Proc. A. Razmadze Math. Inst. 132 (2003), 143–147

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Zaremba's Problem in One Class of Harmonic Functions

(Reported on 18.04.2002)

Different classes of boundary value problems for harmonic functions which at the same time are the real parts of analytic functions from Smirnov classes are studied [1–4]. It is of interest to consider in these classes the problem when an a value of an unknown function is given on one part of the boundary and a value of its derivative in the direction of inner normal (Zaremba's problem) (see [5]). To formulate and study the problem we introduce the most suitable for that case a weight class of harmonic functions of Smirnov type.

Let $\cup$ be a unit circle bounded by the circumference γ and let $\gamma_k = (a_k, b_k)$, $k = \overline{1,m}$ be arcs lying separately on γ. Moreover, let $[a'_k, b'_k]$ be an arc lying on γ_k, $k = \overline{1,m}$ and $\widetilde{\gamma} = \bigcup\limits_{k=1}^{m}([a_k, a'_k] \cup [b'_k, b_k])$. By $c_1, c_2, \dots, c_{2m}$ we denote the points a_k, b_k taken arbitrarily. We consider also the points d_k, $k = \overline{1,n}$ which are different from c_k; note that points $d_{n_1+1} \cdots d_{n_1}$ are on the set, $\Gamma_1 = \bigcup\limits_{k=1}^{m} \gamma_k$, while the points $d_{n_1+1} \cdots d_n$ are on $\Gamma_2 = \gamma \backslash \Gamma_1$. Let m_1 be an integer from the segment $[0, 2m]$, and $p > 1$, $q > 1$. Suppose

$$\omega_1(z) = \prod_{k=1}^{n_1} (z - d_k)^{\alpha_k}, \quad -\frac{1}{p} < \alpha_k < \frac{1}{p'}, \quad p' = \frac{p}{p-1}, \tag{1}$$

$$\omega_2(z) = \prod_{k=1}^{m_1} (z - c_k)^{\nu_k} \prod_{k=m_1+1}^{2m} (z - c_k)^{\lambda_k} \prod_{k=n_1+1}^{n} (z - d_k)^{\beta_k}, \tag{2}$$
$$-\frac{1}{q} < \nu_k < 0, \quad 0 \le \lambda_k < \frac{1}{q'}, \quad -\frac{1}{q} < \beta_k < \frac{1}{q'}.$$

If E is a finite union of closed arcs on γ, then we put $\theta(E) = \{\mu : 0 \le \theta \le 2\pi,\ e^{i\theta} \in E\}$. By $A(E)$ we denote a set of functions, absolutely continuous on $\theta(E)$, and by χ_E a characteristic function of the set E.

We say that a harmonic in the circle $\cup$ function $u(z)$, $z = x + iy = re^{i\varphi}$ belongs to the class $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ if

$$\sup_{0<r<1} \Bigg[\int\limits_{\theta(\Gamma_1)} \big|u(re^{i\theta})\omega_1(re^{i\theta})\big|^p \, d\theta +$$
$$+ \int\limits_{\theta(\Gamma_2)} \Big[\Big|\frac{\partial u}{\partial x}(re^{i\theta})\Big|^q + \Big|\frac{\partial u}{\partial y}(re^{i\theta})\Big|^q\Big] \big|\omega_2(re^{i\theta})\big|^q \, d\theta\Bigg] < \infty. \tag{3}$$

If $\Gamma_1 = \gamma$, $\omega_1 = 1$ this class coincides with the class h_p (see [6], Ch. IX).

2000 *Mathematics Subject Classification:* 35I05, 36I25

Key words and phrases. Harmonic functions of Smirnov's type, Zeremba's problem, Mixed problems

Lemma 1. *If* $u \in h(\Gamma_{ip}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$ *then:*

(a) *there exists the number* $\sigma > 1$ *such that* $u \in h_\sigma$*; hence the function* u *has angular boundary values* $u^+ \in L^\sigma(\gamma)$ *and is representable by the Poisson integral with density* u^+*;*

(b) *if* v *is the function, harmonically conjugate to* u*, then* $v \in h(\Gamma_{1p_1}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p_1 = \frac{p\sigma}{p+\sigma}$*;*

