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GLOBAL GEOMETRIC ASPECTS OF LINEAR CONJUGATION PROBLEMS

G. Khimshiashvili UDC 517.925.7

We discuss certain geometric and topological properties of the collection of elliptic linear conjugation problems in the context of the so-called Grassmanian models for loop groups introduced by Bojarski and further elaborated by Pressley and Segal. We perform some considerations in a wider context of Riemann–Hilbert problems. Special attention is paid to the so-called restricted Grassmanian of a polarized Hilbert space; in particular, we show that it has a natural structure of an infinite-dimensional Kähler manifold and can be endowed with the so-called Fredholm structures defined by the families of Riemann–Hilbert problems. Several topics concerned with generalizations of the classical linear conjugation problem are also considered; in particular, the Fredholm theory of generalized linear conjugation problems with coefficients in a compact Lie group and the structure of elliptic linear conjugation problems over a C^*-algebra are discussed in some detail. Certain natural multidimensional generalizations of linear conjugation problems and Riemann–Hilbert problems are considered and a comprehensive description of generalized Cauchy–Riemann systems possessing elliptic Riemann–Hilbert problems is given in terms of representations of Clifford algebras. Various nonlinear aspects of the Riemann–Hilbert transmission problem are also discussed in the framework of analytic discs attached to totally real submanifolds and hyperholomorphic cells attached to admissible targets

Introduction

As is well known, the classical linear conjugation problem (LCP) for holomorphic functions and the closely related Riemann–Hilbert problem (RHP) have deep and far-reaching connections with many important problems in analysis and geometry (see, e.g., [20–22, 60, 112]). Thus, in addition to a comprehensive analytic theory [112], these problems have a number of natural geometric aspects. In particular, as was suggested in [21] (see also [22]), the totality of elliptic linear conjugation problems permits a visual geometric description in terms of Fredholm pairs of subspaces of an appropriate functional space.

This geometric interpretation allowed studying various global aspects of linear conjugation problems in an abstract setting, which has eventually led to some conceptual developments [23, 24, 80, 81] and nontrivial geometric results about the so-called *Fredholm Grassmanian* [21,23,83,85,144]. Closely related concepts and constructions appeared useful in the geometric theory of loop groups of compact Lie groups [81, 120]. The aim of this paper is to present a coherent exposition of these geometric results and discuss some new developments in the same direction.

We start with a brief recollection of basic facts about the classical linear conjugation problems. We emphasize the factorization of matrix-valued functions on the unit circle (the so-called *Birkhoff factorization* [120], which is sometimes also called *Wiener–Hopf factorization* [33]). In particular, we present explicit formulas for partial indices obtained in [5,33] and describe a recent interpretation of factorization theorems in terms of Hermitian vector bundles suggested by Donaldson [42].

We proceed by studying the geometric model of the set of elliptic linear conjugation problems suggested by Bojarski [21]. The related Grassmanians and operator groups are introduced and their topology is studied. In particular, we determine the homotopy type of the Fredholm Grassmanians, show that they can be endowed with a number of interesting geometric structures, and explain how one can put them in the context of the theory of Fredholm structures. We also explain that such Grassmanians can be

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considered in a more general setting of Grassmanians of Banach algebras [45]. Our exposition of these topics basically relies on results of [21, 25, 45, 57, 81, 120], but we also present some new results.

In the next five sections, we describe some recent developments related to geometric aspects of linear conjugation problems and Riemann–Hilbert problems. They arise from natural generalizations of these classical problems in various directions. Those generalizations can be roughly divided into two groups: linear (Secs. 2–4) and nonlinear generalizations (Secs. 5, 6). Our exposition of these topics is based on the results of [81, 83, 90, 133, 142].

In Sec. 2, we describe the Fredholm theory of the so-called *generalized linear conjugation problem with coefficients in a compact Lie group*, which was studied in the author's papers [81, 83]. Our derivation of the main results is based on the generalized Birkhoff factorization for regular loops on a compact Lie group developed in [120].

It should be noted that most of the results of Sec. 2 can be derived from the classical Fredholm theory by realizing the group of coefficients as a matrix group. However, our invariant approach suggests a more flexible setting and reveals some new aspects of the topic, which do not arise in the classical setting. For example, in this way, one obtains a natural way of constructing Fredholm structures on loop groups and Birkhoff strata, which suggests a number of nontrivial questions and perspectives. In should be noted that similar problems have been treated in recent papers of Freed [57, 58] and Misiolek [109, 110] by substantially different methods.

Another type of generalization of the linear conjugation problem is considered in Sec. 3. This generalization was suggested by the present author in the framework of Hilbert modules over C^*-algebras [83, 85].

It turns out that one can determine the homotopy type of the related Grassmanian in terms of the K-groups of the basic algebra. This result enables one to obtain natural invariants of families of elliptic Riemann–Hilbert problems and extends some aspects of the Birkhoff factorization in the context of C^*-algebras. Its proof requires a considerable portion of the theory of Hilbert modules over C^*-algebras, which was founded in the seminal papers of Fomenko and Mishchenko [108] and Kasparov [74].

Most of the technical results needed for our purposes can be found in the papers of Mishchenko and his school [102, 107, 138]. Closely related results on the structure of Grassmanians over C^*-algebras can be found in recent papers [45, 152], but those authors did not consider relations to Riemann–Hilbert problems.

The general theory of boundary-value problems for elliptic systems suggests a natural formulation of local boundary-value problem for a wide class of first-order systems with constant coefficients [18, 143], which specializes to the Riemann–Hilbert problem in the case of the usual Cauchy–Riemann system. Thus, it is natural to refer to those multidimensional boundary-value problems as *Riemann–Hilbert problems for first-order systems*. As was revealed by recent developments in Clifford analysis [32], this analogy is quite far-reaching in the case of the so-called *generalized Cauchy–Riemann systems* introduced by Stein and Weiss [132], in particular, for systems associated with Euclidean Dirac operators.

A version of Fredholm theory can be developed for multidimensional Riemann–Hilbert problems which satisfy the Shapiro–Lopatinsky condition [133]. Some effective methods of verifying this condition were developed by Stern [133, 134]. This topic suggested several nontrivial problems and it has gained considerable attention in recent decades [10, 133, 143]. Note that the problem of describing generalized Cauchy–Riemann systems possessing elliptic local boundary-value problems appeared sufficiently difficult and remained unsolved for a long time [133]. Its solution eventually became possible [85] due to the recent advances in K-theory [11, 68].

In Sec. 4, we describe some results of Stern [133, 134] and of the author [85, 86], which contribute to the Fredholm theory of multidimensional Riemann–Hilbert problems. In particular, following [133], we derive explicit criteria of Fredholmness and present a comprehensive list of generalized Cauchy–Riemann systems possessing elliptic Riemann–Hilbert problems. This list extends the list presented in [134]; it was obtained by the author using some recent advances in operator K-theory [11, 68]. These results seem interesting in their own right and provide a background for our approach to nonlinear Riemann–Hilbert problems for Cauchy–Riemann systems described in the last section.

A natural nonlinear generalization of linear conjugation problems is the so-called *nonlinear transmission problem* [141]. The theory of nonlinear transmission problems and nonlinear Riemann–Hilbert problems for holomorphic functions is at present a well-developed topic of complex analysis [127,141]. It is naturally connected with the so-called *analytic discs attached to totally real submanifolds* [17,55] and nonlinear singular integral equations [141]. It is worth noting that attached analytic and pseudo-analytic discs play an important role in Gromov's approach to some difficult problems of symplectic geometry [69].

A good understanding of the structure of solutions to nonlinear Riemann–Hilbert problems is important in many aspects of this topic. Especially interesting are the cases where a *target manifold* is globally foliated by the boundaries of attached analytic discs. In Sec. 5, we describe a class of nonlinear transmission problems which possess this property. This class was investigated in [90,142], and it seems to provide a reasonable starting point for investigating multidimensional nonlinear Riemann–Hilbert problems.

