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Non-abelian tensor product and homology of Lie superalgebras

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A B S T R A C T

We introduce the non-abelian tensor product of Lie superalgebras and study some of its properties. We use it to describe the universal central extensions of Lie superalgebras. We present the low-dimensional non-abelian homology of Lie superalgebras and establish its relationship with the cyclic homology of associative superalgebras. We also define the non-abelian exterior product and give an analogue of Miller's theorem, Hopf formula and a six-term exact sequence for the homology of Lie superalgebras.

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1. Introduction

In [1], Brown and Loday introduced the non-abelian tensor product of groups in the context of an application in homotopy theory. Analogous theories of non-abelian tensor product have been developed in other algebraic structures such as Lie algebras [8] and Lie–Rinehart algebras [4]. In [8], Ellis investigated the main properties of the non-abelian tensor product of Lie algebras and its relation to the low-dimensional homology of Lie algebras. In particular, he described the universal central extension of a perfect Lie algebra via the non-abelian tensor product. In [7], the non-abelian exterior product of Lie algebras is introduced and a six-term exact sequence relating low-dimensional homologies is obtained. In [10], using the non-abelian tensor product, Guin defined the non-abelian low-dimensional homology of Lie algebras and compared these groups with the cyclic homology and Milnor additive K-theory of associative algebras.

The theory of Lie superalgebras, also called $\mathbb{Z}_2$-graded Lie algebras, has aroused much interest both in mathematics and physics. Lie superalgebras play a very important role in theoretical physics since they are used to describe supersymmetry in a mathematical framework. A comprehensive description of the mathematical theory of Lie superalgebras is given in [14], containing the complete classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. In the last few years, the theory of Lie superalgebras has experienced a remarkable evolution obtaining many results on representation theory and classification, most of them extending well-known facts on Lie algebras.

In this paper we develop the non-abelian tensor product and the low-dimensional non-abelian homology of Lie superalgebras, generalizing the corresponding notions for Lie algebras, with applications in universal central extensions and homology of Lie superalgebras and cyclic homology of associative superalgebras.

The organization of this paper is as follows: after this introduction, in Section 2 we give some definitions and necessary well-known results for the development of the paper. We also introduce actions and crossed modules of Lie superalgebras. In Section 3 we introduce the non-abelian tensor product of Lie superalgebras, we establish its principal properties such as right exactness and relation with the tensor product of supermodules. We describe the universal central extension of a perfect Lie superalgebra via the non-abelian tensor product (Theorem 4.1). In particular, applying this theorem, we obtain that $\mathfrak{st}(m,n,A)$ is the universal central extension of $\mathfrak{sl}(m,n,A)$, for $m+n \geq 5$, where A is a unital associative superalgebra. We also study nilpotency and solvability of the non-abelian tensor product of Lie superalgebras (Theorem 3.9). Using the non-abelian tensor product, in Section 5 we introduce the low-dimensional non-abelian homology of Lie superalgebras with coefficients in crossed modules. We show that, if the crossed module is a supermodule, then the non-abelian homology is the usual homology of Lie superalgebras. Then we apply this non-abelian homology to relate cyclic homology and Milnor cyclic homology of associative superalgebras, extending the results of [10]. Finally, in the last section we construct the non-abelian exterior product of Lie superalgebras and we use it to obtain Miller's type theorem for free Lie superalgebras, Hopf formula and a six-term exact sequence in the homology of Lie superalgebras.

Conventions and notations

Throughout this paper we denote by $\mathbb{K}$ a unital commutative ring unless otherwise stated. All modules and algebras are defined over $\mathbb{K}$. We write $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and use its standard field structure. We put $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

By *a supermodule* M we mean a module endowed with a $\mathbb{Z}_2$-gradation: $M = M_{\bar{0}} \oplus M_{\bar{1}}$. We call elements of $M_{\bar{0}}$ (resp. $M_{\bar{1}}$) even (resp. odd). Non-zero elements of $M_{\bar{0}} \cup M_{\bar{1}}$ will be called *homogeneous*. For a homogeneous $m \in M_{\bar{\alpha}}$, $\bar{\alpha} \in \mathbb{Z}_2$, its degree will be denoted by $|m|$. We adopt the convention that whenever the degree function occurs in a formula, the corresponding elements are supposed to be homogeneous. By *a homomorphism of supermodules* $f\colon M \to N$ of degree $|f| \in \mathbb{Z}_2$ we mean a linear map satisfying $f(M_{\bar{\alpha}}) \subseteq N_{\bar{\alpha}+|f|}$. In particular, if $|f| = \bar{0}$, then the homomorphism f will be called *of even grade (or even linear map)*.

By *a superalgebra* A we mean a supermodule $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with a bilinear multiplication satisfying $A_{\bar{\alpha}} A_{\bar{\beta}} \subseteq A_{\bar{\alpha}+\bar{\beta}}$, for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$.

2. Preliminaries on Lie superalgebras

In this section we review some terminology on Lie superalgebras and recall notions used in the paper. We mainly follow [2,18], although with some modifications. We also introduce notions of actions and crossed modules of Lie superalgebras.

2.1. Definition and some examples of Lie superalgebras

Definition 2.1. A *Lie superalgebra* is a superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with a multiplication denoted by $[\ ,\ ]$, called bracket operation, satisfying the following identities:

$$\begin{aligned}
[x, y] &= -(-1)^{|x||y|}[y, x], \\
\big[x, [y, z]\big] &= \big[[x, y], z\big] + (-1)^{|x||y|}\big[y, [x, z]\big], \\
[m_{\bar{0}}, m_{\bar{0}}] &= 0,
\end{aligned}$$

for all homogeneous elements $x, y, z \in M$ and $m_{\bar{0}} \in M_{\bar{0}}$.

Note that the last equation is an immediate consequence of the first one in the case 2 has an inverse in $\mathbb{K}$. Moreover, it can be easily seen that the second equation is equivalent to the graded Jacobi identity

$$(-1)^{|x||z|}\big[x, [y, z]\big] + (-1)^{|y||x|}\big[y, [z, x]\big] + (-1)^{|z||y|}\big[z, [x, y]\big] = 0.$$

For a Lie superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$, the even part $M_{\bar{0}}$ is a Lie algebra. Hence, if $M_{\bar{1}} = 0$, then M is just a Lie algebra. A Lie superalgebra M without even part, i. e., $M_{\bar{0}} = 0$, is an *abelian Lie superalgebra*, that is, $[x, y] = 0$ for all $x, y \in M$.

A *Lie superalgebra homomorphism* $f : M \to M'$ is a supermodule homomorphism of even grade such that $f[x, y] = [f(x), f(y)]$ for all $x, y \in M$.

Example 2.2. (i) Any associative superalgebra A can be considered as a Lie superalgebra with the bracket

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

(ii) Let m, n be positive integers and A a unital associative superalgebra. Consider the algebra $\mathcal{M}(m, n, A)$ of all $(m + n) \times (m + n)$-matrices with entries in A and with the usual product of matrices. A $\mathbb{Z}_2$-gradation is defined as follows: homogeneous elements are matrices $E_{ij}(a)$ having the homogeneous element $a \in A$ at the position (i, j) and zero elsewhere, and $|E_{ij}(a)| = |i| + |j| + |a|$, where $|i| = \bar{0}$ if $1 \leq i \leq m$ and $|i| = \bar{1}$ if $m + 1 \leq i \leq m + n$. With this gradation, $\mathcal{M}(m, n, A)$ turns out to be an associative superalgebra. The corresponding Lie superalgebra will be denoted by $\mathfrak{gl}(m, n, A)$.

(iii) Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a supermodule. Then the supermodule $\mathrm{End}_{\mathbb{K}}(V)$ of all linear endomorphisms $V \to V$ (of both degrees 0 and 1) has a structure of an associative superalgebra with respect to composition (see [2]) and hence becomes a Lie superalgebra. In particular, if the ground ring $\mathbb{K}$ is a field, and m, n are dimensions of $V_{\bar{0}}$ and $V_{\bar{1}}$ respectively, then choosing a homogeneous basis of V ordered such that even elements stand before odd, the elements of $\mathrm{End}_{\mathbb{K}}(V)$ can be seen as $(m + n) \times (m + n)$-square matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are respectively $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices with entries in $\mathbb{K}$. The even elements are the matrices with $b = c = 0$ and the odd elements are matrices with $a = d = 0$.

Let M and N be two submodules of a Lie superalgebra P. We denote by $[M, N]$ the submodule of P spanned by all elements $[m, n]$ with $m \in M$ and $n \in N$. A $\mathbb{Z}_2$-graded submodule M is a *graded ideal* of P if $[M, P] \subseteq M$. In particular, the submodule $\mathsf{Z}(P) = \{c \in P : [c, p] = 0 \text{ for all } p \in P\}$ is a graded ideal and it is called the *centre* of P. Clearly if M and N are graded ideals of P, then so is $[M, N]$.

