi_1 \cot \phi_2 P_1 + \sin \phi_1 P_2 \\ &\quad - \frac{\cos \phi_1}{\sin \phi_2} P_3, \end{aligned} \quad (26)$$

$$\eta_3^R = P_1, \quad (27)$$

whose Poisson brackets with the left-invariant vector fields η_a^L vanish $\{\eta_a^L, \eta_b^R\} = 0$.

Now we pass to analysis of another type of symmetry action of the orthogonal group. Lagrangian (4) is invariant under the transformations

$$g \mapsto g' = R^T g R \tag{28}$$

with constant orthogonal matrix $R \in SO(3, \mathbb{R})$. After implementation of the polar decomposition the symmetry transformation reads

$$S' = R^T S R, \quad O' = R^T O R. \tag{29}$$

The orbit space of the action $S \mapsto R^T S R$ of the $SO(3, \mathbb{R})$ group in the space of 3×3 symmetric matrices \mathcal{S} is given as a quotient $\mathcal{S}/SO(3, \mathbb{R})$. The quotient space $\mathcal{S}/SO(3, \mathbb{R})$ is a stratified manifold; orbits with the same isotropy group are collected into *strata* and uniquely parameterized by the set of ordered eigenvalues of the matrix $S: x_1 \leq x_2 \leq x_3$. The strata are classified according to the isotropy groups which are determined by the degeneracies of the matrix eigenvalues:

- (1) *Principal orbit-type stratum*, when all eigenvalues are unequal $x_1 < x_2 < x_3$, with the smallest isotropy group $Z_2 \otimes Z_2$.
- (2) *Singular orbit-type strata* forming the boundaries of the orbit space with
 - (a) two coinciding eigenvalues (e.g., $x_1 = x_2$), when the isotropy group is $SO(2) \otimes Z_2$;
 - (b) all three eigenvalues are equal ($x_1 = x_2 = x_3$); here the isotropy group coincides with the isometry group $SO(3, \mathbb{R})$.

To write down the Hamiltonian describing the motion on the principal orbit stratum, we introduce coordinates along the slices x and along the orbits χ . Namely, since the matrix S is positive definite and symmetric, we use the main-axes decomposition in the form

$$S = R^T(\chi) e^{2X} R(\chi), \tag{30}$$

where $R(\chi) \in SO(3, \mathbb{R})$ is an orthogonal matrix parameterized by three Euler angles $\chi = (\chi_1, \chi_2, \chi_3)$, and the matrix e^{2X} is diagonal $e^{2X} = \text{diag} \|e^{2x_1}, e^{2x_2}, e^{2x_3}\|$. The momenta p_i and p_{χ_i} , canonically conjugated to the eigenvalues x_i and the angles χ_i correspondingly,

$$\{x_i, p_j\} = \delta_{ij}, \quad \{\chi_i, p_{\chi_j}\} = \delta_{ij}, \tag{31}$$

can be found using the condition of canonical invariance of the symplectic 1-form

$$\sum_{i,j=1}^3 P_{ij} \dot{S}_{ij} dt = \sum_{i=1}^3 p_i \dot{x}_i dt + \sum_{i=1}^3 p_{\chi_i} \dot{\chi}_i dt. \tag{32}$$

The original momenta P_{ij} are expressed in terms of the new canonical pairs (x_i, p_i) and (χ_i, p_{χ_i}) as

$$P = R^T e^{-X} \left(\sum_{a=1}^3 \bar{P}_a \bar{\alpha}_a + \sum_{a=1}^3 \mathcal{P}_a \alpha_a \right) e^{-X} R, \tag{33}$$

with

$$\bar{P}_a = \frac{1}{2} p_a, \tag{34}$$

$$\mathcal{P}_a = - \frac{\xi_a^R}{4 \sinh(x_b - x_c)} \tag{35}$$

(cyclic permutation $a \neq b \neq c$).