To our knowledge, only a few papers devoted to multidimensional nonlinear Riemann–Hilbert problems exist, mainly in the case where the nonlinearity enters through a small perturbation of a linear problem [12,146]. It seems natural to ask if it is possible to apply the paradigm of analytic discs attached to totally real submanifolds in a multidimensional setting. This leads to considering *hyperholomorphic cells* with boundaries in a given submanifold.

Obviously, in order that such objects become tractable, it is necessary to choose a submanifold (target manifold) in an appropriate way. A natural approach to this problem based on our results for linear conjugation problems is described in Sec. 6. We can indicate a class of targets which give rise to certain nonlinear Fredholm operators describing the local structure of attached hyperholomorphic cells. A number of seemingly interesting open problems may be formulated in this setting, and we briefly discuss some of them at the end of the section.

In a single paper, it is of course impossible to present a comprehensive exposition of all those topics we touched upon. Additional information on these and other geometric aspects of linear conjugation problems and Riemann–Hilbert problems can be found, e.g., in [7,24,26,27,36,37,41,57,58,63,88,100,141].

The author benefitted a lot from discussions of various aspects of the topic with a number of experts and colleagues. Especially extensive and useful were discussions with Bojarski, who initiated many ideas and concepts of the geometric approach to linear conjugation problems. Some of the results on the global geometric structure of linear conjugation problems and Fredholm Grassmanians were obtained jointly with Bojarski [25]. The results of Sec. 5 were obtained jointly with Wegert and Spitkovsky [142]; the author had numerous discussions with them about various aspects of Birkhoff factorization and nonlinear analysis.

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1. Linear Conjugation Problems and Fredholm Grassmanians

The classical formulation of the linear conjugation problem is related to the decomposition of the extended complex plane $\overline{\mathbb{C}}$ (Riemann sphere) into two complementary domains with a smooth common boundary Γ. In the simplest and the most classical case, one just takes the decomposition

$$\overline{\mathbb{C}} = D_+ \cup \mathbb{T} \cup D_-,$$

where D_+ is the unit disc, $\mathbb{T}$ is the unit circle, and D_- is the complementary domain containing the infinite point ∞ (the north pole of the Riemann sphere). Let $A(D_\pm) = C(\overline{D}_\pm) \cap H(D_\pm)$ denote the set of all complex-valued functions which are continuous in the closure of the corresponding domain and holomorphic inside. The set of vector-valued functions of length $n \geq 1$ with all their components in $A(D_\pm)$ is denoted by $A^n(D_\pm)$.

A fundamental problem of complex analysis, known as the linear conjugation problem or the Riemann–Hilbert transmission problem, is to describe the totality of piecewise-holomorphic (vector-valued) functions $(X_+, X_-) \in A^n(D_+) \times A^n(D_-)$, with the normalizing condition $X_-(\infty) = 0$, such that their boundary values on $\mathbb{T}$ satisfy the transmission (or linear conjugation) condition

$$X_+(t) = G(t)X_-(t) + h(t), \quad \forall t \in \mathbb{T}, \tag{1.1}$$

where $h(t)$ is a given (vector-valued) function and $G(t)$ is a given continuous $(n\times n)$-matrix-valued function on T.

Of course, the same problem can be formulated on any Riemann surface, but we stick here to the zero genus case (Riemann sphere). Solutions may be considered in various functional spaces. For example, the problem can be placed in a Hilbert-space context by working with square-integrable functions, and this is well-suited for studying global geometric aspects of the problem.

Solvability and other properties of this problem are very well understood in various classes of functional spaces (see, e.g., [112]). For example, the problem is a Fredholm problem in appropriate L^2-spaces if the coefficient matrix $G(t)$ is nondegenerate at every point of the unit circle and belongs to some Hölder class. The index of this problem appears to be equal to the winding number (topological degree) of the determinant $\det G(t)$, i.e., it is equal to the increment of the argument of $\det G(t)$ along the unit circle [112] divided by 2π.

One can also express the kernel and cokernel dimension in terms of the so-called *partial indices* of the matrix-valued function $G(t)$. Those are defined in terms of *Birkhoff factorization* of nondegenerate matrix-valued functions on the circle [20, 112].

The famous *Birkhoff factorization theorem* [112, 120] states that a sufficiently regular nondegenerate matrix-valued function on $\mathbb{T}$ can be represented in the form

$$G(t) = G_+(t)\,\mathrm{diag}(z^{\mathbf{k}})G_-(t), \tag{1.2}$$

where matrix-valued functions $G_\pm(t)$ are regular, nondegenerate, and holomorphic in domains $D_\pm$ respectively, $G_-(\infty)$ is the identity matrix, and

$$\mathrm{diag}(z^{\mathbf{k}}) = \mathrm{diag}(z^{k_1}, \dots, z^{k_n}), \quad k_1, \dots, k_n \in \mathbb{Z},$$

is a diagonal matrix-valued function on $\mathbb{T}$ [112, 140].

Integers k_i are called (left) partial indices [112, 140] (or exponents [40, 120]) of the matrix-valued function $G(t)$. For a given matrix-valued function $G(t)$, there can exist different factorizations of the form (1.2) but (left) partial indices are uniquely defined up to the order [140]. Similarly, one can define the right Birkhoff factorization of $G(t)$ and right partial indices. We will only deal with left factorizations since they are well-suited for investigation of linear conjugation problems of the form (1.1).

Partial indices exhibit quite nontrivial behavior. Right partial indices need not be equal to the left indices. However, for sufficiently regular (rational, Hölder) matrix-valued functions, the sum of all left partial indices (*left total index*) is equal to the *right total index*, which is defined similarly. Actually, both the left and right total index are equal to the Fredholm index of the corresponding linear conjugation problem 1.1.

In fact, even for very regular (smooth, rational) matrix-valued functions, their collections of left and right partial indices are practically independent of each other (except the restriction that both total indices should be equal). For example, it was proved in [52] that for each two integer vectors k and l such that $\sum k_i = \sum l_i$, there exists a nondegenerate rational matrix-valued function on the unit circle whose vectors of left and right partial indices are k and l respectively.

At the same time, if the algebraic degrees of the numerator and denominator of each element of a rational matrix-valued function are bounded by a fixed integer N, then one can obtain upper estimates for the modulus of the left and right indices [63]; thus, in this case differences between right and left indices cannot be arbitrary. If one drops regularity requirements and considers almost everywhere nondegenerate matrix-valued functions with bounded measurable coefficients, then each pair of integer vectors can serve as collections of left and right partial indices of such a matrix-valued function [31].

Partial indices are closely related to the properties of holomorphic vector bundles over the Riemann sphere [20, 22, 62]. The problem of calculating (left or right) partial indices of a concrete matrix-valued function is far from trivial since in most cases, they are not topological invariants and one has to take into account the analytic properties of a given matrix-valued function. After several decades of gradual progress, this problem was eventually solved for several important classes of matrix-valued functions [33, 100].

Recently, these results were simplified and generalized in [5]. We reproduce here some results of [5] since they allow effectively calculating partial indices in most of the situations appearing in practice. Thus, the problem of calculating partial indices at present can be considered as an algorithmically solvable problem. Moreover, these results, in addition to their importance and instructive matter in themselves, suggest further interesting problems, some of which are described below.

More precisely, following [5], we present formulas with the aid of which one can calculate the left and right partial indices of continuous matrix-valued functions (or, equivalently, *matrix loops*) and relate them to the splitting type of the corresponding holomorphic vector bundle.

Let Γ be a smooth simple oriented loop in $\mathbb{C}P^1$, which divides $\mathbb{C}P^1$ into two connected domains U_+ and U_-. Assume that $0 \in U_+$ and $\infty \in U_-$. In this situation, one can investigate the linear conjugation problem with a coefficient $f : \Gamma \to GL_n(\mathbb{C})$ of the Hölder class.

To investigate the solvability of such a problem, one needs some effective methods of finding partial indices of the coefficient matrix. Now we describe an algorithm based on results of [5, 33].

Let $f : \varGamma \to \mathrm{GL}(n, \mathbb{C})$ be a continuous and invertible matrix-valued function on the contour $\varGamma$. Its partial indices can be found by using the so-called *power moments* of f [33].

