Proceedings of A. Razmadze
Mathematical Institute
Vol. **159** (2012), 57–64

ON ASSOCIATIVE AND LIE 2-ALGEBRAS

E. KHMALADZE

Abstract. Our goal is to translate the classical adjunction between associative and Lie algebras to the corresponding 2-objects, that is, to categorify it.

რეზიუმე. ჩვენი მიზანია ასოციურ და ლის ალგებრას შორის კლასიკური შეუღლების გადატანა შესაბამისი 2-ობიექტებისათვის, ანუ ამ შეუღლების კატეგორიფიკაცია.

1. Introduction

In recent times categorification of algebraic structures plays a growing role in algebra and geometry. Here categorification means the replacement of the underlying sets in some algebraic structure by categories. For example, instead of regarding a group in the category of sets (which is the ordinary concept of a group), one considers a group object in the category of small categories, and arrives at the notion of a 2-group.

There are many different ways to make the notion of a 2-group precise. A crucial first step was Whitehead's [6] concept of crossed module, formulated in the late 1940s without the aid of category theory. From the 1960s it was clear that crossed modules are essentially the same as categorical groups, called strict 2-groups, since they are categorified groups in which the group laws hold strictly, as equations.

In the categorification of groups, the associativity law is replaced by an isomorphism called the associator, which satisfies a new law. The counterpart of the associative law in the theory of Lie algebras is the Jacobi identity. In a Lie 2-algebra this is replaced by an isomorphism which is called the Jacobiator and satisfies a new law of its own [1]. If the Jacobiator is the identity map, then we arrive to the notion of a strict Lie 2-algebra, which is equivalent to the notion of a crossed module of Lie algebras (see [1, 3]).

Reported on Annual Scientific Conference of A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University (28.11–2.12, 2011).

2010 *Mathematics Subject Classification.* 16E99, 16S30, 16W25.

Key words and phrases. 2-vector space, Lie and associative 2-algebras, crossed module.

In the present note we deal with the categorification of the concept of associative algebra and introduce the notion of an associative 2-algebra. Then we focus on strict associative 2-algebras and describe them as crossed modules of associative algebras (Theorem 3.5). We generalize the well-known adjunction between the categories of Lie and associative algebras to the corresponding strict 2-objects (Theorem 4.6, Corollary 4.7).

2. Preliminaries

2.1. **On 2-vector spaces.** We begin by reviewing some needed material on categorified linear algebra of Baez and Crans [1]. Throughout the paper we fix a ground field $\mathbf{k}$.

Definition 2.2. A 2-vector space is an internal category in the category of vector spaces **Vect**.

Thus, a 2-vector space $\mathcal{V} = (V_1, V_0, s, t, i, \circ)$ consists of a vector space of objects V_0, a vector space of morphisms V_1, source and target maps $s, t : V_1 \to V_0$, an identity-assigning map $i : V_0 \to V_1$, together with a composition map $\circ : V_1 \times_{V_0} V_1 \to V_1$ subject to coherence conditions expressing the axioms of category theory (more details see in [5]). For example, $\mathcal{K} = (\mathbf{k}, \mathbf{k}, 1_{\mathbf{k}}, 1_{\mathbf{k}}, 1_{\mathbf{k}}, \circ)$, where $\circ$ is the multiplication in $\mathbf{k}$, is a 2-vector space.

A linear functor $F : \mathcal{V} \to \mathcal{V}'$ between 2-vector spaces is an internal functor in **Vect**, that is, F consists of linear maps $F_0 : V_0 \to V_0'$ and $F_1 : V_1 \to V_1'$ making certain diagrams commutative, which correspond to the usual laws satisfied by a functor.