In representation (33), we introduced the orthogonal basis for the symmetric 3×3 matrices $\alpha_a = (\bar{\alpha}_a, \alpha_a)$, $a = 1, 2, 3$, with the scalar product

$$\begin{aligned} \text{tr}(\bar{\alpha}_a \bar{\alpha}_b) &= \delta_{ab}, & \text{tr}(\alpha_a \alpha_b) &= 2\delta_{ab}, \\ \text{tr}(\bar{\alpha}_a \alpha_b) &= 0 \end{aligned} \tag{36}$$

and the $SO(3, \mathbb{R})$ right-invariant Killing vectors

$$\begin{aligned} \xi_1^R &= -\sin \chi_1 \cot \chi_2 p_{\chi_1} + \cos \chi_1 p_{\chi_2} \\ &+ \frac{\sin \chi_1}{\sin \chi_2} p_{\chi_3}, \end{aligned} \tag{37}$$

$$\begin{aligned} \xi_2^R &= \cos \chi_1 \cot \chi_2 p_{\chi_1} + \sin \chi_1 p_{\chi_2} \\ &- \frac{\cos \chi_1}{\sin \chi_2} p_{\chi_3}, \end{aligned} \tag{38}$$

$$\xi_3^R = p_{\chi_1}. \tag{39}$$

Thus, after passing to main-axes variables (x_i, p_i) and (χ_i, p_{χ_i}) , the canonical Hamiltonian reads

$$\begin{aligned} H_{GL} &= \frac{1}{8} \sum_{a=1}^3 p_a^2 + \frac{1}{16} \sum_{(abc)} \frac{(\xi_a^R)^2}{\sinh^2(x_b - x_c)} \\ &- \frac{1}{4} \sum_{(abc)} \frac{(R_{am} \eta_m^L + (1/2)\xi_a^R)^2}{\cosh^2(x_b - x_c)}. \end{aligned} \tag{40}$$

Here (abc) means cyclic permutations $a \neq b \neq c$. Hence we conclude that the integrable dynamical system describing a free motion on the principal orbit stratum can be interpreted in the adapted basis as generalized Euler–Calogero–Moser–Sutherland

model. The generalization consists in the introduction of two types of internal dynamical variables ξ and η —“spin” and “isospin” degrees of freedom. From their explicit expressions (see Eqs. (21)–(23) and (37)–(39)) it follows that they satisfy $SO(3, \mathbb{R}) \oplus SO(3, \mathbb{R})$ Poisson bracket algebra:

$$\{\eta_a^L, \eta_b^L\} = -\varepsilon_{abc}\eta_c^L, \tag{41}$$

$$\{\xi_a^R, \xi_b^R\} = \varepsilon_{abc}\xi_c^R, \tag{42}$$

$$\{\eta_a^L, \xi_b^R\} = 0. \tag{43}$$

2.2. Lax pair for the generalized Euler–Calogero–Moser–Sutherland model

In order to find a Lax representation for the generalized Euler–Calogero–Moser–Sutherland model (40) let us consider the integrals of the geodesic motion on the principal orbit stratum. The integrals of motion can be written in Hamiltonian form, following from (7), as

$$J_{ab} = (\pi^T g)_{ab}. \tag{44}$$

The algebra of this integrals realizes on the symplectic level the $GL(n, \mathbb{R})$ algebra

$$\{J_{ab}, J_{cd}\} = \delta_{bc}J_{ad} - \delta_{ad}J_{cb}. \tag{45}$$

After the transformation to scalar and rotational variables (30), the expression for the current J reads

$$J = \frac{1}{2} \sum_{a=1}^3 R^T (p_a \bar{\alpha}_a - i_a \alpha_a - j_a J_a) R, \tag{46}$$

where

$$i_a = \sum_{(abc)} \frac{1}{2} \xi_a^R \coth(x_b - x_c) + \left(R_{am} \eta_m^L + \frac{1}{2} \xi_a^R \right) \tanh(x_b - x_c) \tag{47}$$

and

$$j_a = R_{am} \eta_m^L + \xi_a^R. \tag{48}$$

Using expressions (46) for the integrals J_{ab} the classical equations of motion for generalized Euler–Calogero–Moser–Sutherland model can be rewritten in the