(c) *the function* $\phi(z) = u(z) + iv(z)$ *belongs to the Hardy class* H^σ*, and*

$$\sup_{0<r<1} \int\limits_{\theta(\Gamma_2)} \left|\phi'(re^{i\theta})\right|^q \left|\omega_2(re^{i\theta})\right|^q d\theta < \infty; \tag{4}$$

(d) *if* $u^+ A(\Gamma_2)$*, then the function* $\frac{\partial u}{\partial \varphi}(re^{i\varphi})$ *has angular boundary values* $(\frac{\partial u}{\partial \varphi})^+$ *almost everywhere on* Γ_2 *coinciding with* $\frac{\partial u^+}{\partial \varphi}(e^{i\varphi_0})$, $e^{i\varphi_0} \in \Gamma_2$ *and* $(\frac{\partial u}{\partial \varphi})^+ \in L^q(\Gamma_2; \omega_2)$.

Lemma 2. *If* $u(re^{i\varphi})$ *is representable by the Poisson integral with density* f*, where* $f \in L^p(\Gamma_1 \backslash \widetilde{\gamma}; \omega_1)$, $p > 1$, $f \in A(\Gamma_2 \cup \widetilde{\gamma})$, $f' \in L^q(\Gamma_2; \omega_2)$, $q > 1$, *then* $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$.

Consider the problem: find the function u, satisfying the following conditions:

$$\begin{cases} \Delta u = 0, & u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2)), \ \ p > 1, \ \ q > 1, \\ u^+\big|_{\Gamma_1 \backslash \widetilde{\gamma}} = f, & f \in L^p(\Gamma_1 \backslash \widetilde{\gamma}; \omega_1); \ \ u^+ \in A(\Gamma_2 \cup \widetilde{\gamma}), \\ u^+\big|_{\widetilde{\gamma}} = \psi, & \psi \in A(\widetilde{\gamma}), \ \ \psi' \in L^q(\widetilde{\gamma}; \omega_2); (\frac{\partial u}{\partial n})^+\big|_{\Gamma_2} = g, \ \ g \in L^q(\Gamma_2; \omega_2). \end{cases} \tag{5}$$

A solution, if any, of that problem is, be Lemma 1, representable by the Poisson integral. By virtue of (5), its density u^+ is known on a part Γ_1. To find u^+ on Γ_2, we use the equalities $\frac{\partial u}{\partial n}(re^{i\varphi}) = -\frac{\partial u}{\partial r}(re^{i\varphi}) = -\frac{1}{r}\frac{\partial v}{\partial \varphi}(re^{i\varphi})$ and by means of simple calculations we arrive with respect to $\frac{\partial u^+}{\partial \theta}$ at the equation (in the class $L^q(\Gamma_2; \omega_2)$)

$$\frac{1}{2\pi} \int\limits_{\theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} \, d\theta = \mu(\varphi), \tag{6}$$

$$\mu(\varphi) = -g(e^{i\varphi}) - \frac{1}{2\pi} \int\limits_{\theta(\Gamma_1 \backslash \widetilde{\gamma})} f(\theta) \frac{d\theta}{2\sin^2 \frac{\theta-\varphi}{2}} - \frac{1}{2\pi} \int\limits_{\theta(\widetilde{\gamma})} \psi(\theta) \frac{d\theta}{2\sin^2 \frac{\theta-\varphi}{2}} +$$

$$+ \frac{1}{2\pi} \sum_{k=1}^m \Big[\psi(a_{k+1}) \operatorname{ctg} \frac{\alpha_{k+1} - \varphi}{2} - \psi(b_k) \operatorname{ctg} \frac{\beta_k - \varphi}{2}\Big], \tag{7}$$

$$a_k = e^{i\alpha_k}, \quad b_k = e^{i\beta_k}, \quad a_{m+1} = a_1.$$

Equation (6) is equivalent to

$$\frac{1}{\pi i} \int\limits_{\Gamma_2} \frac{\partial u^+}{\partial \theta} \frac{d\tau}{\tau - e^{i\varphi}} = i\mu(\varphi) + a, \quad a = \frac{1}{2\pi} \sum_{k=1}^m \big[\psi(a_{k+1}) - \psi(b_k)\big], \tag{8}$$

with the additional condition $\frac{1}{2\pi} \int\limits_{\theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} d\theta = a$.

