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ON THE CONFORMAL MAPPING OF SIMPLY CONNECTED DOMAINS WITH NON-JORDAN BOUNDARIES

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Abstract. We consider the domains G with the boundary consisting of the Jordan curve γ and mutually disjoint Jordan rectifiable arcs $\gamma_{A_kB_k}$ lying in G and connecting the points $B_k \in \gamma$ and $A_k \in G$. It is proved that if $z = z(\omega)$ is a conformal mapping of the unit circle onto G, then z' belongs to the Hardy class H^1. When γ and $\gamma_{A_kB_k}$ are either piecewise smooth, or piecewise Lyapunov curves, the representation (8) of the function z' is given which characterizes, in particular, its behaviour in the neighborhood of the points mapped at angular points γ and at the points A_k.

რეზიუმე. განიხილება ისეთი G არე, რომლის საზღვარი შედგება ჟორდანის γ წირისა და იმ ჟორდანის თანაუკვეთ $\gamma_{A_kB_k}$ რკალებისაგან, რომლებიც γ-ზე მდებარე B_k წერტილებს აერთებენ G-ში მდებარე A_k წერტილებთან. დამტკიცებულია, რომ: თუ $z = z(\omega)$ არის ერთეულოვანი წრის G-არეებზე კონფორმულად ამსახველი ფუნქცია, მაშინ z' ეკუთვნის ჰარდის H^1 კლასს. როდესაც γ და $\gamma_{A_kB_k}$ უბან-უბან გლუვი ან უბან-უბან ლიაპუნოვის წირებია, მოცემულია z'-ის (8) წარმოდგენა, რომელიც ახასიათებს ამ ფუნქციის ყოფაქცევას წრეწირის იმ წერტილთა მიდამოებში, რომლებიც აისახებიან γ-ს კუთხურ წერტილებში ან A_k წერტილებში.

Conformal mappings of the unit circle on the domains with Jordan boundaries are well studied (see, e.g., [1]), but the same cannot be said about the mappings on domains with non-Jordan boundaries. In the present paper we consider the simplest cases of such mappings and some of their properties.

1^0. Non-Jordan Curves with Branches. Let γ be a closed rectifiable Jordan curve bounding the domain G. Let $B_1, B_2, \dots B_m$ be the points lying on Γ, and $A_1, A_2, \dots, A_m$ be the points from the set $G \cup \{B_1, B_2, \dots, B_m\}$, assuming the point A_k may coincide only with the point B_k. Consider the Jordan rectifiable curves $\gamma_{A_kB_k}$, $k = \overline{1,m}$ which connect the points A_k and B_k, lie in G and are mutually disjoint, (if $A_j = B_j$, then $\gamma_{A_jB_j}$ is a closed

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curve). Choosing from G the set $\bigcup\limits_{k=1}^{m}\gamma_{A_kB_k}$, we obtain the domain whose set of prime ends consists of the points of the curve γ and of the curves $\gamma^-_{A_kB_k}$ and $\gamma^+_{A_kB_k}$ which are in fact the left and the right sides of the curves $\gamma_{A_kB_k}$. For the sake of brevity, we call this set the non-Jordan curve with branches of the first step, and assume that $\Gamma=\bigcup\limits_{k=1}^{m}\gamma^-_{A_kB_k}\cup\gamma\bigcup\limits_{k=1}^{m}\gamma^+_{A_kB_k}$.

If on the branches $\gamma_{A_kB_k}$ we take the points C_{k_j}, $l=\overline{1,m_k}$ and consider mutually disjoint Jordan arcs $\gamma_{D_{k_j}C_{k_j}}$, $D_{k_j}\subset G\setminus\bigcup\limits_{k=1}^{m}\gamma_{A_kB_k}$ lying in the domain $G\setminus\bigcup\limits_{k=1}^{m}\gamma_{A_kB_k}$ and then cut it along the union of curves $\bigcup\limits_{k=1}^{m}\bigcup\limits_{j=1}^{m_k}D_{k_j}C_{k_j}$, we will obtain the domain having the non-Jordan boundary with branches of the second step. Continuing such constructions, we will get curves with branches of any step.