2-vector spaces and linear functors form a category denoted by **2-Vect**, which can be enriched with some categorified linear algebra structure:

Proposition 2.3 ([1]). *Given 2-vector spaces $\mathcal{V} = (V_1, V_0, s, t, i)$ and $\mathcal{V}' = (V_1', V_0', s', t', i')$ there are 2-vector spaces*

$$\mathcal{V} \oplus \mathcal{V}' = (V_1 \oplus V_1', V_0 \oplus V_0', s \oplus s', t \oplus t', i \oplus i' \oplus \circ'),$$
$$\mathcal{V} \otimes \mathcal{V}' = (V_1 \otimes V_1', V_0 \otimes V_0', s \otimes s', t \otimes t', i \otimes i' \otimes \circ')$$

called direct sum and tensor product of $\mathcal{V}$ and $\mathcal{V}'$. Moreover, $\mathcal{V} \oplus \mathcal{V}'$ is the biproduct in the categorical sense and $\mathcal{V} \otimes \mathcal{V}'$ satisfies the usual universal property of the tensor product.

For any 2-vector space $\mathcal{V}$, there is an isomorphism $l_{\mathcal{V}} : \mathcal{K} \otimes \mathcal{V} \to \mathcal{V}$ defined on objects or on morphisms by $k \otimes x \mapsto kx$. Similarly, there is another ismorphism $r_{\mathcal{V}} : \mathcal{V} \otimes \mathcal{K} \to \mathcal{V}$.

With the tensor product, the category **2-Vect** *becomes a symmetric monoidal category.*

Definition 2.4. Let $\mathcal{V}$ and $\mathcal{V}'$ be 2-vector spaces. A functor $F : \mathcal{V}^n \to \mathcal{V}'$ is n-linear if $F(x_1, \dots, x_n)$ is linear in each argument, where $(x_1, \dots, x_n)$ is an object or morphism of $\mathcal{V}^n$. Given n-linear functors $F, F' : \mathcal{V}^n \to \mathcal{V}'$, a natural transformation $\eta : F \Rightarrow F'$ is n-linear if $\eta_{x_1,\dots,x_n}$ depends linearly on each object x_i, and completely antisymmetric if the arrow part of $\eta_{x_1,\dots,x_n}$ is completely antisymmetric under permutations of the objects.

2.5. **On Lie 2-algebras.** Let us recall the definition of a categorified Lie algebra from [1].

Definition 2.6. A Lie 2-algebra is a 2-vector space $\mathcal{L}$ together with a skew-symmetric bilinear functor $[-,-] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and a completely antisymmetric trilinear natural isomorphism, the Jacobiator, $J_{x,y,z} : [[x,y],z] \to [x,[y,z]] + [[x,z],y]$, satisfying the identity:

$$J_{[w,x],y,z}([J_{w,x,z},y]+1)(J_{w,[x,z],y}+J_{[w,z],x,y}+J_{w,x,[y,z]}) =$$
$$= [J_{w,x,y},z](J_{[w,y],x,z}+J_{w,[x,y],z})([J_{w,y,z},x]+1)([w,J_{x,y,z}]+1).$$

The usual definition of monoidal functor can be modeled to define a homomorphism between Lie 2-algebras. Then one can construct the category **2-Lie** with Lie 2-algebras as objects and homomorphisms between these as morphisms.

A Lie 2-algebra is called *strict* if its Jacobiator is the identity. A homomorphism of Lie 2-algebras F is called strict if F_2 is the identity. Strict Lie 2-algebras and strict homomorphisms form a subcategory **2-SLie** of the category **2-Lie**.

Now we recall from [4] that a *Lie crossed module* (M, P, μ) is a Lie homomorphism $\mu : M \to P$ together with an action of P on M such that

$$\mu({}^{p}m) = [p, \mu(m)], \qquad {}^{\mu(m)}m' = [m, m'] \qquad \text{for} \quad m, m' \in M, \quad p \in P.$$

Here, by an *action* of P on M we mean a $\mathbf{k}$-bilinear map $P \times M \to M$, $(p, m) \mapsto {}^{p}m$, satisfying ${}^{[p,p']}m = {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m)$ and ${}^{p}[m, m'] = [{}^{p}m, m'] + [m, {}^{p}m']$.