Equation (8) has been solved for a particular case with weight ω_2 (when $\nu_k = -\frac{1}{2q}$ $\lambda_k = \frac{1}{2q'}$) has been solved in [7] (pp. 35–46; see also [8], pp. 104–108). Following these works, it is not difficult to solve equation (8) for weight ω_2 for which

$$-\frac{1}{q} < \frac{1}{2} + \nu_k < \frac{1}{q'}, \quad -\frac{1}{q} < \lambda_k - \frac{1}{2} < \frac{1}{q'} \tag{9}$$

If these conditions are fulfilled, equation (8) is, undoubtedly, solvable in the space $L^q(\Gamma_2;\omega_2)$, if $m_1 \le m$ and the solution contains an arbitrary polynomial P_{r-1} of order $r = m - m_1$, satisfying the condition

$$\int\limits_{\theta(\Gamma_2)} R(e^{i\varphi})P_{r-1}(e^{i\varphi})\,d\varphi = 0, \tag{10}$$

where $R(t) = [R(z)]^+$, $t = e^{i\varphi}$, $R(z) = \Pi_1(z)\Pi_2^{-1}(z)$, $\Pi_1(z) = \sqrt{\prod\limits_{k=1}^{m}(z-c_k)}$, $\Pi_2(z) = \sqrt{\prod\limits_{k=m_1+1}^{2m}(z-c_k)}$. If, however, $m_1 > m$, then for equation (8) to be solvable, it is necessary and sufficient that the conditions

$$\int\limits_{\Gamma_2} t^k\big[R(t)\big]^{-1}\big(i\mu(t)+a\big)dt = 0, \quad k = \overline{0, l-1}, \quad l = m_1 - m. \tag{11}$$

be fulfilled. If these conditions are fulfilled, then equation (8) is uniquely solvable. In both cases the solution is given explicitly (in quadratures).

Having solved equation (8) and obtained $\frac{\partial u^+}{\partial\theta}$, we can find the function u^+ on Γ_2. The solution, besides P_{r-1}, contains $2m$ arbitrary constants. Conditions (10) can be fulfilled automatically, if by choosing free constants, according to the condition $u^+ \in A(\Gamma_2^\nu\widetilde{\gamma})$ from (5), we achieve equalities $u^+(b_k) = \psi(b_k)$ and $u^+(a_k) = \psi(a_k)$). If u^+ is known on the whole γ, by using Lemma 2 we can construct the solution of problem (5). As a result we state the following

Theorem. *If for all ω_1 conditions* (1) *and for ω_2 conditions from* (2) *and* (9), *i.e.,*

$$-\frac{1}{q} < \nu_k < \min\Big(0; \frac{1}{q'} - \frac{1}{2}\Big), \quad \max\Big(0, \frac{1}{2} - \frac{1}{q}\Big) \le \lambda_k < \frac{1}{q'},$$

are fulfilled, then for problem (5) *to be solvable:*

(I) *for $m_1 \le m$, it is necessary and sufficient that the conditions*

$$\int\limits_{\beta_k}^{\alpha_k+1} \mathrm{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\theta(\gamma_2)} \frac{i\mu(\tau)+a}{R(r)(r-e^{i\alpha})}\,d\tau\Big]d\alpha = \psi(a_k+1) - \psi(b_k), \quad k = \overline{1,m}; \tag{12}$$

be fulfilled;

(II) *for $m_1 > m$, it is necessary and sufficient that conditions* (11) *and* (12) *be fulfilled;*

(III) *if the above-mentioned conditions are fulfilled, then the solution of problem* (5) *is given by the equality*

$$u(re^{i\varphi}) = u^*(re^{i\varphi}) + u_0(re^{i\varphi}), \tag{13}$$

where

$$u^*(re^{i\varphi}) = \frac{1}{2\pi}\int\limits_{\theta(\Gamma_1\setminus\widetilde{\gamma})} f(\rho)P(r,\theta-\varphi)\,d\theta + \frac{1}{2\pi}\int\limits_{\theta(\widetilde{\gamma})} \psi(\theta)\rho(r,\theta-\varphi)d\theta +$$

$$+\frac{1}{2\pi}\int\limits_{\theta(\Gamma_2)} W_{\Gamma_2}(\theta)P(r,\theta-\varphi)\,d\theta, \tag{14}$$