Definition 1.1 ([5, 33])**.** A power moment of a matrix-valued function $f(t)$ with respect to contour $\varGamma$ is the matrix

$$c_j = \frac{1}{2\pi i} \int\limits_{\varGamma} t^{-j-1} f^{-1}(t) dt, \qquad j \in \mathbb{Z}.$$

Let $k = \operatorname{ind} f(t)$; we consider the family of block Toeplitz matrices T_{-l} of the form

$$\begin{pmatrix} c_l & c_{l-1} & \cdots & c_{-2k} \\ c_{l+1} & c_l & \cdots & c_{-2k+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_0 & c_{-1} & \cdots & c_{-2k-l} \end{pmatrix}, \qquad -2k \le l \le 0.$$

Theorem 1.1 ([5, 33])**.** *The left and right partial indices of the matrix-valued function $f(t)$ are given by the formulas*

$$k_j^{\mathrm{r}} = \operatorname{card} \{ l \mid n + r_{-l-1} - r_{-l} \le j - 1, \ l = 2k, 2k-1, \dots 0 \} - 1,$$
$$k_j^{\mathrm{l}} = 2k + 1 - \operatorname{card} \{ l \ \mid \ r_{-l-1} - r_{-l} \le j - 1, \ l = 2k, 2k-1, \dots, 0 \},$$

where $j = 1, \dots, n$, r_l is the rank of the Toeplitz matrix T_{-l}, and it is assumed that $r - 2l - 1 = 0$ and $l = 2k, 2k-1, \dots, 0$.

Note that if a matrix-valued function $f(t)$ admits analytic continuation to U_+ and has in U_+ p poles $z_1, \dots, z_p$ of multiplicities $\kappa_1, \dots, \kappa_p$, then the matrix-valued function $\tilde{f}(t) = \prod\limits_{j=1}^{p} (z - z_j)^{r_j} f(t)$ is analytic and

$$\operatorname{ind}_{\varGamma} \det \tilde{f}(t) = \operatorname{ind}_{\varGamma} \det f(t) + (\kappa_1 + \kappa_2 + \ldots + \kappa_p) n.$$

For such a matrix-valued function, its left and right partial indices k_j^{l}, k_j^{r}, $j = 1, \dots, n$, can be expressed by the formulas

$$k_j^{\mathrm{r}} = \operatorname{card}\{l \mid n + r_{-l-1} - r_{-l} \le j-1,\ l = 2k, 2k-1, \dots 0\} - \kappa - 1,$$
$$k_j^{\mathrm{l}} = 2(\kappa + n\kappa)k - \kappa + 1 - \operatorname{card}\{l \mid r_{-l-1} - r_{-l} \le j-1,\ l = 2(\kappa + n\kappa), \dots, 0\},$$

where $\kappa = \kappa_1 + \ldots + \kappa_p$ is the total multiplicity of poles of $f(t)$.

Using these results, one can effectively obtain information about the solvability of a given RHP. For this, set

$$\alpha = \sum_{k_j^{\mathrm{r}} < 0} |k_j^{\mathrm{r}}| \quad \text{and} \quad \beta = \sum_{k_j^{\mathrm{r}} > 0} |k_j^{\mathrm{r}}|.$$

As is well known, the numbers α and β are the dimensions of the kernel and cokernel of the Fredholm operator of a given Riemann–Hilbert problem (RHP). It turns out that one can calculate both these dimensions in terms of matrices T_{-k}.

Proposition 1.1 ([5]). *Assume that among the partial indices of the meromorphic function $f(t)$, there are positive as well as negative numbers. Then*

$$\alpha = (\kappa + 1)n - r_{\kappa - 2k}, \quad \beta = k + n - r_{\kappa - 2k}.$$

Obviously, these results allow one to formulate sufficient conditions for solvability and stability of a given RHP and some estimates for the partial indices on various classes of rational matrix-valued functions [5, 33]. It is also worth noting that for (2×2)- and (3×3)-matrices, the partial indices can be calculated in terms of invariants of the corresponding holomorphic vector bundle over the Riemann sphere [62].

Thus, partial indices of a matrix-valued function are sufficiently well understood and their calculation is merely an algorithmic task, so we will not further dwell upon this topic. In the spirit of our geometric approach, it is natural to investigate the properties of a natural decomposition of the loop group defined by partial indices.

For $K = (k_1, k_2, \dots, k_n)$, denote by Ω_K the collection of all matrix-valued functions with the (unordered) set K of partial indices. Ω_K is called the *Birkhoff stratum* of type K. We will usually consider them in the groups of based Hölder loops on $GL(n, \mathbb{C})$.

Let Ω_+ and Ω_- denote the subgroups consisting of boundary values of matrix-valued functions holomorphic in U_+ and U_- respectively (in the latter case, we require that a matrix-valued function be regular at infinity and tend to the identity matrix at infinity). The Banach Lie group $\Omega^+ \times \Omega^-$ acts analytically on Ω by the rule

$$f \xmapsto{\alpha} h_1 f h_2^{-1}, \quad f \in \Omega, \quad h_1 \in \Omega^+, \quad h_2 \in \Omega^-.$$

It is clear that the orbit of the diagonal matrix d_K by the action α is Ω_K.

In [40], it was proved that the stability subgroup H_K of f under the action α consists of pairs (h_1, h_2) of upper-triangular, matrix-valued functions such that the (i, j)th entry in h_1 is a polynomial in z of degree at most $(k_1 - k_2)$ and $f = h_1 f h_2^{-1}$. Hence, the subgroup H_K has the finite dimension

$$\dim H_K = \sum_{k_i \ge k_j} (k_i - k_j + 1).$$

Now choose any pair $(h_1, h_2) \in H_K$ and consider the holomorphic vector bundle on $\mathbb{C}P^1$ obtained by the covering of the Riemann sphere $\mathbb{C}P^1$ by three open sets $\{U^+, U^-, U_3 = \mathbb{C}P^1 \backslash \{0, \infty\}\}$ with transition functions

$$g_{13} = h_1 : U^+ \cap U_3 \to GL(n, \mathbb{C}), \qquad g_{23} = h_2 d_K : U^- \cap U_3 \to GL(n, \mathbb{C}).$$

Denote this bundle by $E \to \mathbb{C}P^1$. The Birkhoff factorization theorem implies that every holomorphic vector bundle splits into a direct sum of line bundles:

$$E \cong E(k_1) \oplus \ldots \oplus E(k_n).$$

Remark 1.1. The possibility of decomposing a holomorphic vector bundle into the direct sum of line bundles was proved by Grothendieck without applying the Birkhoff theorem; hence, there exist two independent proofs of this important fact.

The numbers $k_1, \dots, k_n$ are the Chern numbers of the line bundles $E(k_1), \dots, E(k_n)$. We order them such that $k_1 \geq \dots \geq k_n$. The integer-valued vector $K = (k_1, \dots, k_n) \in \mathbb{Z}^n$ is called the *splitting type* of the holomorphic vector bundle E. It completely defines the holomorphic type of the bundle E.

Taking into account the relations between partial indices $\kappa_1, \dots, \kappa_n$ of the matrix-valued function $f \in \Omega$ and the splitting type of holomorphic vector bundle E, it is easy to see that Birkhoff strata Ω_K numerate holomorphy types of vector bundles over $\mathbb{C}P^1$.

Theorem 1.2 ([20, 66])**.** *There is a one-to-one correspondence between the strata Ω_K and the isomorphism classes of holomorphic vector bundles on $\mathbb{C}P^1$.*

Denote by $O(E)$ the sheaf of germs of holomorphic sections of the bundle E; then the solutions of the linear conjugation problem can be interpreted as elements of the zeroth cohomology group $H^0(\mathbb{C}P^1, O(E))$. Therefore, the number l of the linearly independent solutions equals $\dim H^0(\mathbb{C}P^1, O(E))$. Moreover, the (total) Chern number $c_1(E)$ of the bundle E is equal to index $\det G(t)$. In particular, one obtains a well-known criterion of solvability of a linear conjugation problem.