A *morphism of Lie crossed modules* $(\alpha, \beta) : (M, P, \mu) \to (M', P', \mu')$ is a pair of Lie homomorphisms $\alpha : M \to M'$, $\beta : P \to P'$ such that $\mu'\alpha = \beta\mu$ and $\alpha({}^{p}m) = {}^{\beta(p)}\alpha(m)$ for all $m \in M$, $p \in P$. We denote by **XLie** the category of Lie crossed modules.

Theorem 2.7 ([1, 3])**.** *The categories* **2-SLie** *and* **XLie** *are equivalent.*

3. Associative 2-algebras

Since we have discussed some basic tools of categorified linear algebra, we may introduce categorified version of the concept of associative algebra.

Definition 3.1. An associative 2-algebra is a 2-vector space $\mathcal{A}$ with a bilinear functor, the multiplication, $- \bullet - : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a trilinear natural isomorphism, the associator,

$$A_{a,b,c} : (a \bullet b) \bullet c \to a \bullet (b \bullet c),$$

such that the following coherence condition, relating two ways of using the associator to rebracket the expression $((a \bullet b) \bullet c) \bullet d$, holds:

$$A_{a,b,c\bullet d}A_{a\bullet b,c,d} = (1_a \bullet A_{b,c,d})A_{a,b\bullet c,d}(A_{a,b,c} \bullet 1_d)$$

A homomorphism of associative 2-algebras $\mathcal{A}$ and $\mathcal{A}'$ is a linear functor $F = (F_0, F_1) : \mathcal{A} \to \mathcal{A}'$ of underlying 2-vector spaces together with a bilinear natural transformation $F_2(a, b) : F_0(a) \bullet F_0(b) \to F_0(a \bullet b)$ such that the following coherence condition holds:

$$F_2(a, b \bullet c)\big(1_{F_0(a)} \bullet F_2(b, c)\big)A'_{F_0(a),F_0(b),F_0(c)} = \\ = F_1(a \bullet (b \bullet c))F_2(a \bullet b, c)\big(F_2(a, b) \bullet 1_{F_0(c)}\big).$$

The composite of homomorphisms of associative 2-algebras $F : \mathcal{A} \to \mathcal{A}'$ and $F' : \mathcal{A}' \to \mathcal{A}''$ is given by the usual composition $F'F$ together with $(F'F)_2$ defined as the composition $(F' \circ F_2)F'_2$, where $(F' \circ F_2)$ is the result of whiskering the functor F' by the natural transformation F_2. It is routine to show that the composite of homomorphisms is a homomorphism and this composition is associative. *The identity homomorphism* $1_{\mathcal{A}}$ has the identity functor as its underlying functor, together with an identity natural transformation as $(1_{\mathcal{A}})_2$. Thus, we have the following

Proposition 3.2. *There is a category* **2-Alg** *whose objects are associative* 2*-algebras, morphisms are homomorphisms between associative* 2*-algebras, with composition and identities defined as above. One can go further and make the category* **2-Alg** *into a* 2*-category.*

3.3. **Strict associative 2-algebras and crossed modules.**

Definition 3.4. An associative 2-algebra is strict if its associator is the identity. A homomorphism F of associative 2-algebras is strict if F_2 is the identity.

Strict associative 2-algebras and strict homomorphisms form a subcategory of **2-Alg** denoted by **2-SAlg**. Below we obtain strict associative 2-algebras from crossed modules.