Lax form²

$$\dot{L} = [A, L], \tag{49}$$

where the 3×3 matrices are given explicitly as

$$L = \begin{pmatrix} p_1 & L_3^+ & L_2^- \\ L_3^- & p_2 & L_1^+ \\ L_2^+ & L_1^- & p_3 \end{pmatrix},$$

$$A = \frac{1}{4} \begin{pmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{pmatrix}. \tag{50}$$

Entries A_a and L_a^\pm of matrices (50) are given as

$$L_a^\pm = -\frac{1}{2} \xi_a^R \coth(x_b - x_c) - \left(R_{am} \eta_m^L + \frac{1}{2} \xi_a^R \right) \tanh(x_b - x_c) \pm \left(R_{am} \eta_m^L + \xi_a^R \right) \tag{51}$$

and

$$A_a = \frac{\xi_a^R}{2 \sinh^2(x_b - x_c)} - \frac{R_{am} \eta_m^L + (1/2) \xi_a^R}{\cosh^2(x_b - x_c)}, \tag{52}$$

where (a, b, c) means cyclic permutations of $(1, 2, 3)$.

Below relations to the standard Euler–Calogero–Moser–Sutherland model (5) will be demonstrated.

2.3. Reduction to Euler–Calogero–Moser–Sutherland model

2.3.1. Reduction using discrete symmetries

Now we shall demonstrate how the IIA_3 Euler–Calogero–Moser–Sutherland model arises from the canonical Hamiltonian (10) after projection onto a certain invariant submanifold determined by discrete symmetries. Let us impose the condition of symmetry of the matrices $g \in GL(3, \mathbb{R})$:

$$\psi^{(1)} = g - g^T = 0. \tag{53}$$

In order to find an invariant submanifold, it is necessary to supplement the constraints (53) with the new ones

$$\psi^{(2)} = \pi - \pi^T = 0. \tag{54}$$

² We set here aside the constructions of the Lax pairs with a spectral parameter. The Lax representations with a spectral parameter for the spin Calogero–Moser–Sutherland models associated with the root systems of simple Lie algebras were constructed in [7,8,18].

Indeed, using the Hamilton equations (11) and (12), one can check that the surface defined by the set of constraints $\Psi_A = (\psi^{(1)}, \psi^{(2)})$ represents an invariant submanifold in the $GL(3, \mathbb{R})$ phase space

$$\dot{\psi}^{(1)}|_{\Psi_A=0} = 0, \quad \dot{\psi}^{(2)}|_{\Psi_A=0} = 0, \quad (55)$$

and the dynamics of the corresponding induced system is governed by the reduced Hamiltonian

$$H_{GL(3,\mathbb{R})}|_{\Psi_A=0} = \frac{1}{2} \text{tr}(\pi g)^2. \quad (56)$$

The matrices g and π are now symmetric nondegenerate matrices, and one can be convinced that this expression leads to the Hamiltonian of the IIA_3 Euler–Calogero–Moser–Sutherland model. To prove this, it is necessary to note that the Poisson matrix $C_{AB} = \|\{\psi^{(1)}, \psi^{(2)}\}\|$ is not degenerate and after projection on the invariant submanifold, the canonical Poisson structure is changed according to the Dirac prescription

$$\{F, G\}_D = \{F, G\}_{PB} - \{F, \psi_A\} C_{AB}^{-1} \{\psi_B, G\} \quad (57)$$

for arbitrary functions F and G . The resulting fundamental Dirac brackets are

$$\{g_{ab}, \pi_{cd}\}_D = \frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}). \quad (58)$$

If now we introduce as in Section 2.1 the main-axes variables (x_a, p_a) and (χ_a, p_{χ_a}) instead of the symmetric variables (g_{ab}, π_{ab}) and use representations (30) and (33), then one can convince that projected Hamiltonian (56), up to the time rescaling $t \mapsto 4t$, governs the same dynamics as Hamiltonian (5) of IIA_3 Euler–Calogero–Moser–Sutherland model with the intrinsic spin variables $l_{ij} = \varepsilon_{ijk} \xi_k^R$.