Theorem 1.3 ([123])**.** *A linear conjugation problem has solutions if and only if $c_1(E) \geq 0$ and the number l of linearly independent solutions is*

$$l = \dim H^0(\mathbb{C}P^1, O(E)).$$

Note that, besides being interesting in themselves, the aforementioned explicit formulas for the partial indices suggest further perspectives and problems. For example, consider the set R_N of nondegenerate rational matrix-valued functions on $\mathbf{T}$ with the degrees of the numerator and denominator bounded from above by a certain number N. The explicit formulas for the partial indices imply that the range of the vector of right partial indices of matrices in R_N is finite (cf. [63]). One can wonder what is the maximal possible "distance" between the vectors of right and left partial indices for matrix-valued functions from R_N. Some estimates of such type were obtained in [63].

Furthermore, one may ask what can be the topological types of the intersections $\Omega_K \cap R_N$. For small N, they should admit a complete description. On the other hand, for N large enough, it is natural to conjecture that the topological type should be the same as for Ω_K itself. To the author's knowledge, both these issues remain uninvestigated.

A very interesting result closely related to factorization of matrix-valued functions was recently obtained by Donaldson [42]. The approach used by Donaldson is quite remarkable and seemingly may indicate a new paradigm for studying boundary-value problems in higher dimensions; we reproduce below the setting and main result of [42].

Let E be a holomorphic vector bundle over a complex manifold M with a Hermitian metric h on E. As is well known, there exists a preferred unitary connection induced on E. Taking a local holomorphic trivialization of E by section s_i, one can represent h by a Hermitian matrix $H_{ij} = (s_i, s_j)_h$. Then the connection matrix in this trivialization is $H^{-1}\partial H$, and its curvature is

$$F_H = \overline{\partial}(H^{-1}\partial H) = H^{-1}(\overline{\partial}\partial H - \overline{\partial} H H^{-1}\partial H).$$

Moreover, assume that M has a fixed Kähler metric with the Kähler form ω. Let $\Lambda : \Omega^{1,1}_M \to \Omega^0_M$ be the contraction $\Lambda(\theta) = (\omega, \theta)$. Then the Hermitian Yang–Mills (HYM) equation for the metric H takes the form $\imath\Lambda F_H = 0$. Its solutions are called Hermitian Yang–Mills metrics. If M is (complex) one-dimensional, then Λ is an isomorphism and the HYM connections are just the flat (zero-curvature) connections.

Donaldson assumed that M is the interior of a compact manifold $\overline{M}$ with nonempty boundary ∂M and the Kähler metric ω is smooth and nondegenerate on the boundary. It is also assumed that a holomorphic bundle E extends to the boundary; in this case, we say that E is a holomorphic bundle over $\overline{M}$. The main result of [42] can be considered as a solution of the Dirichlet problem for Hermitian Yang–Mills equations. Namely, for any Hermitian metric f on the restriction of E to ∂M, there exists a unique metric h on E such that $h = f$ on ∂M and $\imath\Lambda F_H = 0$ in M.

This result is already nontrivial if M is the unit disc in $\mathbb{C}$ and E is topologically trivial over its boundary $\mathbb{T}$. As is explained in [42], in this case the result implies a factorization theorem for loops on the unitary group $U(n)$, which is closely related to the Birkhoff factorization. It is also possible to obtain a generalization of this result for holomorphic principal bundles considered in the next section.

From the preceding discussion, it is clear that properties of a concrete linear conjugation problem can be studied by well-developed methods, and we will not further discuss those classical topics in this paper. As was realized relatively recently, the set of all elliptic linear conjugation problems possesses interesting geometric properties and permits nontrivial geometric descriptions in the framework of global analysis (see, e.g., [21, 22, 120]). The same holds for certain natural subsets of this set, e.g., for Birkhoff strata [22, 40].

In this way, there emerged several interesting geometric settings and approaches, which will be our main concern in the sequel. We begin by recalling an abstract geometric model for the set of elliptic linear conjugation problems suggested in [21]. Following an established tradition, we say that a linear conjugation problem is elliptic if it is described by a Fredholm operator (i.e., an operator T with a closed image and finite-dimensional kernel and cokernel [47]). The index of a linear conjugation problem is defined as the index of the corresponding Fredholm operator [47, 112], i.e., the difference between the dimension of the kernel and the codimension of the image:

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T.$$

For simplicity and brevity, we choose the framework of operators acting in Hilbert spaces. However, most of our constructions and results remain valid for a wide class of Banach spaces. As will be shown in the sequel, they can also be generalized in the context of Hilbert modules over C^*-algebras [85].

Let H be a complex Hilbert space and M and N be its closed infinite-dimensional subspaces.

Definition 1.2 ([21, 75]). A pair $\mathbf{P} = (M, N)$ is called the Fredholm pair (FP) if $M + N$ is a closed subspace of finite codimension $b_{\mathbf{P}}$ and $\dim(M \cap N) = a_{\mathbf{P}}$ is also finite. In this case, the difference $a_{\mathbf{P}} - b_{\mathbf{P}} = i(M, N)$ is called the index of the Fredholm pair $\mathbf{P}$.

The concept of Fredholm pair was introduced in 1960's by Kato [75], who established, in particular, that such pairs and their indices are stable with respect to continuous deformations of the subspaces in question. For a precise formulation of this property, see [21, 75].

In order to characterize Fredholm pairs, certain classes of bounded linear operators in H were introduced in [21]. Let $L(H)$ denote the algebra of bounded linear operators in H and $GL(H)$ denote the group of operators possessing a bounded inverse. Let J be a fixed two-sided ideal in $L(H)$. For example, one can take the (unique, closed, two-sided) ideal K of compact (completely continuous) operators or the subideal K_0 consisting of finite-rank operators.

For a given operator $S \in L(H)$, let $C(S, J)$ denote the subalgebra of operators $A \in L(H)$ such that the commutator $[A, S] = AS - SA$ belongs to the ideal J. The intersection $C(S, J) \cap GL(H)$ is denoted by $GL(S, J)$; clearly, it is a subgroup of $GL(H)$ (not necessarily closed).

As was explained in [21], the classical singular integral operators and linear conjugation problems can be interpreted as elements of the algebra $C(P, K)$, where P is an orthogonal projector with infinite-dimensional image and kernel. Many topological properties of such operators and related Grassmanians remain valid if one changes the ideal K by certain subideal J as above.

Definition 1.3 ([21]). Let P be an orthogonal projection on a closed subspace in H such that $\dim \operatorname{im} P = \dim \ker P = \infty$. The algebra $C(P, J)$ is called the algebra of abstract singular operators associated with ideal J, $K_0 \subset J \subset K$.

In the sequel, we will be mainly interested in the group of invertible (abstract) singular operators $GL(P, J)$. If M is a closed linear subspace of H and $A \in GL(H)$ an invertible operator in H, then $A(M)$ denotes the image of M under A and we think of it as a subspace M rotated by A. Let P_M denote any projection onto M, i.e., the range of P_M is M and $(P_M)^2 = P_M$. Of course, there exist many projectors with the given range M. In a Hilbert space, the condition that P_M is self-adjoint (or orthogonal) specifies it uniquely, but we do not assume that P_M is orthogonal. We consider the complementary projections $P = P_M$ and $Q = \mathrm{Id} - P$ and present a useful characterization of Fredholm pairs which was obtained in [21].

Theorem 1.4 ([21])**.** *A pair (M, N) of closed subspaces of a Hilbert space is a Fredholm pair if and only if it has the form $(M, A(\ker P))$ for some projection P as above and some operator $A \in G(P, K)$. The operator $\Phi \in L(H)$ defined by the formula*

$$\Phi(x) = Px + AQx \tag{1.3}$$

is a Fredholm operator with $\operatorname{ind}\Phi = i(M, N)$*. Any operator of this form in $L(H)$ is a Fredholm operator.*

It turns out that in many problems, it becomes necessary to consider the set of all Fredholm pairs with a fixed first subspace. In other words, one chooses a closed infinite-dimensional and infinite-codimensional subspace M and considers the so-called Fredholm Grassmanian consisting of all subspaces N such that (M, N) is a Fredholm pair (cf. [120, Chap. 7]). This is actually just a "leaf" in the Grassmanian of all Fredholm pairs, and one can represent the whole Grassmanian as a fibration with the fiber isomorphic to this leaf.