Let M be a Lie superalgebra and $D \in \mathrm{End}_{\mathbb{K}}(M)$. We say that D is a *derivation* if for all $x, y \in M$

$$D([x, y]) = [D(x), y] + (-1)^{|D||x|} [x, D(y)].$$

We denote by $\big(\mathrm{Der}_{\mathbb{K}}(M)\big)_{\bar{\alpha}}$ the set of homogeneous derivations of degree $\bar{\alpha} \in \mathbb{Z}_2$. One verifies that the supermodule of derivations

$$\mathrm{Der}_{\mathbb{K}}(M) = \big(\mathrm{Der}_{\mathbb{K}}(M)\big)_{\bar{0}} \oplus \big(\mathrm{Der}_{\mathbb{K}}(M)\big)_{\bar{1}}$$

is a subalgebra of the Lie superalgebra $\mathrm{End}_{\mathbb{K}}(M)$.

2.2. Actions and crossed modules of Lie superalgebras

Definition 2.3. Let P and M be two Lie superalgebras. By an *action* of P on M we mean a $\mathbb{K}$-bilinear map of even grade,

$$P \times M \to M, \quad (p, m) \mapsto {}^{p}m,$$

such that

(i) ${}^{[p,p']}m = {}^{p}({}^{p'}m) - (-1)^{|p||p'|}\,{}^{p'}({}^{p}m)$,
(ii) ${}^{p}[m, m'] = [{}^{p}m, m'] + (-1)^{|p||m|}[m, {}^{p}m']$,

for all homogeneous $p, p' \in P$ and $m, m' \in M$.

The action is called *trivial* if ${}^{p}m = 0$ for all $p \in P$ and $m \in M$.

For example, if M is a graded ideal and P is a subalgebra of a Lie superalgebra Q, then the bracket in Q induces an action of P on M.

Note that the action of P on M is the same as a Lie superalgebra homomorphism $P \to \mathrm{Der}_{\mathbb{K}}(M)$.

Remark. If M is an abelian Lie superalgebra enriched with an action of a Lie superalgebra P, then M has a structure of a supermodule over P (P-supermodule, for short) (see e.g. [18]), that is, there is a $\mathbb{K}$-bilinear map of even grade $P \times M \to M$, $(p, m) \mapsto pm$, such that

$$[p, p']m = p(p'm) - (-1)^{|p||p'|}p'(pm),$$

for all homogeneous $p, p' \in P$ and $m \in M$.

Note that a P-supermodule M is the same as a $\mathbb{K}$-supermodule M together with a Lie superalgebra homomorphism $P \to \mathrm{End}_{\mathbb{K}}(M)$.

Definition 2.4. Given two Lie superalgebras M and P with an action of P on M, we can define the *semidirect product* $M \rtimes P$ with the underlying supermodule $M \oplus P$ endowed with the bracket given by

$$[(m, p), (m', p')] = ([m, m'] + {}^{p}m' - (-1)^{|m||p'|}({}^{p'}m), [p, p']).$$

Now we are ready to introduce the following notion of crossed modules of Lie superalgebras (see also [23, Definition 5]).

Definition 2.5. A *crossed module of Lie superalgebras* is a homomorphism of Lie superalgebras $\partial \colon M \to P$ with an action of P on M satisfying

(i) $\partial({}^{p}m) = [p, \partial(m)]$,
(ii) ${}^{\partial(m)}m' = [m, m']$,

for all $p \in P$ and $m, m' \in M$.

Example 2.6. There are some standard examples of crossed modules:

(i) The inclusion $M \hookrightarrow P$ of a graded ideal M of a Lie superalgebra P is a crossed module of Lie superalgebras.
(ii) If P is a Lie superalgebra and M is a P-supermodule, the trivial map $0 \colon M \to P$ is a crossed module of Lie superalgebras.
(iii) A central extension of Lie superalgebras $\partial \colon M \twoheadrightarrow P$ (i.e., $\mathsf{Ker}\,\partial \subseteq \mathsf{Z}(M)$) is a crossed module of Lie superalgebras. Here the action of P on M is given by ${}^{p}m = [\tilde{m}, m]$, where $\tilde{m} \in M$ is any element of $\partial^{-1}(p)$.
(iv) The homomorphism of Lie superalgebras $\partial \colon M \to \mathrm{Der}_{\mathbb{K}}(M)$ which sends $m \in M$ to the inner derivation $\mathsf{ad}(m) \in \mathrm{Der}_{\mathbb{K}}(M)$, defined by $\mathsf{ad}(m)(m') = [m, m']$, together with the action of $\mathrm{Der}_{\mathbb{K}}(M)$ on M given by ${}^{D}m = D(m)$, is a crossed module of Lie superalgebras.

Lemma 2.7. *Let $\partial \colon M \to P$ be a crossed module of Lie superalgebras. Then the following conditions are satisfied:*

(i) *The kernel of ∂ is in the centre of M.*
(ii) *The image of ∂ is a graded ideal of P.*
(iii) *The Lie superalgebra $\mathsf{Im}\,\partial$ acts trivially on the centre $\mathsf{Z}(M)$, and so trivially on $\mathsf{Ker}\,\partial$. Hence $\mathsf{Ker}\,\partial$ inherits an action of $P/\mathsf{Im}\,\partial$ making $\mathsf{Ker}\,\partial$ a $P/\mathsf{Im}\,\partial$-supermodule.*

Proof. This is an immediate consequence of Definition 2.5. □

2.3. Free Lie superalgebra and enveloping superalgebra of a Lie superalgebra

Definition 2.8. The *free Lie superalgebra* on a $\mathbb{Z}_2$-graded set $X = X_{\bar{0}} \cup X_{\bar{1}}$ is a Lie superalgebra $\mathrm{F}(X)$ together with a degree zero map $i \colon X \to \mathrm{F}(X)$ such that if M is any Lie superalgebra and $j \colon X \to M$ is a degree zero map, then there is a unique Lie superalgebra homomorphism $h \colon \mathrm{F}(X) \to M$ with $j = h \circ i$.

The existence of free Lie superalgebras is guaranteed by an analogue of Witt's theorem (see [18, Theorem 6.2.1]). In the sequel we need the following construction of the free Lie superalgebra.

Construction 2.9. *Let $X = X_{\bar{0}} \cup X_{\bar{1}}$ be a $\mathbb{Z}_2$-graded set. Denote by* $\operatorname{mag}(X)$ *the free magma over the set X. The free superalgebra on X, denoted by* $\operatorname{alg}(X)$*, has as elements the finite sums $\sum_i \lambda_i v_i$, where $\lambda_i \in \mathbb{K}$ and x_i are elements of* $\operatorname{mag}(X)$ *and the multiplication in* $\operatorname{alg}(X)$ *extends the multiplication in* $\operatorname{mag}(X)$*. Note that the grading is naturally defined in* $\operatorname{alg}(X)$*. The free Lie superalgebra* $\mathrm{F}(X)$ *is the quotient* $\operatorname{alg}(X)/I$*, where I is the graded ideal generated by the elements*

$$xy + (-1)^{|x||y|} yx,$$
$$(-1)^{|x||z|}\big(x(yz)\big) + (-1)^{|y||x|}\big(y(zx)\big) + (-1)^{|z||y|}\big(z(xy)\big),$$
$$x_{\bar{0}} x_{\bar{0}},$$

for all homogeneous $x, y, z \in X$ and $x_{\bar{0}} \in X_{\bar{0}}$.

Definition 2.10. The *universal enveloping superalgebra* of a Lie superalgebra M is a pair $(\mathrm{U}(M), \sigma)$, where $\mathrm{U}(M)$ is a unital associative superalgebra and $\sigma \colon M \to \mathrm{U}(M)$ is an even linear map satisfying

$$\sigma[x, y] = \sigma(x)\sigma(y) - (-1)^{|x||y|}\sigma(y)\sigma(x), \tag{1}$$

for all homogeneous $x, y \in M$, such that the following universal property holds: for any other pair (A, σ'), where A is a unital associative superalgebra and $\sigma' \colon M \to A$ is an even linear map satisfying (1), there is a unique superalgebra homomorphism $f \colon \mathrm{U}(M) \to A$ such that $f \circ \sigma = \sigma'$.