2.3.2. Reduction due to continuous symmetry

Let us now derive a reduced Hamiltonian system employing certain continuous symmetries of the model. For Hamiltonian (10) all angular variables are gathered in the three left-invariant vector fields η_a^L and thus the corresponding right-invariant fields η_a^R (25)–(27) are integrals of motion:

$$\{\eta_a^R, H_{GL}\} = 0. \quad (59)$$

The surface in the phase space determined by the constraints

$$\eta_a^R = 0 \quad (60)$$

defines an invariant submanifold. These constraints obey the algebra $\{\eta_a^R, \eta_b^R\} = \varepsilon_{abc} \eta_c^R$ and according to the Dirac terminology [19,20] are first class constraints; this means that after projection on the constraint shell (60), the corresponding cyclic coordinates disappear from the projected Hamiltonian. To prove this one can use the relation $\eta_a^R = O_{ab} \eta_b^L$ between the left- and the right-invariant Killing vector fields. Then, after projection to the constraint surface (60), Hamiltonian (40) reduces to

$$H_{GL}|_{\eta_a^R=0} = \frac{1}{8} \sum_a^3 p_a^2 + \frac{1}{4} \sum_{(abc)} \frac{(\xi_a^R)^2}{\sinh^2 2(x_b - x_c)}. \quad (61)$$

After rescaling of the variables $2x_a \mapsto x_a$, one can be convinced that the derived Hamiltonian coincides with the Euler–Calogero–Moser–Sutherland Hamiltonian (5), where the intrinsic spin variables are $l_{ij} = \varepsilon_{ijk} \xi_k^R$.

As it was outlined in Section 2.2 apart from the integrals η_a^R system (10) possesses integrals (46). Using these integrals one can choose different invariant submanifold and to derive the corresponding reduced system. Here we would like only to mention that after performing reduction to the surface defined by the vanishing integrals $j_a = 0$, we again arrive at the Euler–Calogero–Moser–Sutherland system.

3. Geodesic motion on the singular orbit strata

In the previous sections we have investigated the geodesic motion on the principal orbit stratum, i.e., under the supposition that the symmetric matrix S in the polar representation (15) has three different eigenvalues. We now turn our attention to dynamical system corresponding to the geodesic motion on the singular orbit strata. For the sake of technical simplicity, we restrict ourselves to invariant submanifold of the phase space defined by $\eta^R = 0$ and consider a geodesic motion on the singular orbit stratum with two coinciding eigenvalues of the matrix S in the case of $GL(3, \mathbb{R})$ group. Below we use two alternative methods. At first a special parameterizations of the subspace of 3×3 symmetric matrices with two coinciding eigenvalues are exploited and as a result we find that the Hamiltonian system describing the geodesic motion on the 4-dimensional singular orbit stratum is

IIA₂ Calogero–Moser–Sutherland model with particle mass ratio $m_1 : m_2 = 2 : 1$. Afterwards, based on the observation that the singular orbits of the configuration space represent the boundary of Principle orbit, using appropriate limiting procedure we derive from Hamiltonian (40) again a mass-deformed IIA₂ Calogero–Moser–Sutherland model with the same particle mass ratio $m_1 : m_2 = 2 : 1$.

3.1. Mass-deformed Calogero–Moser–Sutherland model via explicit parameterizations of the singular orbit stratum

The singular orbits have continuous isotropy groups and this leads to the modification of geodesic motion. For the case we are interesting in, $GL(3, \mathbb{R})$ group and two equal eigenvalues of the symmetric matrix S , it is $SO(2) \otimes Z_2$.