This definition permits several useful modifications, which we present following [120]. Consider a complex Hilbert space decomposed into the orthogonal direct sum $H = H_+ \oplus H_-$ and choose a positive number s. For further use, we need a family of subideals in $K(H)$, which is defined as follows (cf. [57]).

Recall that for any bounded operator $A \in L(H)$, the product A^*A is a nonnegative self-adjoint operator; therefore, it has a well-defined square root $|A| = (A^*A)^{1/2}$ (see, e.g., [128]). If A is compact, then A^*A is also compact and $|A|$ has a discrete sequence of eigenvalues

$$\mu_1(A) \geq \mu_2(A) \geq \cdots$$

tending to zero. $\mu_n(A)$ are called singular values of A. For finite $s \geq 1$, one can consider the expression (sth norm of A)

$$\|A\|_s = \left[\sum_{j=1}^{\infty} (\mu_j(A))^s\right]^{1/s} \tag{1.4}$$

and define the sth Schatten ideal K_s as the collection of all compact operators A with finite sth norm (s-summable operators) [128].

Using elementary inequalities, it is easy to verify that K_s is really a two-sided ideal in $L(H)$. These ideals are not closed in $L(H)$ with its usual norm topology, but if one endows K_s with the sth norm as above, then K_s becomes a Banach space [128]. Two special cases are well-known: K_1 is the ideal of trace class operators and K_2 is the ideal of Hilbert–Schmidt operators. For $s = 2$, the above norm is called the Hilbert–Schmidt norm of A and it is well known that $K_2(H)$ endowed with this norm becomes a Hilbert space (see, e.g., [128]). Obviously, $K_1 \subset K_s \subset K_r$ for $1 < s < r$, and we obtain a chain of ideals starting with K_1. For convenience, we set $K_\infty = K$ and obtain an increasing chain of ideals K_s with $s \in [1, \infty]$.

Of course, one can introduce similar definitions for linear operators acting between two different Hilbert spaces, e.g., for operators from one subspace M to another subspace N of a fixed Hilbert space H. In particular, we can consider the classes $K_s(H_\pm, H_\mp)$. Let us also denote by $F(M, N)$ the space of all Fredholm operators from M to N.

Definition 1.4 ([120])**.** The sth Fredholm Grassmanian of a polarized Hilbert space H is defined as follows:

$$\mathrm{Gr}_F^s(H) = \{W \subset H : \pi_+|W \text{ is an operator from } F(W, H_+),$$
$$\pi_-|W \text{ is an operator from } K_s(W, H_-)\}.$$

These Grassmanians are of major interest to us. Actually, many of their topological properties (e.g., the homotopy type) are independent of the number s appearing in the definition. On the other hand, more subtle properties like manifold structures and characteristic classes of Gr_F^s depend on s in a quite essential way. As follows from the discussion in [57], this is a delicate issue and we circumvent it by properly choosing the context.

As follows from the results of [120], it is especially convenient to work with the Grassmanian $\mathrm{Gr}_F^2(H)$ defined by the condition that the second projection π_- restricted to W is a Hilbert–Schmidt operator. Following [120], we denote it by $\mathrm{Gr}_r(H)$ and call it the *restricted Grassmanian* of H.

Fredholm Grassmanians appear to have interesting analytic and topological properties. It turns out that Grassmanian Gr_F^s can be turned into a Banach manifold modeled on Schatten ideal K_s. In particular, $\mathrm{Gr}_r(H)$ has the natural structure of a Hilbert manifold modeled on the Hilbert space $K_2(H)$ [120]. All these Grassmanians have the same homotopy type (see Theorem 1.5 below). Moreover, certain natural subsets of Grassmanians Gr_F^s can be endowed with so-called Fredholm structures [47], which suggests in particular that one can define various global topological invariants of $\mathrm{Gr}_F^s(H)$.

Definition 1.3 also yields a family of subgroups $GL^s = GL(\pi_+, K_s)$ of $GL(\pi_+, K)$ $(s \geq 1)$. For our purposes, the subgroup $GL(\pi_+, K_2)$ acting on $\mathrm{Gr}_r(H)$ is especially important.

Definition 1.5 ([120])**.** The restricted linear group $GL_r(H)$ is defined as the subgroup of $GL(\pi_+, K)$ consisting of all operators A such that the commutator $[A, \pi_+]$ belongs to the Hilbert–Schmidt class $K_2(H)$.

From the very definition, it follows that GL^s acts on Gr^s and by merely an examination of the proof of Theorem 1.4 given in [21] (cf. also [120, Chap. 7]) one finds out that these actions are transitive. In order to give the most convenient description of the isotropy subgroups of these actions, we follow the presentation of [120] and introduce the subgroup $U^s(H) = U(H) \cap GL^s(H)$ consisting of all unitary operators in GL^s. For $s = 2$, this subgroup is denoted by U_r. Now the description of isotropy groups is available by the same way of reasoning as was applied in [120] for $s = 2$.

Proposition 1.2. *The subgroup $U^s(H)$ acts transitively on $\mathrm{Gr}^s(H)$ and the isotropy subgroup of the subspace H_+ is isomorphic to $U(H_+) \times U(H_-)$.*

The existence of a polar decomposition for a bounded operator on H implies that the subgroup $U^s(H)$ is a retract of GL^s and it is straightforward to obtain similar conclusions for the actions of GL^s.

Corollary 1.1. *The group GL^s acts transitively on the Grassmanian $\mathrm{Gr}^s(H)$ and the isotropy groups of this action are contractible.*

Thus, such an action obviously defines a fibration with contractible fibers, and it is well known that for such fibrations, the total space (GL^s) and the base (Gr^s) are homotopy equivalent [47].

Corollary 1.2. *For any $s \geq 1$, the Grassmanian Gr^s and the group GL^s have the same homotopy type. In particular, GL_r is homotopy equivalent to Gr_r.*

Remark 1.2. As we will see in the next section, all the groups $GL(\pi_+, J)$ have the same homotopy type for any ideal J between K_0 and K. In particular, this is true for every Schatten ideal K_s. Thus, all above groups and Grassmanians have the same homotopy type.

Now we are ready to have a closer look at the topology of Gr^s and GL^s, which will be our main concern in the rest of this section.

The homotopy type of GL_r and Gr_r is described in the following statement, which was obtained in [81, 120, 144]. This gives the answer to a question posed in [21]. The proof presented below follows [120].

Theorem 1.5. *For any $s \in [1, \infty]$, the homotopy groups of the group GL^s and Fredholm Grassmanian Gr^s are given by the formulas*

$$\pi_0 \cong \mathbb{Z}; \quad \pi_{2k+1} \cong \mathbb{Z}, \quad \pi_{2k+2} = 0, \quad k \geq 0. \tag{1.5}$$

Proof. By virtue of Corollary 1.2, it is sufficient to determine the homotopy type of GL^s, which we denote simply by G. For this, let us consider a fibration

$$p_1 : GL^s \to F(H_+, H_+) \times K_s(H_+, H_-)$$

defined in the following way.

Write any element (operator) $A \in GL^s$ as a (2×2)-matrix of operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to the given polarization of H (thus, a is a bounded operator from to H_+ to H_+, etc.).

Then define $p_1(A)$ as the first column of this matrix, i.e., $p_1(A) = (a, c)$. It is obvious that the image $X = \Im p_1$ is an open subset of the target space. Now introduce a subgroup $G_1 \subset G$ of elements of G defined by upper-triangular matrices of the form

$$\begin{pmatrix} I_+ & b \\ 0 & d \end{pmatrix},$$

where I_+ denotes the identity operator on H_+.