where in which

$$\rho(r,x)=\frac{1-r^2}{1+r^2-2r\cos x},$$

$$W_{\Gamma_2}(\theta)=\int\limits_{\beta_i}^{\theta}\chi_{\theta(\Gamma_2)}(\alpha)\,\mathrm{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]\,d\alpha+B_k,$$

$$r=m-m_1,`$$

$$B_k=\psi(a_{k+1})-\int\limits_{\beta_1}^{\alpha_{k+1}}\chi_{\theta(\Gamma_2)}(\alpha)\,\mathrm{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]\,d\alpha, \tag{15}$$

$$u_0(z)=\begin{cases}0, & \textit{for}\quad m_1\ge m,\\ \frac{1}{2\pi}\int_0^{2\pi}W^*_{\Gamma_2}(\theta)P(r,\theta-\varphi)\,d\theta, & \textit{for}\quad m_1<m;\end{cases} \tag{16}$$

here

$$W^2_{\Gamma_2}(\theta)=\int\limits_{\beta_1}^{\theta}\chi_{\theta(\Gamma_2)}(\alpha)\,\mathrm{Re}\Big[R(e^{i\alpha})P_{r-1}(e^{i\alpha})\Big]d\alpha-\int\limits_{\beta_k}^{\alpha_{k+1}}\mathrm{Re}\,\big[R(e^{i\alpha})P_{r-1}(re^{i\alpha})\big]\,d\alpha,$$

$$e^{i\mu}\in(b_k^{a_{k+1}})-p_{r-1}(e^{i\beta})=\sum_{j=0}^{r-1}(x_j+iy_j)e^{ij\theta},$$

where x_j, y_j, $j=\overline{0,r-1}$ *is the solution of the system*

$$\begin{aligned}&\sum_{j=0}^{r-1}\int\limits_{\beta_k}^{\alpha_{k+1}}\big[x_j\,\mathrm{Re}\,R(e^{i\theta})\cos j\theta-y_j\,\mathrm{Im}\,R(e^{i\theta})\sin j\theta\big]\,d\theta=0,\\ &\sum_{j=0}^{r-1}\int\limits_{\beta_k}^{\alpha_{k+1}}\big[x_j\,\mathrm{Im}\,R(e^{i\theta})\cos j\theta+y_i\,\mathrm{Re}\,R(e^{i\theta})\sin j\theta\big]\,d\theta=0,\end{aligned} \tag{17}$$

If ν *is the rank of the matrix of that system, then the solution* $(x_0\cdots x_{r-1}y_0\cdots y_{r-1})$ *contains* $2(m-m_1)-\nu$ *arbitrary parameters.*

Similarly to (5) one can formulate and solve Zaremba's problem (using conformal mapping) for simply connected domains bounded by the Ljapunov curve.

References

1. V. Kokilashvili and V. Paatashvili, On the Riemann-Hilbert problem in the domain with a non-smooth boundary. *Georgian Math. J.* **4**(1997), No. 3, 279–302.
2. V. Kokilashvili and V. Paatashvili, Neumann problem in a class of harmonic functions in domains with a piecewise Ljapunov boundary. *Mem. Differential Equations Math. Phys.* **12**(1997), 114–121.
3. G. Khuskivadze, V. Kokilashvili and V. Paatashvili, Boundary value problems of analytic and harmonic functions in domains with non-smooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14**(1998), 1–195.
4. R. Duduchava and D. Silbermann, Boundary value problems in domains with peaks. *Mem. Differential Equations Math. Phys.* **21**(2000), 1–122.
5. S. Zaremba, Sur un probleme mixte relatif a l'equation de Laplace. *Bulletin de l'Acadimic des sciences de Cracovie des sciences mathematiques et naturelles serie*

(*A*). 1910, 313–344; Russian transl.: *Uspekhi Mat. Nauk* **1**(1946), No. 3–4(13–14), 125–146.

6. G. M. Goluzin, Geometric theory of functions of a complex variable. (Russian) *Nauka, Moskva,* 1966.
7. B. Khvedelidze, Linear discontinuous boundary value problems of function theory, singular integral equations, and some of their applications. (Russian) *Trudy Math. Inst. Razmadze Akad. Nauk Gruz. SSR* **23**(1958), 3–158.
8. B. Khvedelidze, The method of Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one complex variable. (Russian) *Itogi Nauki i Tekhniki* **7**(1975), 5–162, *J. Sov. Math.* **7**(1977), 309–414.

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