Now we need to recall (see e.g. [21]) that, given two supermodules M and N, the tensor product of modules $M \otimes_{\mathbb{K}} N$ has a natural supermodule structure with $\mathbb{Z}_2$-grading given by

$$(M \otimes_{\mathbb{K}} N)_{\bar{\alpha}} = \bigoplus_{\bar{\beta}+\bar{\gamma}=\bar{\alpha}} (M_{\bar{\beta}} \otimes_{\mathbb{K}} N_{\bar{\gamma}}).$$

In particular, the tensor power $M^{\otimes n}$, $n \geq 2$, has the induced $\mathbb{Z}_2$-grading. Hence the tensor algebra $T(M)$ has the $\mathbb{Z}_2$-grading extending that of M. We call $T(M)$ *the tensor superalgebra.*

Construction 2.11. *Let M be a Lie superalgebra and $T(M)$ the tensor superalgebra over the underlying supermodule of M. Consider the two-sided ideal $J(M)$ of $T(M)$ generated by all elements of the form*

$$m \otimes m' - (-1)^{|m||m'|} m' \otimes m - [m, m'],$$

for all homogeneous $m, m' \in M$. Then the quotient $\mathrm{U}(M) = T(M)/J(M)$ is a unital associative superalgebra. By composing the canonical inclusion $M \to T(M)$ with the canonical projection $T(M) \to \mathrm{U}(M)$ we get the canonical even linear map $\sigma: M \to \mathrm{U}(M)$. Then the pair $(\mathrm{U}(M), \sigma)$ is the universal enveloping superalgebra of M (see [2]).

Note that, as in the Lie algebra case, the universal enveloping superalgebra turns out to be a very useful tool for the representation theory of Lie superalgebras. In particular, by the universal property, it follows that a Lie supermodule over a Lie superalgebra M is the same as a $\mathbb{Z}_2$-graded (left) $\mathrm{U}(M)$-module (see [21, Chapter 1]).

Let us consider $\mathbb{K}$ with $\mathbb{Z}_2$-grading concentrated in degree zero, that is, with $\mathbb{K}_{\bar{1}} = 0$. Then the trivial map from a Lie superalgebra M into $\mathbb{K}$ gives rise to a unique homomorphism of superalgebras $\varepsilon: \mathrm{U}(M) \to \mathbb{K}$. The kernel of ε, denoted by $\Omega(M)$, is called the *augmentation ideal* of M. Obviously, $\Omega(M)$ is just the graded ideal of $\mathrm{U}(M)$ generated by $\sigma(M)$.

2.4. Homology of Lie superalgebras

Now we briefly recall from [18,22] the definition of homology of Lie superalgebras.

The *Grassmann algebra* of a Lie superalgebra P, denoted by $\bigwedge_{\mathbb{K}}(P)$, is defined to be the quotient of the tensor superalgebra $T(P)$ of P by the ideal generated by the elements

$$x \otimes y + (-1)^{|x||y|} y \otimes x,$$

for all homogeneous $x, y \in P$. Note that $\bigwedge_{\mathbb{K}}(P) = \bigoplus_{n>0} \bigwedge_{\mathbb{K}}^n(P)$, where $\bigwedge_{\mathbb{K}}^n(P)$ is the image of $P^{\otimes n}$ in $\bigwedge_{\mathbb{K}}(P)$, has an induced P-supermodule structure given by

$$x(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{|x| \sum_{k<i} |x_k|} (x_1 \wedge \cdots \wedge [x, x_i] \wedge \cdots \wedge x_n).$$

Let M be a P-supermodule and consider the chain complex $(C_*(P, M), d_*)$ defined by $C_n(P, M) = \bigwedge_{\mathbb{K}}^n(P) \otimes_{\mathbb{K}} M$, for $n \geq 0$, with boundary maps $d_n: C_n(P, M) \to C_{n-1}(P, M)$ defined on generators by

$$\begin{aligned} d_n(x_1 \wedge \cdots \wedge x_n \otimes y) = & \sum_{i=1}^{n} (-1)^{i + |x_i| \sum_{k>i} |x_k|} (x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes x_i y) \\ & + \sum_{i<j} (-1)^{i+j+|x_i| \sum_{k<i} |x_k| + |x_j| \sum_{l<j} |x_l| + |x_i||x_j|} \\ & \times ([x_i, x_j] \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n \otimes y). \end{aligned}$$

The n-th *homology of the Lie superalgebra* P with coefficients in the P-supermodule M, $\mathrm{H}_n(P,M)$, is the n-th homology of the chain complex $(C_*(P,M),d_*)$, i.e.

$$\mathrm{H}_n(P,M)=\frac{\mathsf{Ker}\,d_n}{\mathsf{Im}\,d_{n+1}}.$$

If $\mathbb{K}$ is regarded as a trivial P-supermodule, we write $\mathrm{H}_n(P)$ for $\mathrm{H}_n(P,\mathbb{K})$.

In the case when the ground ring $\mathbb{K}$ is a field, there is a relation between Tor functor and the homology (see [18]) given by

$$\mathrm{H}_n(P,M)\cong \mathsf{Tor}_n^{\mathrm{U}(P)}(\mathbb{K},M).$$

By analogy to Lie algebras (see e.g. [11]), we have the following isomorphisms

$$H_0(P,M)\cong \mathsf{Coker}\big(\Omega(P)\otimes_{\mathrm{U}(P)}M\longrightarrow M\big), \tag{2}$$

$$H_1(P,M)\cong \mathsf{Ker}\big(\Omega(P)\otimes_{\mathrm{U}(P)}M\longrightarrow M\big). \tag{3}$$

3. Non-abelian tensor product of Lie superalgebras

In this section we introduce a non-abelian tensor product of Lie superalgebras, which generalizes the non-abelian tensor product of Lie algebras [8], and study its properties.

3.1. Construction of the non-abelian tensor product

Definition 3.1. Let M and N be two Lie superalgebras with actions on each other. Let $X_{M,N}$ be the $\mathbb{Z}_2$-graded set of all symbols $m\otimes n$, where $m\in M_{\bar 0}\cup M_{\bar 1}$, $n\in N_{\bar 0}\cup N_{\bar 1}$ and the $\mathbb{Z}_2$-gradation is given by $|m\otimes n|=|m|+|n|$. We define the *non-abelian tensor product* of M and N, denoted by $M\otimes N$, as the Lie superalgebra generated by $X_{M,N}$ and subject to the relations:

(i) $\lambda(m\otimes n)=\lambda m\otimes n=m\otimes\lambda n$,

(ii) $(m+m')\otimes n=m\otimes n+m'\otimes n$, where m,m' have the same grade,
$m\otimes(n+n')=m\otimes n+m\otimes n'$, where n,n' have the same grade,

(iii) $[m,m']\otimes n=m\otimes {}^{m'}n-(-1)^{|m||m'|}(m'\otimes {}^{m}n)$,
$m\otimes[n,n']=(-1)^{|n'|(|m|+|n|)}({}^{n'}m\otimes n)-(-1)^{|m||n|}({}^{n}m\otimes n')$,

(iv) $[m\otimes n,m'\otimes n']=-(-1)^{|m||n|}({}^{n}m\otimes {}^{m'}n')$,

for every $\lambda\in\mathbb{K}$, $m,m'\in M_{\bar 0}\cup M_{\bar 1}$ and $n,n'\in N_{\bar 0}\cup N_{\bar 1}$.

Let us remark that if $m=m_{\bar 0}+m_{\bar 1}$ is any element of M and $n=n_{\bar 0}+n_{\bar 1}$ is any element of N, then under the notation $m\otimes n$ we mean the sum

$$m_{\bar 0}\otimes n_{\bar 0}+m_{\bar 0}\otimes n_{\bar 1}+m_{\bar 1}\otimes n_{\bar 0}+m_{\bar 1}\otimes n_{\bar 1}.$$

If $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$ then $M \otimes N$ is the non-abelian tensor product of Lie algebras introduced and studied in [8] (see also [12]).

Definition 3.2. Actions of Lie superalgebras M and N on each other are said to be *compatible* if

(i) $({}^{n}m)n' = -(-1)^{|m||n|}[{}^{m}n, n']$,
(ii) $({}^{m}n)m' = -(-1)^{|m||n|}[{}^{n}m, m']$,

for all $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$.

For example, if M and N are two graded ideals of some Lie superalgebra, the actions induced by the bracket are compatible.