Lemma 1.1. *The subgroup G_1 is contractible.*

Proof. Note first that in this representation, the operator d is always invertible; in other words, the set of possible d's appearing in the last formula is exactly $GL(H_-)$. The operator b can be an arbitrary operator in $K_s(H_-, H_+)$. Thus, the subgroup G_1 as a topological space is homeomorphic to the product $GL(H_-) \times K_s(H_-, H_+)$. By the Kuiper theorem [93], the first factor is contractible and the second factor, being a vector space, is also contractible. Thus, we conclude that G_1 is contractible. ☐

Now it is straightforward to verify the following assertion.

Lemma 1.2. *$p_1(A) = p_1(A')$ if and only if there exists a $T \in G_1$ such that $A = A'T$.*

Thus, we conclude that X is the homogeneous space G/G_1, which is obviously a fibration with fibers isomorphic to G_1. As was already explained in the previous section, this implies that G is homotopy equivalent to X.

Now consider the mapping $\pi_1 : X \to F(H_+, H_+)$ defined as the restriction of the first projection, i.e., $\pi_1(a, c) = a$. We want to show that this is also a surjective mapping with contractible fibers. Then, by the same reasoning as above, we are able to conclude that G is homotopy equivalent to $F(H_+, H_+)$. Since it is well known that the homotopy groups of the latter space are exactly those as were given in the statement of the theorem, this would complete the proof.

Thus, we see that it remains to verify the following two lemmas.

Lemma 1.3. *Each $a \in F(H_+, H_+)$ can appear as an upper left corner element of a (2×2)-matrix above.*

Lemma 1.4. *For each $a \in F(H_+, H_+)$, the set of all c's such that $(a, c)^*$ can appear as the first column of a matrix representing an element of G coincides with the set of all $c \in K_s(H_+, H_-)$ such that $c|\ker a$ is injective. The set of all such c is a contractible subset in $K_s(H_+, H_-)$.*

The first of these two lemmas follows from a well-known procedure of regularizing of a Fredholm operator. One takes any embedding c of $\ker a$ into H_- and takes b to be a finite rank operator from H_- onto $(\operatorname{im} a)^{\perp}$. Then one can obtain an appropriate d by taking any epimorphism of H_- onto $\ker b$ with the kernel $\operatorname{im} c$. It can be easily verified that this really defines an operator in $GL(\pi_+, K_0)$; therefore, this construction does the job simultaneously for all ideals K_s with $s \geq 1$ and the first lemma is proved.

Moreover, from this argument it becomes obvious that the only restriction on c in order that it could "accompany" a given a in GL^s is that it maps $\ker a$ injectively into H_- (again no matter which ideal K_s is considered). On the other hand, if c appears as the lower left corner element of such a matrix, then its kernel is trivial.

The last statement of Lemma 1.3 follows from the fact that the set of all such c is obviously homeomorphic to the set of all n-tuples of linearly independent vectors (i.e., n-frames) in H_-, where $n = \dim \ker a$. As is well known, all spaces of frames are contractible [47], and we obtain the desired conclusion. This completes the proof of the theorem. □

As was shown in [120], the *restricted Grassmanian* $\mathrm{Gr}^2(H)$ also has a remarkable structure of a cellular complex (CW-complex), which is closely related to the so-called partial indices [20] and gives a visual interpretation of certain phenomena discussed in [20, 21]. Moreover, Fredholm Grassmanians can be turned into differentiable manifolds, which allows one to construct an analogue of the Morse theory and recover in this way the cellular structure obtained from the partial indices [83, 120]. Now we describe a simple explicit way of introducing differentiable manifold structures on Fredholm Grassmanians Gr^s following the exposition of this topic in [120].

Theorem 1.6. *For any finite $s \geq 1$, the Grassmanian $\mathrm{Gr}^s(H)$ is a differentiable manifold modeled on the Banach space $K_s(H)$.*

Proof. First, we construct a natural atlas on Gr^s (cf. [120] for $s = 2$). Note that the graph of every s-summable operator $w : H_+ \to H_-$ belongs to Gr^s. Since the sum of a Fredholm operator and an s-summable operator is a Fredholm operator, one concludes that for every $W \in \mathrm{Gr}^s$, the graph of any s-summable operator from W to $W^{\perp}$ also belongs to Gr^s. Such graphs constitute an open subset $U_W \in \mathrm{Gr}_r$ consisting of all W' such that the orthogonal projection $W' \to W$ is an isomorphism. Obviously this open subset is in a one-to-one correspondence with the space $K_s(W, W^{\perp})$ of s-summable operators from W to $W^{\perp}$, which defines an atlas on Gr^s.

Now we describe the explicit form of the transition diffeomorphisms of this atlas and verify that this atlas really defines a structure of a differentiable manifold, i.e., differentials $D(g_i \circ g_j^{-1})(p)$ are bounded operators in $K_s(H)$. This would obviously complete the proof.

Let U_V and U_W be the open sets in Gr^s corresponding to the spaces $H_1 = K_s(V, V^{\perp})$ and $H_2 = K_s(W, W^{\perp})$. Let us show that the images H_{12} and H_{21} of the intersection $U_V \cap U_W$ in these spaces are open and the corresponding "change of coordinates" $H_{12} \to H_{21}$ is continuously differentiable.

Let us consider the identity transformation of H as the operator

$$V \oplus V^{\perp} \to W \oplus W^{\perp}$$

and write it in the form of the (2×2)-matrix of operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to these direct-sum decompositions, where a is an operator from V to W, etc. (cf. the proof of Theorem 1.5).

The fact that both V and W belong to Gr^s easily implies that the diagonal terms a and d are Fredholm operators and b and c are operators of the class K_s. Now assume that the subspace $L \in U_V \cap U_W$ is simultaneously the graph of operators $T_1 : V \to V^\perp$ and $T_2 : W \to W^\perp$. Then the operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ T_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ T_2 \end{pmatrix} q$$

coincide as operators from V to $W \oplus W^\perp$ for some isomorphism $q : V \to W$. This implies that

$$T_2 = (c + dT_1)(a + bT_1)^{-1}. \tag{1.6}$$

The last relation obviously shows that T_2 is a continuous function of T_1 on the open set $H_{12} = \{T_1 \in H_1 : a + bT_1 \text{ is invertible}\}$.

This means that the atlas U_W really defines on Gr^s a structure of a topological manifold, and it remains to verify that the differentials of coordinate changes in this atlas exist and they are bounded linear operators as operators in $K_s(H)$. For this, let us calculate the differential of T_2 as a function of T_1. By a standard application of the Leibniz rule for operator-valued functions, one obtains

$$DT_2(T_1) = d(a + bT_1)^{-1} - (c + dT_1)(a + bT_1)b(a + bT_1)^{-1}.$$

Now one can make a straightforward examination of the linear operator in $K_s(H)$ defined as the multiplication by the right-hand side of this formula, using the Neumann series for the inverse $(a + bT_1)^{-1}$, and verify that it defines a bounded linear operator on $K_s(H)$. Thus, this atlas really defines a differentiable manifold structure on Gr^s and the proof is complete. □

Remark 1.3. In the case $s = 2$, the same atlas defines a *holomorphic Hilbert manifold* structure on Gr_r (modeled on the Hilbert space $K_2(H)$ with its Hilbert–Schmidt norm) (cf. [120]).

Actually, the smooth structures on Fredholm Grassmanians could be derived from general results on the geometry of Banach spaces (see [45,129]). We prefer to present the direct proof in the case of a Hilbert space since, to our mind, it is simpler and more instructive than more general considerations in [45,129]. However, for the sake of completeness and with a view toward possible generalizations (one of which may be found in Sec. 3), we also present a short discussion of general results on Grassmanians considered as spaces of projections in Banach algebras. Most of the general notions needed below are currently standard, and our exposition closely follows [45]. It might be added that there are a number of other topics in operator theory which are concerned with the geometry of spaces of projections [35,103,148].