The linear space of the real symmetric $n \times n$ matrices with two coinciding eigenvalues has a real dimension [21]

$$\dim S(n) - \dim S(2) + 1. \tag{62}$$

Hence, we are able to parameterize such a subspace of $GL(3, \mathbb{R})$ group by four real independent parameters

$$S_{ab} = e^{2x} \delta_{ab} - 2e^{x+y} \sinh(x - y) n_a n_b, \tag{63}$$

where n_a is a unit 3-dimensional vector

$$n_a = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta). \tag{64}$$

We infer from the expression for the bi-invariant metric on the $GL(n, \mathbb{R})$ group that the metric induced on the 4-dimensional singular orbit stratum parameterizing according to (63) is

$$\begin{aligned} \text{tr}(S^{-1} dS)^2 &= 8 dx^2 + 4 dy^2 \\ &+ 8 \sinh^2(x - y) (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned} \tag{65}$$

Therefore the Lagrangian $L = (1/2) \text{tr}(S^{-1} \dot{S})^2$ on the singular orbit stratum can be written as

$$L = 4\dot{x}^2 + 2\dot{y}^2 + 4 \sinh^2(x - y) \dot{n}^2, \tag{66}$$

where

$$\dot{n}^2 = \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2. \tag{67}$$

The Legendre transformation gives the canonical Hamiltonian

$$H_{GL(3, \mathbb{R})}^{(2)} = \frac{1}{16} p_x^2 + \frac{1}{8} p_y^2 + \frac{l^2}{16 \sinh^2(x - y)}, \tag{68}$$

where

$$l^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}. \tag{69}$$

Now taking into account that l^2 is a constant of motion we convince that the geodesic motion with respect to the bi-invariant metric on the 4-dimensional singular orbit stratum corresponds to 2-particle mass-deformed Calogero–Moser–Sutherland model with particle mass ratio $m_1 : m_2 = 2 : 1$. Following this interpretation of the Hamiltonian system (68) in terms of particles, one can say that the motion on this singular orbit stratum corresponds to some “gluing” of two particles and formation of a bound particle with double mass. It is apparent that using the center-mass coordinates the obtained 4-dimensional Hamiltonian system (68) can be reduced to 2-dimensional integrable model.

3.2. Mass-deformed Calogero–Moser–Sutherland model via limiting procedure from the free motion on the principal orbit stratum

As it was mentioned above the singular orbits with two coinciding eigenvalues form the boundary of the principle orbit stratum. Based on this observation, now we would like to extend Hamiltonian (40), given on the principal orbit stratum, to its boundary by introducing the constraints that force the dynamics specially to the neighborhood of the boundary and then use a limiting procedure. Since Hamiltonian (40) has a singularities along the singular orbits, we impose the following constraints on the phase space variables:

$$\chi^{(1)} = x_1 - x_2 - \epsilon, \quad \phi^{(1)} = \xi_3 - \epsilon^{1+\alpha}, \tag{70}$$

with positive small parameter $\epsilon \ll 1$ and constant $\alpha \geq 1$. When ϵ goes to zero the system tends to the singular orbit stratum with two coinciding eigenvalues. Let us at first find the dynamical consistence of constraints (70) considering instead of Hamiltonian (40)

the modified Hamiltonian³

$$H' = \frac{1}{8} \sum_{a=1}^3 p_a^2 + \frac{1}{16} \sum_{(abc)} \xi_a^2 V(x_b - x_c) + u(x_1 - x_2 - \epsilon) + \lambda(\xi_3 - \epsilon^{1+\alpha}), \tag{71}$$

with Lagrangian multipliers u and λ and $V(x) = \sinh^{-2}(x)$. The conservation in time of the constraints $\chi^{(1)}$ and $\phi^{(1)}$ leads to the new constraints

$$\chi^{(2)} = p_1 - p_2, \quad \phi^{(2)} = \xi_2, \tag{72}$$

while from the maintenance of (72), the Lagrangian multipliers can be found:

$$u = \frac{1}{16} \left[\frac{1}{2} \xi_1^2 V'(x_2 - x_3) - \frac{1}{2} \xi_2^2 V'(x_1 - x_3) - \xi_3^2 V'(x_1 - x_2) \right], \tag{73}$$