First we recall from [2] that a *crossed module of algebras* (R, A, ρ) is an algebra homomorphism $\rho : R \to A$, together with an action of A on R such that

$$\rho(a \cdot r) = a\rho(r), \quad \rho(r \cdot a) = \rho(r)a, \quad \rho(r) \cdot r' = rr' = r \cdot \rho(r'),$$

for all $a \in A$, $r, r' \in R$. Here an *action* of A on R is an A-bimodule structure on R, $A \times R \to R$, $(a, r) \mapsto a \cdot r$, $R \times A \to R$, $(r, a) \mapsto r \cdot a$, satisfying

$$a \cdot (rr') = (a \cdot r)r', \quad (r \cdot a)r' = r(a \cdot r'), \quad (rr') \cdot a = r(r' \cdot a).$$

A *morphism* $(\alpha, \beta) : (R, A, \rho) \to (R', A', \rho')$ of crossed modules is a pair of homomorphisms $(\alpha : R \to R', \beta : A \to A')$ such that $\rho'\alpha = \beta\rho$,

$\alpha(a \cdot r) = \beta(a) \cdot \alpha(r)$ and $\alpha(r \cdot a) = \alpha(r) \cdot \beta(a)$. Let us denote the category of crossed modules of algebras by **XAlg**.

Theorem 3.5. *The categories* **2-SAlg** *and* **XAlg** *are equivalent.*

Proof. Given an associative 2-algebra $(A_1, A_0, s, t, i, \circ)$, the corresponding crossed module is (R, A, ρ), where $R = \operatorname{Ker} s$, $A = A_0$, $\mu = t\,|_{\operatorname{Ker} s}$, and the action of A on R is given by $a \cdot r = i(a) \bullet r$ and $r \cdot a = r \bullet i(a)$, for $a \in A$, $r \in R$.

Conversely, given a crossed module of associative algebras (R, A, ρ), the corresponding associative 2-algebra is defined to be $(A_1, A_0, s, t, i, \circ)$, where $A_0 = A$ and A_1 is the the semi-direct product $R \rtimes A$ in which multiplication is given by

$$(r, a) \bullet (r', a') = (rr' + a \cdot r' + r \cdot a', aa').$$

The source, target and identity-assigning maps are defined by $s(r, a) = a$, $t(r, a) = \rho(r) + a$ and $i(a) = (0, a)$, respectively. The morphisms (r, a) and (r', a') are composable if $a' = \rho(r) + a$ and their composite is $(r', \rho(r) + a) \circ (r, a) = (r + r', a)$.

The remaining details are straightforward and we omit them. □

4. Adjunction Between Strict Associative and Strict Lie 2-algebras

Any associative algebra A becomes a Lie algebra with the Lie bracket $[a, b] = ab - ba$. Let $\mathbb{L} : \mathbf{Alg} \to \mathbf{Lie}$ denote the functor sending an algebra A to its Lie algebra $\mathbb{L}(A)$. Let $\mathbb{U} : \mathbf{Lie} \to \mathbf{Alg}$ denote its left adjoint functor, sending a Lie algebra L to its universal enveloping algebra $\mathbb{U}(L)$. In this section we construct natural generalizations of $\mathbb{L}$ and $\mathbb{U}$ for the categories **XAlg** and **XLie** such that the similar adjunction will be preserved.

4.1. **Liezation of crossed modules of associative algebras.** We can associate to a crossed module of associative algebras (R, A, ρ) the Lie crossed module $(\mathbb{L}(R), \mathbb{L}(A), \mathbb{L}(\rho))$ with the action of $\mathbb{L}(A)$ on $\mathbb{L}(R)$ given by ${}^a r = a \cdot r - r \cdot a$ (see [2]). This assignment defines a functor $\mathbb{XL} : \mathbf{XAlg} \to \mathbf{XLie}$ which is a natural generalization of the functor $\mathbb{L} : \mathbf{Alg} \to \mathbf{Lie}$ in the following sense. There are two ways regarding an algebra A (resp. a Lie algebra P) as a crossed module of algebras (resp. Lie crossed module), via the trivial map $0 : 0 \to A$ (resp. $0 : 0 \to P$) and via the identity map $1_A : A \to A$ (resp. $1_P : P \to P$). There are full embeddings