Let A be a complex Banach algebra with the identity 1 and $G(A)$ the Banach Lie group of units (invertible elements) of A. Let $P(A)$ be the set of idempotents in A. There is a natural affine involution of A sending x to $1 - x$, which maps $P(A)$ into itself, and a natural partial order $\prec$ on $P(A)$ (namely, $p \prec q$ if $qp = p$), which induces an equivalence relation $\sim$ on $P(A)$ ($p \sim q$ if and only if $pq = q$ and $qp = p$).

Definition 1.6 ([45,148])**.** The set of equivalence classes $P(A)/\sim$, considered as a topological space with the quotient topology defined by the quotient map $\Im : P(A) \to \mathrm{Gr}(A)$, is called the Grassmanian of A and is denoted $\mathrm{Gr}(A)$.

If A is an algebra of linear operators in a vector space, then obviously $p \sim q$ if and only if p and q have the same image and, therefore, $\mathrm{Gr}(A)$ can be considered as a generalization of the Grassmanian of subspaces isomorphic to a given subspace. For $p \in P(A)$, denote by $\mathrm{Sim}(p, A)$ the orbit of p under the action of $G(A)$ by inner automorphisms $a \mapsto gag^{-1}$ and define $\mathrm{Gr}(p, A)$ as the image of $\mathrm{Sim}(p, A)$ under the quotient map $\Im$. For $p \in P(A)$, let $G(p) = G(pAp)$ be the group of invertible elements in the algebra pAp. After this preparation, the main result of [45] can be formulated as follows.

Theorem 1.7 ([45])**.** *The Grassmanian* $\mathrm{Gr}(p, A)$ *is a complex Banach manifold modeled on the space* $(1 - p)Ap$*. Furthermore,* $\mathrm{Gr}(A)$ *is also a complex Banach manifold and the action of* $G(A)$ *on* $\mathrm{Gr}(A)$ *induced by the inner automorphic action of* $G(A)$ *on* A *is holomorphic.*

Now it is possible to identify a Fredholm Grassmanian with a subset of $\mathrm{Gr}(p, L(H))$ for a splitting projection p in a Hilbert space and verify that the complex manifold structure provided by the preceding theorem coincides with the structure defined by the above-constructed atlas.

The restricted Grassmanian $\mathrm{Gr}^2(H)$ possesses interesting geometric properties which appear useful in physical applications [104, 105, 120, 130]. For this reason, we now describe some important differential-geometric properties of $\mathrm{Gr}_r = \mathrm{Gr}^2(H)$, which can be established by the same methods as above. First, we collect some of its properties as a particular case of the results already obtained.

Corollary 1.3. *The restricted Grassmanian* Gr_r *is a homogeneous space under the actions of* U_r *and* GL_r*. If* W *is a closed complex subspace of* H *and* P_W *is the orthoprojector on* W*, then* $W \in \mathrm{Gr}_r$ *if and only if the difference* $P_W - P_+$ *is a Hilbert–Schmidt operator.*

The first statement was already proved. To prove the second statement, note that both conditions given in it obviously imply that W has infinite dimension and codimension. Thus, there exist $U \in U(H)$ such that $W = U \cdot H_+$. Then, of course, $P_W = UP_+U^{-1}$, and it follows by a direct calculation that $P_W - P_+$ is a Hilbert–Schmidt operator if and only if $[U, P_+]$ is a Hilbert–Schmidt operator; this exactly means that $U \in U_r$.

Remark 1.4. This corollary indicates a way of obtaining an explicit formula for the index of the Fredholm pair (H_+, W). Indeed, for projectors differing by a trace-class operator, such a formula in terms of the trace was given in [9]. It would be interesting and instructive to show that in the case of a subspace W associated with a linear conjugation problem, such an index formula in terms of traces can be rewritten as the classical index formula of Muskhelishvili [112].

Now we can list the basic geometric properties of Gr_r.

Theorem 1.8 (cf. [120, 130])**.** (1) *The restricted Grassmanian* $\mathrm{Gr}_r(H)$ *is a complex-analytic manifold modeled on the separable Hilbert space* $L^2(H_+, H_-)$*.*

(2) *The actions of* $GL_r(H)$ *and* U_r *on* Gr_r *are complex analytic and real analytic respectively.*

(3) *The linear isotropy representation at the point* H_+ *is described by the map*

$$\mathrm{Ad} : G_+ \to GL(L^2(H_+, H_-)), \quad \mathrm{Ad}\, T \cdot S = T_{11} \circ S \circ T_{22}^{-1}, \quad S \in L^2(H_+, H_-).$$

(4) *The connected components of* Gr_r *consist of subspaces* W *with a fixed value of* $\mathrm{ind}\, P_+|W$*; in other words, they are given by the sets*

$$G_r^k = \{W : \mathrm{ind}(P_+|\, W : W \to H_+) = k\}, \quad \kappa \in \mathbb{Z}.$$

(5) *There exist natural holomorphic embeddings of the total Grassmanians*

$$G(\mathbb{C}^{2N}) = \bigcup_{n=0}^{2N} G_n(\mathbb{C}^{2N})$$

into Gr_r *such that their images* $\mathrm{Gr}_r(N)$ *form an increasing sequence of subsets and the union* $\bigcup_{N\geq 1} \mathrm{Gr}_r(N) = \mathrm{Gr}_r(\infty)$ *is dense in* Gr_r*. Moreover, for all* N *and* k*, the intersection* $\mathrm{Gr}_r(N) \cap G_r^k$ *is biholomorphic to* $G_{N+k}(\mathbb{C}^{2N})$*.*

Corollary 1.4 ([120])**.** *All holomorphic functions on* Gr_r *are locally constant.*

Proof. Since U_r acts by holomorphic transformations, it is sufficient to consider a function f which is holomorphic on the connected component G_r^0 of Gr_r containing H_+. Now, given two points W_1 and W_2 in $G_r^0 \cap \mathrm{Gr}_r(\infty)$, there exists N such that $W_1, W_2 \in G_r^0 \cap \mathrm{Gr}_r(N)$. Since the latter is biholomorphic to a connected compact complex manifold $G_N(\mathbb{C}^{2N})$, the function f takes the same values on W_1 and W_2. Thus, f is constant on $G_r^0 \cap \mathrm{Gr}_r(\infty)$ and the result follows from the density of the latter set in G_r^0. $\square$

As was shown by several authors (see [57, 120, 130, 149]), the restricted Grassmanian Gr_r possesses remarkable differential-geometric properties. Most of them follow from the important observation that Gr_r carries a natural U_r-invariant Kähler structure, which we describe following [120, 130].

Recall that the so-called *Schwinger term* [130] is defined as a bilinear form on the Lie algebra u_r of U_r given by

$$s(A, B) = tr(A_{12}B_{21} - B_{12}A_{21}),$$

where $A, B \in u_r$ are represented as (2×2)-matrices with respect to a fixed polarization. Define a real-valued antisymmetric bilinear form on u_r by setting

$$\tilde{\Omega}(A, B) = (-\imath)s(A, B).$$

One can verify that the bilinear form $\tilde{\Omega}$ vanishes on the subalgebra $u(H_+) \times u(H_-)$ and is invariant under the linear isotropy representation of $U(H_+) \times U(H_-)$. Hence it descends to a form Ω_+ on

$$u_r/u(H_+) \times u(H_-) \cong L^2(H_+, H_-) \cong T_{H_+} \mathrm{Gr}_r,$$

which is invariant under $U(H_+) \times U(H_-)$.

Note that there also exists a natural complex structure J_+ on $T_{H_+} \mathrm{Gr}_r$, which is also $U(H_+)\times U(H_-)$-invariant, namely, $J_+T = \imath T$ for all $T \in L^2(H_+, H_-)$. This, in the usual way, produces a $U(H_+)\times U(H_-)$-invariant Kähler structure on Gr_r.

The main properties of the restricted Grassmanian which can be formulated in this context are collected in the following statement.