2^0. Belonging to the Hardy class H^1 of a derivative of conformal map of a circle onto the domain having the non-Jordan boundary with branches. Let G be an arbitrary simply connected domain. Denote by $E^p(G)$, $p>0$, the Smirnov's class, i.e., the set of analytic in G functions ϕ for which

$$\sup_r\int\limits_{\Gamma_r}\big|\phi(z)\big|^p|dz|<\infty, \tag{1}$$

where Γ_r is the image of the circumference of radius r under the conformal mapping of the unit circle $U=\{\omega:|\omega|<1\}$ onto G.

If $G=\cup$, then $E^p(G)$ is the class of Hardy H^p.

If G is the domain bounded by the rectifiable Jordan curve, and $z=z(\omega)$ is the conformal mapping of U onto G, then $z'\in H^1$ (see, e.g., [2], p. 405). This statement remains valid for domains having boundaries with branches.

Theorem 1. *If G is the domain bounded by the non-Jordan curve with branches of any finite step, and $z=z(\omega)$ is the conformal mapping of the circle $\cup$ onto G, then $z'\in H^1$.*

Proof. As it will be seen from the proof, without loss of generality, we can restrict ourselves to the case in which the boundary of the domain G is the non-Jordan curve Γ obtained by the Jordan curve γ and the branch γ_{AB}.

Consider on γ two sequences of points $\{B'_n\}$ and $\{B''_n\}$ converging to B, such that the arc $B'_nB''_n$ contains the point B, and $B'_1B''_1\supset B'_2B''_2\supset\cdots B'_nB''_n\supset\cdots$. We connect these points with the point A by means of smooth mutually disjoint curves $\gamma_{AB'_n}$, $\gamma_{AB''_n}$ in such a way that: (i) the domains G_n bounded by the curves $(\gamma\setminus B'_nB''_n)\cup\gamma_{AB'_n}\cup\gamma_{AB''_n}$ form an increasing sequence i.e., $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$; (ii) sequences of the curves $\{\gamma_{AB'_n}\}$ and $\{\gamma_{AB''_n}\}$ converge uniformly to the curve γ_{AB}.

It can be easily verified that the domains G_n converge to G as to the kernel (for the notion of a kernel, see, e.g., [2], p. 56).

Let z_0 be some point from G, $z = z(\omega)$, $z(0) = z_0$, $z'(0) > 0$ be the conformal mapping of U onto G, and let $z_n = z_n(\omega)$, $z_n(0) = z_0$, $z'_n(0) > 0$, $n \in \mathbb{N}$, be the conformal mapping of the same circle onto G_n.

According to the Caratheodory theorem ([2], p. 56), the sequence of functions z_n converges uniformly in U to the function z.

Since the boundary of the domain G_n is the Jordan curve, the function z_n is continuous in $\overline{U}$, and $z'_n \in H^1$ ([2], p. 405). Therefore

$$z_n(\omega) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{z_n(s)d\zeta}{\zeta - \omega}, \quad |\omega| < r < 1 \tag{2}$$

and hence

$$z'_n(\omega) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{z_n(\zeta)d\zeta}{(\zeta - \omega)^2}. \tag{3}$$

As far as the sequence of functions z_n converges on the circumference $|\zeta| = r$ uniformly to the function z, it follows from (3) that

$$\lim_{n\to\infty} z'_n(\omega) = \\ = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{z(\zeta)d\zeta}{(\zeta - \omega)^2} = \Big(\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{z(\zeta)d\zeta}{\zeta - \omega}\Big)' = z'(\omega), \quad |\omega| < r. \tag{4}$$

On the other hand, due to the fact that $z'_n \in H^1$, we have

$$\int\limits_{|\omega|=r} \big|z'_n(\omega)\big||d\omega| \le \int\limits_{|\omega|=1} \big|z'_n(\omega)\big||d\omega| = \operatorname{mes} F_r G_n = |\gamma| + |\gamma_{AB'_n}| + |\gamma_{AB''_n}|, \tag{5}$$

where $|\gamma|$, $|\gamma_{AB'_n}|$, $|\gamma_{AB''_n}|$ are lengths of the curves γ, $\gamma_{AB'_n}$, $\gamma_{AB''_n}$ respectively. Since $\{\gamma_{AB'_n}\}$, $\{\gamma_{AB''_n}\}$ converge uniformly to γ_{AB}, therefore starting from some n_0, we can assume that $|\gamma_{AB'_n}| < |\gamma_{AB}| + 1$, $|\gamma_{AB''_n}| < |\gamma_{AB}| + 1$ and it follows from (5) that for any $r < 1$ the inequality

$$\int\limits_{|\omega|=r} \big|z'_n(\omega)\big||d\omega| \le M = |\gamma| + 2\big(|\gamma_{AB}| + 1\big) \tag{6}$$

is valid.