$$\lambda = \frac{1}{8} \xi_3 [V(x_2 - x_3) - V(x_1 - x_2)]. \tag{74}$$

Hence we conclude that the constraint surface defined by (70) and (72) represents an invariant submanifold for Hamiltonian (71). Because these constraints are second class in the Dirac terminology [11,19,20] we are able to replace the Poisson brackets by the Dirac ones according to (57) and let the constraint functions to vanish. One can easily verify that for the canonical variables (x, p) the corresponding nonzero fundamental Dirac brackets are

$$\{x_i, p_j\}_D = \frac{1}{2}, \quad i, j = 1, 2, \tag{75}$$

$$\{x_3, p_3\}_D = 1,$$

while for the angular variables we have

$$\{\xi_a^R, \xi_b^R\}_D = 0, \quad a, b = 1, 2, 3. \tag{76}$$

Projecting the Hamiltonian H' to the constraint shell and then taking the limit $\epsilon \rightarrow 0$ we obtain

$$H^{(2)} := \lim_{\epsilon \rightarrow 0} H_T|_{CS} = \frac{1}{4} p_1^2 + \frac{1}{8} p_3^2 + \frac{\xi_1^2}{16 \sinh^2(x_1 - x_3)}. \tag{77}$$

The quantity ξ_1^2 in (77) is a constant of motion, that is the reminiscent of conserved total momentum $\xi_1^2 +$

$\xi_2^2 + \xi_3^2$ for the Euler–Calogero–Moser–Sutherland Hamiltonian (61). So, using the appropriate limiting procedure from the principal orbit stratum we arrive at mass-deformed Calogero–Moser–Sutherland model corresponding to the singular stratum, labeled by two coinciding eigenvalues of 3×3 symmetric matrix. Finally, we establish a relation between (77) and Hamiltonian (68) derived before. To achieve this it is necessary to rescale the momentum $p_1 \mapsto 2^{-1} p_1$ so that variables (x_1, p_1, x_3, p_3) obey canonical Poisson bracket relations instead of the Dirac brackets (75), identify variables $x := x_1, y := x_3$, and constants $l^2 = \xi_1^2$. As a result we arrive at Hamiltonian system which coincides with the mass-deformed IIA₃ Euler–Calogero–Moser–Sutherland model, derived in the previous section using explicit parameterizations of the induced metric on singular orbit stratum.

4. Concluding remarks

Nowadays we have revival of the interest to matrix models (see, e.g., [22]) connected with the search of relations between the supersymmetric Yang–Mills theory and integrable systems (for a modern review see [23]). As it has been shown in the recent paper [24] the Euler–Calogero–Moser–Sutherland model with certain external potential describes the gauge invariant long-wavelength approximation of the $SU(2)$ Yang–Mills field theory [25]. In the context of the consideration of higher-dimensional gauge groups it is interesting to explore the mechanics on the general linear group manifold. In the present Letter we have considered the simplest version of geodesic motion on $GL(n, \mathbb{R})$ group manifold and analyze the dynamics in the context of isometries of the bi-invariant metric. Namely, we intensively exploit the slice structure of $GL(n, \mathbb{R})$ based on the existence of the principal orbit stratum and singular orbit strata of the $SO(n, \mathbb{R})$ group action. We demonstrated that the free motion on the principal orbit stratum corresponds to the integrable many-body system of free particles on a line with two types of internal variables called “spin” and “isospin”, which is a generalization of the Euler–Calogero–Moser–Sutherland model. To clarify its relation to the known integrable models we have implemented two different types of reduction: due to discrete symmetry and due to continuous symme-

³ Here we again restrict consideration by the case $\eta_a^R = 0$.

try. In both cases we derived IIA_n Euler–Calogero–Moser–Sutherland model. Concerning the singular orbit strata, it was shown that in this case the corresponding dynamical system is the certain deformation of the Calogero–Moser–Sutherland model in a sense of description of particles with different masses. The masses of the particles are not arbitrary, their mass ratios depend on the degeneracy of the given singular orbit stratum. As example was considered the case of $GL(3, \mathbb{R})$ group, restricted to the singular orbit stratum with two coinciding eigenvalues. In this case the reduced system coincides with the IIA_2 Calogero–Moser–Sutherland model with particle mass ratio $m_1 = 2m_2$.

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