$$\mathbb{I}_0, \mathbb{I}_1 : \mathbf{Alg} \longrightarrow \mathbf{XAlg} \qquad (\text{resp. } \mathbb{I'}_0, \mathbb{I'}_1 : \mathbf{Lie} \longrightarrow \mathbf{XLie})$$

defined by $\mathbb{I}_0(A) = (0, A, 0)$, $\mathbb{I}_1(A) = (A, A, 1_A)$ (resp. $\mathbb{I'}_0(P) = (0, P, 0)$, $\mathbb{I'}_1(P) = (P, P, 1_P)$). Then, we have the following commutative diagram, for $i = 0$ or 1,

$$\begin{array}{ccc} \mathbf{Alg} & \xrightarrow{\mathbb{I}_i} & \mathbf{XAlg} \\ \downarrow{\scriptstyle \mathbb{L}} & & \downarrow{\scriptstyle \mathbb{XL}} \\ \mathbf{Lie} & \xrightarrow{\mathbb{I}'_i} & \mathbf{XLie} \end{array}$$

4.2. **Universal enveloping crossed module.** Now we construct a left adjoint to $\mathbb{XL}$.

Given a Lie crossed module $\mu : M \to P$, consider Lie homomorphisms $M \rtimes P \underset{t}{\overset{s}{\rightrightarrows}} P$, $s(m,p) = m$, $t(m,p) = \mu(m) + p$. Applying the functor $\mathbb{U}$ and dividing $\mathbb{U}(M \rtimes P)$ by $X = \operatorname{Ker} \mathbb{U}(s) \operatorname{Ker} \mathbb{U}(t) + \operatorname{Ker} \mathbb{U}(t) \operatorname{Ker} \mathbb{U}(s)$, we obtain a diagram of associative algebras

$$\mathbb{U}(M \rtimes P)/X \underset{\overline{\mathbb{U}}(t)}{\overset{\overline{\mathbb{U}}(s)}{\rightrightarrows}} \mathbb{U}(P) \ ,$$

where $\overline{\mathbb{U}}(s)$ and $\overline{\mathbb{U}}(t)$ are induced by $\mathbb{U}(s)$ and $\mathbb{U}(t)$, respectively. Define $\mathbb{XU}(M, P, \mu)$ as the crossed module of associative algebras

$$\mathbb{XU}(M, P, \mu) = \big(\operatorname{Ker} \overline{\mathbb{U}}(s), \mathbb{U}(P), \overline{\mathbb{U}}(t) \mid_{\operatorname{Ker} \overline{\mathbb{U}}(s)} \big).$$

Definition 4.3. Given a Lie crossed module (M, P, μ), the crossed module of associative algebras $\mathbb{XU}(M, P, \mu)$ is called the universal enveloping crossed module of (M, P, μ).

The universal enveloping crossed module construction provides a functor $\mathbb{XU}$: **XLie** $\to$ **XAlg**, which is natural generalization of $\mathbb{U}$: **Lie** $\to$ **Alg**. In particular, we have

Proposition 4.4. *There is a commutative diagram*

$$\begin{array}{ccc} \mathbf{Lie} & \xrightarrow{\mathbb{I}'_0} & \mathbf{XLie} \\ \downarrow{\scriptstyle \mathbb{U}} & & \downarrow{\scriptstyle \mathbb{XU}} \\ \mathbf{Alg} & \xrightarrow{\mathbb{I}_0} & \mathbf{XAlg} \end{array}$$

and a natural isomorphism of functors $\mathbb{I}_1 \circ \mathbb{U} \approx \mathbb{XU} \circ \mathbb{I}'_1$.

4.5. **The adjunction.** The following result is a natural generalization of the well-known classical adjunction between the categories **Lie** and **Alg**.