Proposition 3.3. *Let M and N be Lie superalgebras acting compatibly on each other. Then there is a natural isomorphism of Lie superalgebras*

$$M \otimes N \cong \frac{M \otimes_{\mathbb{K}} N}{D(M,N)},$$

where $D(M,N)$ is the submodule of the supermodule $M \otimes_{\mathbb{K}} N$ generated by the elements

(i) $[m, m'] \otimes n - m \otimes {}^{m'}n + (-1)^{|m||m'|}(m' \otimes {}^{m}n)$,
(ii) $m \otimes [n, n'] - (-1)^{|n'|(|m|+|n|)}({}^{n'}m \otimes n) + (-1)^{|m||n|}({}^{n}m \otimes n')$,
(iii) $({}^{n}m) \otimes ({}^{m}n)$, *with* $|m| = |n|$,
(iv) $(-1)^{|m||n|}({}^{n}m) \otimes ({}^{m'}n') + (-1)^{(|m|+|n|)(|m'|+|n'|)+|m'||n'|}({}^{n'}m') \otimes ({}^{m}n)$,
(v) $\displaystyle\circlearrowleft_{(m,n),(m',n'),(m'',n'')} (-1)^{(|m|+|n|)(|m''|+|n''|)+|m||n|+|m'||n'|}[{}^{n}m, {}^{n'}m'] \otimes ({}^{m''}n'')$,

for all $m, m', m'' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $n, n', n'' \in N_{\bar{0}} \cup N_{\bar{1}}$, where $\circlearrowleft_{x,y,z}$ denotes the cyclic summation with respect to x, y, z.

Proof. There is a Lie superalgebra structure on the supermodule $(M \otimes_{\mathbb{K}} N)/D(M,N)$ given on generators by the following bracket

$$[m \otimes n, m' \otimes n'] = -(-1)^{|m||n|}({}^{n}m \otimes {}^{m'}n'),$$

for all $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$ and extended by linearity. It is routine to check that this bracket is compatible with the defining relations of $(M \otimes_{\mathbb{K}} N)/D(M,N)$ and it indeed defines a Lie superalgebra structure. Then the canonical homomorphism $M \otimes N \to (M \otimes_{\mathbb{K}} N)/D(M,N)$, $m \otimes n \mapsto m \otimes n$, is an isomorphism. □

The proof of the following proposition is a routine calculation.

Proposition 3.4. *Let M and N be two Lie superalgebras acting compatibly on each other.*

(i) *The following morphisms*

$$\mu : M \otimes N \to M, \quad m \otimes n \mapsto -(-1)^{|m||n|}({}^{n}m),$$
$$\nu : M \otimes N \to N, \quad m \otimes n \mapsto {}^{m}n,$$

are Lie superalgebra homomorphisms.

(ii) *There are actions of M and N on $M \otimes N$ given by*

$${}^{m'}(m \otimes n) = [m', m] \otimes n + (-1)^{|m||m'|} m \otimes ({}^{m'}n),$$
$${}^{n'}(m \otimes n) = ({}^{n'}m) \otimes n + (-1)^{|n||n'|} m \otimes [n', n],$$

for $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$ and extended by linearity. Moreover, with these actions μ and ν are crossed modules of Lie superalgebras.

We will denote by $[M, N]^M$ (resp. $[M, N]^N$) the image of μ (resp. ν), which by Lemma 2.7(ii) is a graded ideal of M (resp. N) generated by the elements of the form ${}^{n}m$ (resp. ${}^{m}n$) for $m \in M$ and $n \in N$. Note that by Lemma 2.7(iii) $\mathsf{Ker}(\mu)$ (resp. $\mathsf{Ker}(\nu)$) is an $M/[M, N]^M$-supermodule (resp. $N/[M, N]^N$-supermodule).

3.2. Some properties of the non-abelian tensor product

The obvious analogues of Brown and Loday results [1] hold for Lie superalgebras. In the following two propositions immediately below we show that sometimes the non-abelian tensor product of Lie superalgebras can be expressed in terms of the tensor product of supermodules.

Proposition 3.5. *Let M and N be Lie superalgebras acting on each other. Then the canonical map $M \otimes_{\mathbb{K}} N \to M \otimes N$, $m \otimes n \mapsto m \otimes n$, is an even, surjective homomorphism of supermodules. In addition, if M and N act trivially on each other, then $M \otimes N$ is an abelian Lie superalgebra and there is an isomorphism of supermodules*

$$M \otimes N \cong M^{\mathrm{ab}} \otimes_{\mathbb{K}} N^{\mathrm{ab}},$$

where $M^{\mathrm{ab}} = M/[M, M]$ and $N^{\mathrm{ab}} = N/[N, N]$.

Proof. It is straightforward by the identities (iv), (iii) of Definition 3.1. □

Proposition 3.6. *Let P be a Lie superalgebra and M a P-supermodule considered as an abelian Lie superalgebra acting trivially on P. Then there is an isomorphism of supermodules*

$$P \otimes M \cong \Omega(P) \otimes_{\mathrm{U}(P)} M.$$

Proof. By Proposition 3.3 there is an isomorphism of supermodules

$$P \otimes M \cong \frac{P \otimes_{\mathbb{K}} M}{W},$$

where W is the submodule of $P \otimes_{\mathbb{K}} M$ generated by all elements of the form

$$[p, p'] \otimes m - p \otimes p'm + (-1)^{|p||p'|} p' \otimes pm$$

for all $p, p' \in P_{\bar{0}} \cup P_{\bar{1}}$ and $m \in M_{\bar{0}} \cup M_{\bar{1}}$. Now by using Construction 2.11 and by repeating the respective part of the proof of [3, Proposition 13], it is easy to see that there is an isomorphism of supermodules

$$\frac{P \otimes_{\mathbb{K}} M}{W} \cong \Omega(P) \otimes_{\mathrm{U}(P)} M,$$

which completes the proof. □

The non-abelian tensor product of Lie superalgebras is symmetric, in the sense of the following proposition.

Proposition 3.7. *The Lie superalgebra homomorphism*

$$M \otimes N \to N \otimes M, \quad m \otimes n \mapsto -(-1)^{|m||n|}(n \otimes m),$$

is an isomorphism.

Proof. This can be checked readily. □

Let us consider the category $\mathtt{SLie}^2_{\mathbb{K}}$ whose objects are ordered pairs of Lie superalgebras (M, N) acting compatibly on each other, and the morphisms are pairs of Lie superalgebra homomorphisms $(\phi\colon M \to M', \psi\colon N \to N')$ which preserve the actions, i.e., $\phi({}^n m) = {}^{\psi(n)}\phi(m)$ and $\psi({}^m n) = {}^{\phi(m)}\psi(n)$. For such a pair (ϕ, ψ) we have a homomorphism of Lie superalgebras $\phi \otimes \psi\colon M \otimes N \to M' \otimes N'$, $m \otimes n \mapsto \phi(m) \otimes \psi(n)$. Therefore, $\otimes$ is a functor from $\mathtt{SLie}^2_{\mathbb{K}}$ to the category of Lie superalgebras.

Given an exact sequence in $\mathtt{SLie}^2_{\mathbb{K}}$

$$(0,0) \longrightarrow (K, L) \xrightarrow{(i,j)} (M, N) \xrightarrow{(\phi,\psi)} (P, Q) \longrightarrow (0,0), \tag{4}$$

by Proposition 3.4(ii) there is a Lie superalgebra homomorphism $M \otimes L \to L$ and an action of N on $K \otimes N$. Thus, there is an action of $M \otimes L$ on $K \otimes N$, so we can form the semidirect product $(K \otimes N) \rtimes (M \otimes L)$, and we have the following obvious analogue of [8, Proposition 9].

Proposition 3.8. *Given the short exact sequence (4), there is an exact sequence of Lie superalgebras*

$$(K \otimes N) \rtimes (M \otimes L) \xrightarrow{\alpha} M \otimes N \xrightarrow{\phi \otimes \psi} P \otimes Q \longrightarrow 0.$$

In particular, given a Lie superalgebra M and a graded ideal K of M, there is an exact sequences of Lie superalgebras

$$(K \otimes M) \rtimes (M \otimes K) \to M \otimes M \to (M/K) \otimes (M/K) \to 0. \tag{5}$$

3.3. Nilpotency, solvability and Engel of the non-abelian tensor product

The results from [20] on nilpotency, solvability and Engel of the non-abelian tensor product on Lie algebras can be easily extended to the case of Lie superalgebras. The notions of nilpotency and solvability of Lie superalgebras are given in [18]. As they are very similar to the respective notions for Lie algebras, we omit them. We say that a Lie superalgebra M is *n-Engel* if it satisfies $\mathsf{ad}(x)^n = 0$ for all $x \in M$. The proof of the following result is similar to the proof of [20, Theorem 2.2].