Passing to the limit in (6) and taking into account (4), we obtain

$$\int\limits_{|\omega|=r} \big|z'(\omega)\big||d\omega| \le M, \quad r < 1 \tag{7}$$

and hence $z' \in H^1$. □

3^0. On a derivative of conformal mapping of the circle onto the domain bounded by the non-Jordan boundary with piecewise smooth branches. If boundary branches of the domain G_n are piecewise smooth curves, then the domains G_n constructed in Section 2^0 have Jordan boundaries which are piecewise smooth curves. In this case we have representations of functions z'_n describing, in particular, their behaviour in the neighbourhood of angular points (see [1], [3]–[7]). On the basis of the above-said, the passage to the limit allows us to establish representations for z' in the case of domains which are bounded by non-Jordan curves with branches of any step. To avoid cumbersome notation and concentrate on the essence of the problem, we restrict ourselves to the consideration of the simplest case.

Theorem 2. *Let $z = z(\omega)$ be the conformal mapping of a unit circle onto a simply connected domain with a non-Jordan boundary $\Gamma = \gamma^-_{AB} \cup \gamma \cup \gamma^+_{AB}$, where γ_{AB} is a smooth curve, γ_{AB} is a piecewise smooth curve with one angular point B, and the sizes of angles at that point with respect to the domain G are equal to $\pi\alpha$, $\pi\beta$, $0 \le \alpha \le 2$, $0 \le \beta \le 2$, $\alpha + \beta \le 2$. Then on the unit circumference ℓ there exist the points c, d and e such that*

$$z'(\omega) = (\omega - c)^{\alpha-1}(\omega - d)^{\beta-1}(\omega - e)z_0(\omega), \tag{8}$$

where

$$z_0(\omega) = \exp\left\{\frac{1}{2\pi}\int\limits_{\ell}\frac{\varphi(t)\,dt}{t-\omega}\right\}, \quad \ell = \{t : |t| = 1\}, \tag{9}$$

with the continuous function φ, dependent of Γ, and hence

$$z_0^{\pm 1}(\omega) \subset \bigcap_{p>1} H^p. \tag{10}$$

But if γ and γ_{AB} are the Lyapunov curves, and $\alpha > 0$, $\beta > 0$, then $z_0(\omega)$ is the function of the Hölder class which is different from zero in the closed circle $\overline{U}$.

Proof. Let $\gamma_{AB'_n}$ and $\gamma_{AB''_n}$ be the curves constructed when proving Theorem 1, satisfying the conditions (i)–(ii) with the additional condition that at the points B'_n and B''_n they make with γ the angles of sizes $\alpha\pi$ and $\beta\pi$, respectively, while at the point A the angle of size 2π. These curves can be chosen so as the angular functions of the curves $\Gamma_n = \gamma_{AB'_n} \cup \gamma \cup \gamma_{AB''_n}$ to converge uniformly to the angular function of the curve $\Gamma = \gamma^-_{AB} \cup \gamma \cup \gamma^+_{AB}$ (for the definition of the angular function, see [5], p. 138).

Let G_n be the domain with the boundary set $(\gamma \setminus (B'_n B''_n)) \cup \gamma_{AB'_n} \cup \gamma_{AB''_n}$, $z_n = z_n(\omega)$ be the conformal mapping of U onto G_n, and $\omega_n = \omega_n(z)$ be the inverse mapping.