Theorem 4.6. *The functor* $\mathbb{XU}$ *is left adjoint to the Liezation functor* $\mathbb{XL}$.

Proof. Given a morphism $(\alpha, \beta) \in \mathrm{Hom}_{\mathbf{XLie}}\big((M, P, \mu), \mathbb{XL}(R, A, \rho)\big)$, consider the diagram of Lie algebras

$$\begin{array}{ccc} M \rtimes P & \underset{t}{\overset{s}{\rightrightarrows}} & P \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathbb{L}(R \rtimes A) & \underset{\tau}{\overset{\sigma}{\rightrightarrows}} & \mathbb{L}(A) \ , \end{array}$$

where s, t (similarly σ, τ) are defined as above and $\alpha'(m,p) = (\alpha(m), \beta(p))$. Since $\mathbb{U}$ is left adjoint to $\mathbb{L}$, we easily deduce that there is an induced commutative diagram of algebras

$$\begin{array}{ccc} \mathbb{U}(M \rtimes P) & \xrightarrow{\mathbb{U}(s)} & \mathbb{U}(P) \\ {\scriptstyle \alpha'^{*}}\downarrow & & \downarrow{\scriptstyle \beta^{*}} \\ R \rtimes A & \xrightarrow{\ \sigma\ } & A \ , \end{array}$$

and a similar diagram holds with $\mathbb{U}(s)$ replaced by $\mathbb{U}(t)$ and σ replaced by τ. Since $\operatorname{Ker}\sigma \operatorname{Ker}\tau = \operatorname{Ker}\tau \operatorname{Ker}\sigma = 0$, we have one more commutative diagram of algebras

$$\begin{array}{ccc} \mathbb{U}(M \rtimes P)/X & \underset{\mathbb{U}(t)}{\overset{\mathbb{U}(s)}{\rightrightarrows}} & \mathbb{U}(P) \\ {\scriptstyle \alpha'^{*}}\downarrow & & \downarrow{\scriptstyle \beta^{*}} \\ R \rtimes A & \underset{\tau}{\overset{\sigma}{\rightrightarrows}} & A \ , \end{array}$$

which corresponds to a uniquely defined morphism from $\mathrm{Hom}_{\mathbf{XAlg}}\big(\mathbb{XU}(M, P, \mu), (R, A, \rho)\big)$. The inverse assignment is obvious. □

Corollary 4.7. *There is an adjunction between the categories* **2-SAlg** *and* **2-SLie** *generalizing the classical one between* **Alg** *and* **Lie**.

Acknowledgements

The author was supported by Shota Rustaveli National Science Foundation, grant DI/12/5–103/11 and Volkswagen Foundation, grant 85989.

References

1. J. C. Baez and A. S. Crans, Higher-dimensional algebra VI. Lie 2-algebras. *Theory Appl. Categ.* **12** (2004), 492–538.
2. G. Donadze, N. Inassaridze, E. Khmaladze and M. Ladra, Cyclic homologies of crossed modules of algebras. *Noncomm. Geometry* (accepted in 2011).
3. Y. Fregier and F. Wagemann, On Hopf 2-algebras. *Int. Math. Res. Not. IMRN* **15** (2011), 3471–3501.
4. C. Kassel and J. -L. Loday, Extensions centrales d'algebres de Lie. *Ann. Inst. Fourier* (*Grenoble*) **32** (1982), 119–142.

5. S. Mac Lane, Categories for the working mathematician. 2nd edition, *Springer, Berlin*, 1998.
6. J. H. C. Whitehead, Combinatorial homotopy II. *Bull. Amer. Math. Soc.* **55** (1949), 453–496.

(Received 03.02.2012)

Author's addresses:

A. Razmadze Mathemetical Institute
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia

Departamento de Matemática Aplicada I
Universidad de Vigo, E. E. Forestal, 36005, Pontevedra,
Spain
E-mail: e.khmal@gmail.com