Theorem 3.9. *Let M and N be two Lie superalgebras acting compatibly on each other. Then,*

(i) *If $[M,N]^M$ is nilpotent, then $M \otimes N$ and $[M,N]^N$ are nilpotent too. Moreover, if the nilpotency class of $[M,N]^M$ is $\mathrm{cl}([M,N]^M)$, then*

$$([M,N]^M) \leq \mathrm{cl}(M \otimes N) \leq \mathrm{cl}([M,N]^M) + 1,$$
$$\mathrm{cl}([M,N]^N) \leq \mathrm{cl}([M,N]^M) + 1.$$

(ii) *If $[M,N]^M$ is solvable, then $M \otimes N$ and $[M,N]^N$ are solvable too. Moreover, if the derived length of $[M,N]^M$ is $\ell([M,N]^M)$, then*

$$\ell([M,N]^M) \leq \ell(M \otimes N) \leq \ell([M,N]^M) + 1,$$
$$\ell([M,N]^N) \leq \ell([M,N]^M) + 1.$$

(iii) *If $[M,N]^M$ is Engel, then $M \otimes N$ and $[M,N]^N$ are Engel too. Moreover, if $[M,N]^M$ is n-Engel, then $M \otimes N$ and $[M,N]^N$ are $(n+1)$-Engel.*

4. Universal central extensions of Lie superalgebras

Now we use the non-abelian tensor product of Lie superalgebras to describe universal central extensions of Lie superalgebras. Recall that a central extension $\mathfrak{u}: U \twoheadrightarrow P$ is universal if for any other central extension $f: M \twoheadrightarrow P$ there is a unique homomorphism $\theta: U \to M$ such that $f \circ \theta = \mathfrak{u}$. It is shown in [19] that a Lie superalgebra P admits a universal central extension if and only if P is perfect, i.e. $P = [P, P]$.

It follows from Proposition 3.4 and Lemma 2.7(i) that the homomorphism $\mathfrak{u}: P \otimes P \twoheadrightarrow [P, P]$, $\mathfrak{u}(p \otimes p') = [p, p']$, is a central extension of the Lie superalgebra $[P, P]$.

Theorem 4.1. *If P is a perfect Lie superalgebra, then the central extension $\mathfrak{u}: P \otimes P \twoheadrightarrow P$ is the universal central extension.*

Proof. Let $f: M \twoheadrightarrow P$ be a central extension of P. Since $\mathsf{Ker}\, f$ is in the centre of M, we get a well-defined homomorphism of Lie superalgebras $\theta: P \otimes P \to M$ given by $\theta(p \otimes p') = [m_p, m_{p'}]$, where m_p and $m_{p'}$ are any preimages of p and p', respectively. Obviously $\theta \circ f = \mathfrak{u}$. Since P is perfect, then by relation (iv) of Definition 3.1, so is $P \otimes P$. Then by [19, Lemma 1.4] the homomorphism θ is unique. □

Remark. If P is a perfect Lie superalgebra, then $\mathrm{H}_2(P) \approx \mathsf{Ker}(P \otimes P \xrightarrow{\mathfrak{u}} P)$, since the kernel of the universal central extension is isomorphic to the second homology $\mathrm{H}_2(P)$ (see [19]).

It is a classical result that the universal central extension of the Lie algebra $\mathfrak{sl}(n, A)$, where A is a unital associative algebra, is the Steinberg algebra $\mathfrak{st}(n, A)$, when $n \geq 5$ (see e.g. [16]). Recently, in [5,9], this result has been extended to Lie superalgebras. Below, using the non-abelian tensor product of Lie superalgebras, we propose an alternative proof of the same result.

First we recall from [5] that, given a unital associative superalgebra A, the *Lie superalgebra* $\mathfrak{sl}(m, n, A)$, $m + n \geq 3$, is defined to be the subalgebra of the Lie superalgebra $\mathfrak{gl}(m, n, A)$ (see Example 2.2 (ii)) generated by the elements $E_{ij}(a)$, $1 \leq i \neq j \leq m + n$, $a \in A_{\bar{0}} \cup A_{\bar{1}}$. It is shown in [5, Lemma 3.3] that $\mathfrak{sl}(m, n, A)$ is a perfect Lie superalgebra. This guarantees the existence of the universal central extension of $\mathfrak{sl}(m, n, A)$.

The Steinberg Lie superalgebra $\mathfrak{st}(m, n, A)$ is defined for $m + n \geq 3$ to be the Lie superalgebra generated by the homogeneous elements $F_{ij}(a)$, where $1 \leq i \neq j \leq m + n$, $a \in A$ is a homogeneous element and the $\mathbb{Z}_2$-grading is given by $|F_{ij}(a)| = |i| + |j| + |a|$, subject to the following relations:

$$a \mapsto F_{ij}(a) \text{ is a } \mathbb{K}\text{-linear map},$$

$$[F_{ij}(a), F_{jk}(b)] = F_{ik}(ab), \text{ for distinct } i, j, k,$$

$$[F_{ij}(a), F_{kl}(b)] = 0, \text{ for } j \neq k, i \neq l.$$

Theorem 4.2. *(See [5].) If $m + n \geq 5$, then the canonical epimorphism*

$$\mathfrak{st}(m,n,A) \twoheadrightarrow \mathfrak{sl}(m,n,A), \quad F_{ij}(a) \mapsto E_{ij}(a),$$

is the universal central extension of the perfect Lie superalgebra $\mathfrak{sl}(m,n,A)$.

Proof. We claim that there is an isomorphism of Lie superalgebras

$$\mathfrak{st}(m,n,A) \cong \mathfrak{st}(m,n,A) \otimes \mathfrak{st}(m,n,A).$$

Indeed, one can readily check that the maps

$$\mathfrak{st}(m,n,A) \longrightarrow \mathfrak{st}(m,n,A) \otimes \mathfrak{st}(m,n,A), \quad F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1) \text{ for } k \neq i,j,$$
$$\mathfrak{st}(m,n,A) \otimes \mathfrak{st}(m,n,A) \longrightarrow \mathfrak{st}(m,n,A), \quad F_{ij}(a) \otimes F_{kl}(b) \mapsto [F_{ij}(a), F_{kl}(b)],$$

are well-defined homomorphisms of Lie superalgebras if $m+n \geq 5$, and they are inverses to each other. Since $\mathfrak{st}(m,n,A)$ is a perfect Lie superalgebra, then Theorem 4.1 and [19, Corollary 1.9] complete the proof. □

5. Non-abelian homology of Lie superalgebras

The low-dimensional non-abelian homology of Lie algebras with coefficients in crossed modules was defined in [10] and it was extended to all dimensions in [12]. In this section we extend to Lie superalgebras the construction of zero and first non-abelian homologies. We also relate the non-abelian homology of Lie superalgebras with the cyclic homology of associative superalgebras studied in [13,15].

5.1. Construction of the non-abelian homology and some properties

Let P be a Lie superalgebra. We denote by $\texttt{Cross}(P)$ the category of crossed modules of Lie superalgebras over P (crossed P-modules, for short), whose objects are crossed modules $(M,\partial) \equiv (\partial\colon M \to P)$ and a morphism from (M,∂) to (N,∂') is a Lie superalgebra homomorphism $f\colon M \to N$ such that $f({}^{p}m) = {}^{p}f(m)$ for all $p \in P$, $m \in M$ and $\partial' \circ f = \partial$. By an exact sequence $(L,\partial'') \xrightarrow{f} (M,\partial) \xrightarrow{g} (N,\partial')$ in $\texttt{Cross}(P)$ we mean that the sequence of Lie superalgebras $L \xrightarrow{f} M \xrightarrow{g} N$ is exact.

Lemma 5.1. *Given a short exact sequence in $\texttt{Cross}(P)$*

$$0 \to (L,\partial'') \xrightarrow{f} (M,\partial) \xrightarrow{g} (N,\partial') \to 0,$$

the morphism $\partial''\colon L \to P$ is trivial and L is an abelian Lie superalgebra.

Proof. Clearly $\partial'' = \partial' \circ g \circ f = 0$ and $[l, l'] = {}^{\partial''(l)}l' = 0$, for all $l, l' \in L$. $\square$

If (M, ∂) and (N, ∂') are two crossed P-modules, then the Lie superalgebras M and N act compatibly on each other via the action of P. Thus, we can construct the non-abelian tensor product of Lie superalgebras $M \otimes N$. Moreover, we have an action of P on $M \otimes N$ defined by ${}^p(m \otimes n) = {}^pm \otimes n + (-1)^{|p||m|}m \otimes {}^pn$, and straightforward computations show that $\eta\colon M \otimes N \to P$, $m \otimes n \mapsto [\partial(m), \partial'(n)]$, is a crossed P-module.