For the function z_n, the following representation (see, e.g., [4] and [6], and also [5], Ch. III) is valid:

$$z_n'(\omega) = (\omega - c_n)^{\alpha-1}(\omega - d_n)^{\beta-1}(\omega - e_n)z_{n,0}(\omega), \tag{11}$$

and $c_n = \omega_n(B_n')$, $d_n = \omega_n(B_n'')$, $e_n = \omega_n(A)$, $z_{n,0}(\omega) = \exp\left\{\frac{1}{2\pi}\int\limits_e \frac{\varphi_n(t)dt}{t-\omega}\right\}$, $\varphi_n(t)$ is the continuous on ℓ real function which represents the difference of the angular function of the curve Γ_n and the piecewise continuous function $\delta_n(t)$ (see, e.g., [5], p. 138).

Since the sequence of angular functions of the curves Γ_n converges uniformly to the angular function of the curve Γ, it is not difficult to state that the function $\delta_n(t)$ is likewise the same, and therefore the sequence $\varphi_n(t)$ converges uniformly to the continuous on ℓ function $\varphi(t)$. Moreover, from the sequence of points $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ lying on the circumference ℓ one can choose sequences converging, say, to the points c, d and e. Now, passing in (11) to the limit and applying Caratheodory's theorem, we find that

$$z'(\omega) = (\omega - e)^{\alpha-1}(\omega - d)^{\beta-1}(\omega - e)z_0(\omega) \tag{12}$$

where z_0 is the function given by equality (9). For the functions written in the form of (9) with the continuous function φ we obtain inclusions (10) (see [2], p. 401).

When γ_{AB} is the Lyapunov arc and γ is the piecewise Lyapunov curve having angular point in B with a non-zero angle (i.e., when $\alpha > 0$, $\beta > 0$), we take the curves $\gamma_{AB_n'}$ and $\gamma_{AB_n''}$ so as the sequences of functions $\beta_n(t)$ to converge uniformly to the function $\varphi(t)$ which, by our assumptions regarding γ and γ_{AB}, belongs to the Hölder class on ℓ. □

4^0. Remarks.

1. It can be easily shown that $z(e) = A$, $\lim\limits_{\omega\to c} z(\omega) = B$, $\lim\limits_{\omega\to d} z(\omega) = B$.

2. If $A = B$, i.e., if γ_{AB} is the closed curve, then (8) takes the form

$$z'(\omega) = (\omega - c)^{\alpha-1}(\omega - d)^{\beta-1}z_0(\omega). \tag{8'}$$

3. If at the point B the curve γ is smooth, then $\alpha + \beta = 1$, and in the representation (8) we have $\beta = 1 - \alpha$. In particular, when γ_{AB} meets γ under the zero angle ($\alpha = 0$), the representation (8) takes the form

$$z'(\omega) = (\omega - c)^{-1}(\omega - e)z_0(\omega).$$

4. The investigation of the function z' can be performed in the following natural way.

Let $\Gamma = \gamma_{AB}^- \cup \gamma \cup \gamma_{AB}^+$ and $C \neq B$ be some point on γ. Consider a simple smooth curve γ_{AC} lying in G and not intersecting γ_{AB}. The curve $\gamma_{AB} \cup \gamma_{AC}$ divides the domain G by two parts G^- and G^+. If $\omega = \omega(z)$ is inverse function of $z = z(\omega)$, then $\omega(G^-)$ and $\omega(G^+)$ represent the domains

U^- and U^+ in the circle U, are bounded by the piecewise smooth curves. Using the corresponding result from [7], we obtain

$$z'(\omega) = (\omega - c)^{\alpha-1} z_{0,1}(\omega), \ \ \omega \in U^-, \ \ z_{0,1} \in \bigcap_{p>1} E^p(U^-), \tag{13}$$

$$z'(\omega) = (\omega - d)^{\beta-1} z_{0,2}(\omega), \ \ \omega \in U^+, \ \ z_{0,2} \in \bigcap_{p>1} E^p(U^+). \tag{14}$$

Drawing the cuts of somewhat different character, we can get a local estimate z' in some subdomain U closely adjoining to the point $\omega(A)$.

Relying on this fact and also on (13) and (14) which characterize local behaviour of estimate z', we could have endeavored to get conclusions for (8) and (10) of global nature. But this would require additional study. We have given preference to the proof of Theorem 2 in which, besides inclusions (10), we have obtained explicit representation of the function z_0 by means of formula (9).

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