Proposition 5.2. *Let (M, ∂) be a crossed P-module. There is a right exact functor $(M \otimes -)\colon \mathtt{Cross}(P) \to \mathtt{Cross}(P)$ given, for any crossed P-module (N, ∂'), by*

$$(M \otimes -)(N, \partial') = (M \otimes N, \eta).$$

Proof. It is an immediate consequence of Proposition 3.8 $\square$

Definition 5.3. Let (M, ∂) be a crossed P-module. We define the zero and first non-abelian homologies of P with coefficients in M by setting

$$\mathcal{H}_0(P, M) = \mathsf{Coker}\,\nu \qquad \text{and} \qquad \mathcal{H}_1(P, M) = \mathsf{Ker}\,\nu,$$

where $\nu\colon P \otimes M \to M$, $p \otimes m \mapsto {}^pm$, is the Lie superalgebra homomorphism as in Proposition 3.4.

If we consider the crossed P-module (P, id_P) we have that

$$\mathcal{H}_0(P, P) = \frac{P}{[P, P]} \cong \mathrm{H}_1(P).$$

In addition, if P is perfect, by Theorem 4.1 we have that $\mathcal{H}_1(P, P) \cong \mathrm{H}_2(P)$.

The zero and first non-abelian homologies generalize respectively the zero and first homologies of Lie superalgebras in the sense of the following proposition.

Proposition 5.4. *Let the ground ring $\mathbb{K}$ be a field. Let P be a Lie superalgebra and M a P-supermodule thought as a crossed P-module $(M, 0)$. Then there are isomorphisms of super vector spaces*

$$\mathcal{H}_0(P, M) \cong \mathrm{H}_0(P, M) \quad \text{and} \quad \mathcal{H}_1(P, M) \cong \mathrm{H}_1(P, M).$$

Proof. This is a direct consequence of Proposition 3.6 and the isomorphisms (2) and (3). $\square$

Proposition 5.5. *Given a short exact sequence in* $\mathtt{Cross}(P)$

$$0 \to (L, 0) \to (M, \partial) \to (N, \partial') \to 0$$

we have an exact sequence of supermodules

$$\mathcal{H}_1(P, L) \to \mathcal{H}_1(P, M) \to \mathcal{H}_1(P, N) \to \mathcal{H}_0(P, L) \to \mathcal{H}_0(P, M) \to \mathcal{H}_0(P, N) \to 0.$$

Proof. The proof is an immediate consequence of the snake lemma applied to the diagram obtained from Proposition 5.2

$$\begin{array}{ccccccccc} & & P \otimes L & \longrightarrow & P \otimes M & \longrightarrow & P \otimes N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

□

5.2. Application to the cyclic homology of associative superalgebras

Now we recall from [15] and [13] the definition of cyclic homology of associative superalgebras. Let A be an associative superalgebra and $(C'_*(A), d'_*)$ denote its Hochschild complex, that is $C'_n(A) = A^{\otimes_{\mathbb{K}}(n+1)}$ and the boundary map $d_n: C'_n(A) \to C'_{n-1}(A)$ is given by

$$\begin{aligned} d'_n(a_0 \otimes \cdots \otimes a_n) = & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^{n+|a_n|(|a_0|+\cdots+|a_{n-1}|)} a_n a_0 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Now the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts on $A^{\otimes_{\mathbb{K}}(n+1)}$ via

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^{n+|a_n|\sum_{k<n}|a_k|} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

where $t_n = 1 + (n+1)\mathbb{Z} \in \mathbb{Z}/(n+1)\mathbb{Z}$. For each $n \geq 0$, consider the quotient $C_n(A) = A^{\otimes_{\mathbb{K}}(n+1)}/\,\mathsf{Im}(1 - t_n)$ which is the module of coinvariants of $C'_n(A)$ under the $\mathbb{Z}/(n+1)\mathbb{Z}$-action. Then d'_n induces a well-defined map $d_n: C_n(A) \to C_{n-1}(A)$ and there is an induced chain complex $(C_*(A), d_*)$, which is called the Connes complex of A. Its homologies are, by definition, the *cyclic homologies of the associative superalgebra* A, denoted by $\mathrm{HC}_n(A)$, $n \geq 0$.

Easy calculations show that, given an associative superalgebra A, $\mathrm{HC}_1(A)$ is the kernel of the homomorphism of supermodules

$$(A \otimes_{\mathbb{K}} A)/\,\mathrm{I}(A) \to [A, A], \quad a \otimes b \mapsto ab - (-1)^{|a||b|} ba,$$

where $[A, A]$ is the graded submodule of A generated by the elements $ab - (-1)^{|a||b|}ba$ and $\mathrm{I}(A)$ is the graded submodule of the supermodule $A \otimes_{\mathbb{K}} A$ generated by the elements

$$a \otimes b + (-1)^{|a||b|} b \otimes a,$$
$$ab \otimes c - a \otimes bc + (-1)^{|c|(|a|+|b|)} ca \otimes b,$$

for all homogeneous $a, b, c \in A$.

Now let us consider A as a Lie superalgebra (see Example 2.2(i)). Then there is a Lie superalgebra structure on $(A \otimes_{\mathbb{K}} A)/\,\mathrm{I}(A)$ given by

$$[a \otimes b, a' \otimes b'] = [a, b] \otimes [a', b']$$

for all $a, a', b, b' \in A$. We denote this Lie superalgebra by $\mathrm{V}(A)$. In fact, $\mathrm{V}(A)$ is the quotient of the non-abelian tensor product $A \otimes A$ by the graded ideal generated by the elements $x \otimes y + (-1)^{|x||y|} y \otimes x$ and $xy \otimes z - x \otimes yz + (-1)^{|z|(|x|+|y|)} zx \otimes y$, for all homogeneous $x, y, z \in A$.

Proposition 5.6. *Let A be a Lie superalgebra. Then the following assertions hold:*

(i) *There are compatible actions of the Lie superalgebras A and $\mathrm{V}(A)$ on each other.*
(ii) *The map $\mu\colon \mathrm{V}(A) \to A$ given by $x \otimes y \mapsto [x, y]$, together with the action of A on $\mathrm{V}(A)$, is a crossed module of Lie superalgebras.*
(iii) *The action of A on $\mathrm{V}(A)$ induces the trivial action of A on $\mathrm{HC}_1(A)$.*
(iv) *There is a short exact sequence in the category $\mathtt{Cross}(A)$*

$$0 \to (\mathrm{HC}_1(A), 0) \to (\mathrm{V}(A), \mu) \to ([A, A], i) \to 0,$$

where $i\colon [A, A] \to A$ is the inclusion.

Proof. (i) The action of A on $\mathrm{V}(A)$ is induced by the action of A on $A \otimes A$ given in Proposition 3.4(ii), that is

$$\begin{aligned} {}^{a}(x \otimes y) &= [a, x] \otimes y + (-1)^{|a||x|} x \otimes [a, y] \\ &= ax \otimes y + (-1)^{|a|(|x|+|y|)} x \otimes ya - (-1)^{|x||a|} x \otimes ay - (-1)^{|x||a|} xa \otimes y \\ &= a \otimes xy - (-1)^{|x||y|} a \otimes yx \\ &= a \otimes [x, y], \end{aligned}$$

whilst the action of $\mathrm{V}(A)$ on A is defined by

$$^{x \otimes y} a = \big[[x, y], a\big]$$

for all homogeneous $a, x, y \in A$. Straightforward calculations show that these are indeed (compatible) actions of Lie superalgebras.

(ii) Since the crossed module of Lie superalgebras $A \otimes A \to A$, $x \otimes y \mapsto [x, y]$, given in Proposition 3.4, vanishes on the elements of the form $x \otimes y + (-1)^{|x||y|} y \otimes x$ and

$xy \otimes z - x \otimes yz + (-1)^{|z|(|x|+|y|)} zx \otimes y$, then μ is well defined and obviously it is a crossed module of Lie superalgebras.

(iii) If $\sum_i \lambda_i(x_i \otimes y_i) \in \mathrm{HC}_1(A)$, i.e. $\sum_i \lambda_i[x_i, y_i] = 0$, then for all $a \in A$ we have

$$ {}^{a}\Big(\sum_i \lambda_i(x_i \otimes y_i)\Big) = \sum_i \lambda_i(a \otimes [x_i, y_i]) = a \otimes \sum_i \lambda_i[x_i, y_i] = 0. $$

(iv) This is an immediate consequence of the assertions above. □

By Proposition 5.5 we have the following exact sequence of supermodules

$$ \begin{array}{c} \mathcal{H}_1\big(A, \mathrm{HC}_1(A)\big) \longrightarrow \mathcal{H}_1\big(A, \mathrm{V}(A)\big) \longrightarrow \mathcal{H}_1(A, [A, A]) \longrightarrow \\ \longrightarrow \mathcal{H}_0\big(A, \mathrm{HC}_1(A)\big) \longrightarrow \mathcal{H}_0\big(A, \mathrm{V}(A)\big) \longrightarrow \mathcal{H}_0(A, [A, A]) \longrightarrow 0. \end{array} \tag{6} $$

Below, we will calculate some of the terms of this exact sequence. At first, by analogy to the Dennis–Stein generators [6], we give a definition of the first Milnor cyclic homology for associative superalgebras.

Definition 5.7. Let A be an associative superalgebra. We define the first Milnor cyclic homology $\mathrm{HC}_1^M(A)$ of A to be the quotient of the supermodule $A \otimes_{\mathbb{K}} A$ by the graded ideal generated by the elements

$$ \begin{aligned} & a \otimes b + (-1)^{|a||b|} b \otimes a, \\ & ab \otimes c - a \otimes bc + (-1)^{|c|(|a|+|b|)} ca \otimes b, \\ & a \otimes bc - (-1)^{|b||c|} a \otimes cb, \end{aligned} $$

for all homogeneous $a, b, c \in A$.

It is clear that if A is supercommutative, that is, $ab = (-1)^{|a||b|} ba$, for all homogeneous $a, b \in A$, then $\mathrm{HC}_1(A) \cong \mathrm{HC}_1^M(A)$.

Lemma 5.8. *We have the following equalities and isomorphisms*

(i) $\mathcal{H}_0\big(A, \mathrm{HC}_1(A)\big) = \mathrm{HC}_1(A)$,
(ii) $\mathcal{H}_1\big(A, \mathrm{HC}_1(A)\big) \cong A/[A, A] \otimes_{\mathbb{K}} \mathrm{HC}_1(A)$,
(iii) $\mathcal{H}_0(A, [A, A]) = [A, A]/\big[A, [A, A]\big]$,
(iv) $\mathcal{H}_0\big(A, \mathrm{V}(A)\big) \cong \mathrm{HC}_1^M(A)$.

Proof.

(i) Since A acts trivially on $\mathrm{HC}_1(A)$, we have that $\mathsf{Coker}\big(A \otimes \mathrm{HC}_1(A) \to \mathrm{HC}_1(A)\big) = \mathrm{HC}_1(A)$.
(ii) Since $\mathrm{HC}_1(A)$ is abelian, by Proposition 3.5 we have that $\mathsf{Ker}\big(A \otimes \mathrm{HC}_1(A) \to \mathrm{HC}_1(A)\big) \cong A/[A,A] \otimes_{\mathbb{K}} \mathrm{HC}_1(A)$.
(iii) and (iv) are straightforward. □

It follows that the exact sequence (6) can be written as in the following theorem.

Theorem 5.9. *If A is a unital associative superalgebra. Then there is an exact sequence of supermodules*

$$\frac{A}{[A,A]} \otimes_{\mathbb{K}} \mathrm{HC}_1(A) \longrightarrow \mathcal{H}_1\big(A, \mathrm{V}(A)\big) \longrightarrow \mathcal{H}_1(A,[A,A]) \longrightarrow \mathrm{HC}_1(A) \longrightarrow \mathrm{HC}_1^M(A) \longrightarrow \frac{[A,A]}{[A,[A,A]]} \longrightarrow 0.$$

Corollary 5.10. *If A is perfect as a Lie superalgebra, we have an exact sequence*

$$0 \to \mathcal{H}_1\big(A, \mathrm{V}(A)\big) \to \mathrm{H}_2(A) \to \mathrm{HC}_1(A) \to 0,$$

where $\mathrm{H}_2(A)$ is the usual second homology of the Lie superalgebra A. If in addition $\mathrm{H}_2(A) = 0$, then all terms of the exact sequence in the previous theorem are trivial.

Proof. Since A is perfect we know that $\mathcal{H}_1(A,A) \cong \mathrm{H}_2(A)$, $A/[A,A] \otimes_{\mathbb{K}} \mathrm{HC}_1(A) = 0$ and the map $A \otimes \mathrm{V}(A) \to \mathrm{V}(A)$ is surjective. □

6. Non-abelian exterior product of Lie superalgebras

In this section we extend to Lie superalgebras the definition of the non-abelian exterior product of Lie algebras introduced in [8]. Then we use it to derive the Hopf formula for the second homology of a Lie superalgebra and to construct a six-term exact homology sequence of Lie superalgebras.

6.1. Construction of the non-abelian exterior product

Let P be a Lie superalgebra and (M, ∂) and (N, ∂') two crossed P-modules. We consider the actions of M and N on each other via P.

Lemma 6.1. *Let $M \square N$ be the graded submodule of $M \otimes N$ generated by the elements*

(a) $m \otimes n + (-1)^{|m'||n'|} m' \otimes n'$, *where* $\partial(m) = \partial'(n')$ *and* $\partial(m') = \partial'(n)$,

(b) $m_{\bar{0}} \otimes n_{\bar{0}}$, *where* $\partial(m_{\bar{0}}) = \partial'(n_{\bar{0}})$,

with $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ *and* $n_{\bar{0}} \in N_{\bar{0}}$. *Then,* $M \square N$ *is a graded ideal in the centre of* $M \otimes N$.

Proof. Given an element $m \otimes n + (-1)^{|m'||n'|} m' \otimes n'$ of the form (a), suppose that $|m'| = |n|$, then we have

$$\begin{aligned}
[x \otimes y, m \otimes n + (-1)^{|m'||n'|} m' \otimes n'] &= -(-1)^{|x||y|} ({}^{y}x) \otimes \left({}^{m}n + (-1)^{|m'||n'|} ({}^{m'}n')\right) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes \left({}^{\partial(m)}n + (-1)^{|m'||n'|} ({}^{\partial(m')}n')\right) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes \left({}^{\partial'(n')}n + (-1)^{|m'||n'|} ({}^{\partial'(n)}n')\right) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes \left([n', n] + (-1)^{|n||n'|} [n, n']\right) \\
&= 0.
\end{aligned}$$

This is also true when $|m'| \neq |n|$. Indeed, if $|m'| \neq |n|$, since ∂, ∂' are even maps, the equality $\partial(m) = \partial'(n')$ holds if and only if $\partial(m) = 0 = \partial'(n')$. Now take an element $m_{\bar{0}} \otimes n_{\bar{0}}$ of the form (b). Then we have

$$\begin{aligned}
[x \otimes y, m_{\bar{0}} \otimes n_{\bar{0}}] &= -(-1)^{|x||y|} ({}^{y}x) \otimes ({}^{m_{\bar{0}}}n_{\bar{0}}) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes ({}^{\partial(m_{\bar{0}})}n_{\bar{0}}) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes ({}^{\partial(n_{\bar{0}})}n_{\bar{0}}) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes [n_{\bar{0}}, n_{\bar{0}}] \\
&= 0,
\end{aligned}$$

for any $x \otimes y \in M \otimes N$. This completes the proof. □

Definition 6.2. Let P be a Lie superalgebra and (M, ∂) and (N, ∂') two crossed P-modules. The *non-abelian exterior product* $M \wedge N$ of the Lie superalgebras M and N is defined by

$$M \wedge N = \frac{M \otimes N}{M \square N}.$$

The equivalence class of $m \otimes n$ will be denoted by $m \wedge n$.

Note that if $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$ then $M \wedge N$ coincides with the non-abelian exterior product of Lie algebras [8].

Reviewing Section 3, one can easily check that most of results on the non-abelian tensor product are fulfilled for the non-abelian exterior product. In particular, there are homomorphisms of Lie superalgebras $M \wedge N \to M$, $M \wedge N \to N$ and actions of M

and N on $M \wedge N$, induced respectively by the homomorphisms and actions given in Proposition 3.4. It is also satisfied the isomorphism $M \wedge N \cong N \wedge M$. Further, given a short exact sequence of Lie superalgebras $0 \to K \to M \to P \to 0$, as an exterior analogue of the exact sequence (5), we get the following exact sequence of Lie superalgebras

$$K \wedge M \to M \wedge M \to P \wedge P \to 0. \tag{7}$$

Given a Lie superalgebra M, since $\mathsf{id}\colon M \to M$ is a crossed module, we can consider $M \wedge M$. It is the quotient of $M \otimes M$ by the following relations

$$m \wedge m' = -(-1)^{|m||m'|} m' \wedge m,$$
$$m_{\bar{0}} \wedge m_{\bar{0}} = 0,$$

for all $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $m_{\bar{0}} \in M_{\bar{0}}$. In the particular case when M is perfect, it is easy to see that $M \square M = 0$, so $M \wedge M \cong M \otimes M$ and in Theorem 4.1 we can replace $M \otimes M$ by $M \wedge M$.

6.2. *A six term exact homology sequence*

In [7], the non-abelian exterior product of Lie algebras is used to construct a six-term exact sequence of homology of Lie algebras. In this section we will extend these results to Lie superalgebras.

First of all, we prove an analogue of Miller's theorem [17] on free Lie superalgebras extending the similar result obtained in [7] for Lie algebras.

Proposition 6.3. *Let $F = \mathrm{F}(X)$ be the free Lie superalgebra on a graded set X. Then the homomorphism $F \wedge F \to F$, $x \wedge y \mapsto xy$ is injective.*

Proof. Let us prove that $[F, F] \cong F \wedge F$. Using the same notations as in Construction 2.9, we define a map $\phi\colon \mathrm{alg}(X) * \mathrm{alg}(X) \to F \wedge F$ by $\sum_i \lambda_i x_i y_i \mapsto \sum_i \lambda_i (x_i \wedge y_i)$, where $\mathrm{alg}(X) * \mathrm{alg}(X)$ is the free product of superalgebras. It is easy to see that ϕ is a $\mathbb{K}$-superalgebra homomorphism since $[x \wedge y, x' \wedge y'] = xy \wedge x'y'$. The ideal I is contained in $\mathrm{alg}(X) * \mathrm{alg}(X)$ and by using the defining relations of $F \wedge F$ it is not difficult to check that ϕ vanishes on I. So we have an induced map from $[F, F]$ to $F \wedge F$, which is inverse to the homomorphism $F \wedge F \to [F, F]$, $x \wedge y \mapsto xy$. $\square$

Let P be a Lie superalgebra and take the quotient supermodule $(P \wedge_{\mathbb{K}} P)/\operatorname{\mathsf{Im}} d_3$, where $d_3\colon \bigwedge^3_{\mathbb{K}}(P) \to \bigwedge^2_{\mathbb{K}}(P)$ is the boundary map in the homology complex $(C_*(P, \mathbb{K}), d_*)$. Here $\mathbb{K}$ is considered as a trivial P-module. We define a bracket in $(P \wedge_{\mathbb{K}} P)/\operatorname{\mathsf{Im}} d_3$ by setting

$$[x \wedge y, x' \wedge y'] = [x, y] \wedge [x', y']$$

for all $x, y \in P$. As a particular case of the exterior analogue of Proposition 3.3 we have

Lemma 6.4. *There is an isomorphism of Lie superalgebras*

$$\frac{P \wedge_{\mathbb{K}} P}{\mathsf{Im}\, d_3} \approx P \wedge P.$$

Corollary 6.5.

(i) *For any Lie superalgebra P there is an isomorphism of supermodules*

$$H_2(P) \cong \mathsf{Ker}(P \wedge P \to P).$$

(ii) *$H_2(F) = 0$ if F is a free Lie superalgebra.*

(iii) *(Hopf formula) Given a free presentation $0 \to R \to F \to P \to 0$ of a Lie superalgebra P, there is an isomorphism of supermodules*

$$H_2(P) \cong \frac{R \cap [F, F]}{[F, R]}.$$

Proof.

(i) This follows immediately from Lemma 6.4.

(ii) This is a consequence of (i) and Proposition 6.3.

(iii) Since $F \wedge F \cong [F, F]$, using the exact sequence (7), we have

$$P \wedge P \cong \frac{[F, F]}{[F, R]}.$$

Then Lemma 6.4 completes the proof. □

Theorem 6.6. *Let M be a graded ideal of a Lie superalgebra P. Then there is an exact sequence*

$$\mathsf{Ker}(P \wedge M \to P) \to \mathrm{H}_2(P) \to \mathrm{H}_2(P/M) \to \frac{M}{[P, M]} \to \mathrm{H}_1(P) \to \mathrm{H}_1(P/M) \to 0.$$

Proof. By using the exact sequence (7) we have the following commutative diagram of Lie superalgebras with exact rows

$$\begin{array}{ccccccccc}
 & & M \wedge P & \longrightarrow & P \wedge P & \longrightarrow & \dfrac{P}{M} \wedge \dfrac{P}{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & \dfrac{P}{M} & \longrightarrow & 0.
\end{array}$$

Since $\mathsf{Coker}(M \wedge P \cong P \wedge M \to M) \cong M/[P, M]$ and $\mathsf{Coker}(P \wedge P \to P) \cong P/[P, P] \cong \mathrm{H}_1(P)$, then the assertion follows by using snake lemma and Corollary 6.5(i). □

In particular, if P is a Lie algebra and M is an ideal of P, then this sequence coincides with the six-term exact sequence in the homology of Lie algebras obtained in [7].

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References

[1] R. Brown, J.-L. Loday, Van Kampen theorems for diagrams of spaces, Topology 26 (3) (1987) 311–335, with an appendix by M. Zisman.
[2] C. Carmeli, L. Caston, R. Fioresi, Mathematical Foundations of Supersymmetry, EMS Ser. Lect. Math., European Mathematical Society (EMS), Zürich, 2011.
[3] J.M. Casas, M. Ladra, Perfect crossed modules in Lie algebras, Comm. Algebra 23 (5) (1995) 1625–1644.
[4] J.L. Castiglioni, X. García-Martínez, M. Ladra, Universal central extensions of Lie–Rinehart algebras, arXiv:1403.7159, 2014.
[5] H. Chen, J. Sun, Universal central extensions of $sl_{m|n}$ over $\mathbb{Z}/2\mathbb{Z}$-graded algebras, J. Pure Appl. Algebra 219 (2015) 4278–4294.
[6] R.K. Dennis, M.R. Stein, K_2 of discrete valuation rings, Adv. Math. 18 (2) (1975) 182–238.
[7] G.J. Ellis, Nonabelian exterior products of Lie algebras and an exact sequence in the homology of Lie algebras, J. Pure Appl. Algebra 46 (2–3) (1987) 111–115.
[8] G.J. Ellis, A nonabelian tensor product of Lie algebras, Glasg. Math. J. 33 (1) (1991) 101–120.
[9] X. García-Martínez, M. Ladra, Universal central extensions of $\mathfrak{sl}(m, n, a)$ of small rank over associative superalgebras, arXiv:1405.4035, 2014.
[10] D. Guin, Cohomologie des algèbres de Lie croisées et K-théorie de Milnor additive, Ann. Inst. Fourier (Grenoble) 45 (1) (1995) 93–118.
[11] P.J. Hilton, U. Stammbach, A Course in Homological Algebra, 2nd ed., Grad. Texts in Math., vol. 4, Springer-Verlag, New York, 1997.
[12] N. Inassaridze, E. Khmaladze, M. Ladra, Non-abelian homology of Lie algebras, Glasg. Math. J. 46 (2) (2004) 417–429.
[13] K. Iohara, Y. Koga, Second homology of Lie superalgebras, Math. Nachr. 278 (9) (2005) 1041–1053.
[14] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1) (1977) 8–96.
[15] C. Kassel, A Künneth formula for the cyclic cohomology of $\mathbf{Z}/2$-graded algebras, Math. Ann. 275 (4) (1986) 683–699.
[16] C. Kassel, J.-L. Loday, Extensions centrales d'algèbres de Lie, Ann. Inst. Fourier (Grenoble) 32 (4) (1982) 119–142.
[17] C. Miller, The second homology group of a group; relations among commutators, Proc. Amer. Math. Soc. 3 (1952) 588–595.

[18] I.M. Musson, Lie Superalgebras and Enveloping Algebras, Grad. Stud. Math., vol. 131, American Mathematical Society, Providence, RI, 2012.
[19] E. Neher, An introduction to universal central extensions of Lie superalgebras, in: Groups, Rings, Lie and Hopf Algebras, St. John's, NF, 2001, in: Math. Appl., vol. 555, Kluwer Acad. Publ., Dordrecht, 2003, pp. 141–166.
[20] A.R. Salemkar, H. Tavallaee, H. Mohammadzadeh, B. Edalatzadeh, On the non-abelian tensor product of Lie algebras, Linear Multilinear Algebra 58 (3–4) (2010) 333–341.
[21] M. Scheunert, The Theory of Lie Superalgebras. An Introduction, Lecture Notes in Math., vol. 716, Springer-Verlag, Berlin, 1979.
[22] J. Tanaka, On homology and cohomology of Lie superalgebras with coefficients in their finite-dimensional representations, Proc. Japan Acad. Ser. A Math. Sci. 71 (3) (1995) 51–53.
[23] T. Zhang, Z. Liu, Omni-Lie superalgebras and Lie 2-superalgebras, Front. Math. China 9 (5) (2014) 1195–1210.