Theorem 1.9 ([130]). *The restricted Grassmanian is a Hermitian symmetric space; in particular, it is geodesically complete. The geodesic exponential map* Exp *at the point H_+ is given by*

$$\pi \circ \exp : T_{H_+} \mathrm{Gr}_r \to \mathrm{Gr}_r,$$

where exp *is the exponential map of U_r and π is the projection $U_r \to \mathrm{Gr}_r$. The tensor of Riemann curvature of* Gr_r *is completely fixed by its value at the point H_+, where it can be given by a certain Toeplitz-like operator.*

Remark 1.5. Comparing the above formulas and the formulas defining the Kähler metric on the based loop group investigated by Freed [57], one can observe that they are completely similar. This similarity is not of course occasional, the link between both structures being given by the Grassmanian embedding of a based loop group [120]. Actually, several interesting geometric structures (smooth, Riemannian, Kähler, Fredholm, characteristic classes, curvatures, and fundamental classes of Birkhoff strata) were independently introduced on loop groups (see [57,78,109]) and Fredholm Grassmanians (see [83,120,130]). This circumstance suggests an interesting program of comparing these structures and showing that they are related by the Grassmanian embedding. Informally, this can be called the "LG-Gr correspondence principle." For some of these structures, like the Kähler structures above, the "LG-Gr correspondence principle" is quite easy to verify, but the differential-geometric aspects are rather delicate.

We restrict ourselves to this brief remark because the realization of this program requires a separate exposition. Our aim is just to show that Fredholm Grassmanians and loop groups can be fruitfully studied using several independent approaches, each of which is rich enough to deserve an investigation of its own.

In this spirit, we proceed by mentioning that another fruitful framework for discussing various geometric properties of Fredholm Grassmanians emerges from the theory of Fredholm structures [47]. As was observed in [81, 83], certain dense subsets of these Grassmanians can be endowed with natural Fredholm structures.

This fact seems remarkable since a Fredholm structure on an infinite-dimensional manifold enables one to introduce nontrivial global geometric and topological invariants of this manifold. The reason for this circumstance is that Fredholm Grassmanians are closely related to loop groups of compact Lie groups [120], and such loop groups can be endowed with some natural Fredholm structures [57, 80, 81].

Our discussion of this issue is based on the results of [57, 81], but we present them with a view toward Fredholm Grassmanians.

For simplicity, we only consider the classical case corresponding to the loop group of the unitary group U_n. Recall that Riemann–Hilbert problems for arbitrary compact Lie groups were studied in [81]. Some results of [81] are presented in the next section. The discussion below is actually applicable for arbitrary compact Lie groups.

Recall that a Fredholm structure on an (infinite-dimensional) Banach manifold M modeled on a Banach space E is defined by an atlas (U_i, g_i) on M such that for any point $p \in g_j(U_i \cap U_j)$, the differential (Frechet derivative) of the transition diffeomorphism $D(g_i \circ g_j^{-1})(p)$ is an invertible operator of the form "identity + compact" [48].

The appearance of a natural Fredholm structure on an infinite-dimensional Banach manifold is a remarkable event, as such structures possess various interesting global geometric and topological invariants (curvatures, characteristic classes, etc.) [47]. An important result of Elworthy and Tromba states that a Fredholm structure on M can be constructed from a Fredholm mapping $M \to H$ with zero index and also from certain smooth families of zero-index Fredholm operators parametrized by the points of M [49].

These facts were used in [80, 83] for constructing Fredholm structures on loop groups. This was done by using the families of Fredholm operators parametrized by regular loops. By virtue of the results of [80, 83], to each regular loop f one can assign a Fredholm operator associated with the linear conjugation problem R_f defined by loop f (i.e., f is the coefficient of the problem R_f). The following statement follows from the results of [80, 83] combined with the main result of [49].

Theorem 1.10. *To any (complex) linear representation γ of U_n, one can associate a Fredholm structure F_γ on the group L^1U_n of H^1-loops on U_n.*

Here the loop group L^1U_n is endowed with the usual H^1-norm [109]. It is easy to verify that with this norm it becomes a Hilbert Lie group. We do not reproduce details of the argument in [80, 83] because they were performed within the framework of the Fredholm theory of generalized Riemann–Hilbert problems, which involves a lot of technicalities irrelevant to the main subject of this paper.

A nice geometric explanation of the existence of Fredholm structures on loop groups can be given using some recent results of Misiolek [109, 110]. The main result of [109] yields, in particular, that the exponential map of the group LG of H^1-loops on a compact Lie group G is a Fredholm map of zero index. This local result allows one to obtain a Fredholm atlas on the loop group by merely taking the inverse of the exponential map at identity and spreading this chart to any point of LG by left shifts.

More precisely, there exists a local chart (U, ϕ) at the unit of LG such that $\phi : U \to V$ is a diffeomorphism on some open subset in the space of loops LA on the Lie algebra A of group G, and for any $x \in U$, the differential $D\phi_(x)$ is an invertible operator of the form "identity + compact." The same result of Misiolek implies that differentials of left shifts L_g by elements of LG are also of the same form "identity + compact." Let us construct a chart (U_g, ϕ_g) at $g \in LG$ by setting $U_g = L_g(U)$ and $\phi_g = \phi \circ (L_g)^{-1}$. This obviously yields an atlas on LG, and it is easy to verify that differentials of the transition mappings of this atlas also belong to the Fredholm subgroup. In this way, one obtains a Fredholm structure corresponding to the adjoint representation of a loop group.

This argument was worked out jointly with Misiolek in August of 2001 during a Banach Center workshop on nonlinear differential equations. Details and applications will appear in a forthcoming paper of Khimshiashvili and Misiolek.

Actually, Fredholm structures on loop groups come from various sources. An interesting geometric way of constructing Fredholm structures on loop groups was suggested by Freed [57].

The construction used by Freed reveals certain differential geometric aspects of loop groups which are obviously interesting from the standpoint of linear conjugation problems. For this reason, we briefly discuss some ingredients of his construction and their relation to the concepts used in the theory of linear conjugation problems.

Now let ΩG denote the group of based (i.e., the number 1 maps to the identity of G) smooth loops on a compact Lie group G with Lie algebra A. Recall that for any real number s, one can define the (Sobolev) H_s-metric on ΩG by the formula

$$(X, Y)_{H_s} = \int_T (\nabla^s X(x), Y(x))_A dx, \quad X, Y \in \Omega A,$$

where ∇ denotes the Laplace operator d^*d on T and $(\cdot, \cdot)_A$ denotes the inner product on A given by minus the Killing form on A [120].

The Hilbert-space completion of the smooth loops in this inner product is denoted by $\Omega_s A$. As is well known, H_s-loops are continuous for $s > 1/2$ [120]. In this range, one also obtains in the standard way [120] the corresponding completions $\Omega_s G$ which are Hilbert manifolds modeled on $\Omega_s A$ [57, 120]. As was proved in [58], loop groups $\Omega_s G$ are *Hilbert Lie groups* for $s > 1/2$. In fact, in many aspects, it is also important to consider the $H_h = H_{1/2}$ metric (h for "one-half") on ΩG.

As was shown in [57], this metric is a *homogeneous Kähler metric* [94]. The corresponding Kähler structure on ΩG is most easily described by exhibiting its complex and symplectic structures and then observing that the metric defined by those is exactly the H_h-metric.

The almost complex structure on ΩG is obvious from the decomposition of its complexified tangent space Lie algebra $\Omega A_{\mathbb{C}} = M_+ \oplus M_-$ into the direct sum of two subspaces consisting of loops with only positive (negative) Fourier coefficients, i.e., we consider M_+ as the holomorphic tangent space and M_- as the antiholomorphic tangent space. Alternatively, one can define a J-operator on the tangent space ΩA, that is, an operator whose square is equal to -1.

Denote by D the operator $d/d\phi = \imath z d/dz$. Note that its kernel is trivial on the space of based loops ΩA and the operator $|D|$ is the square root of the positive Laplacian $-d^2/d\phi^2$. This implies that $J = D/|D|$ has its square equal to -1. One concludes that ΩG is a complex manifold by applying an infinite-dimensional version of the Newlander–Nirenberg theorem [94]. This is possible since the torsion tensor of this almost complex structure vanishes [57]. The same conclusion can be derived from the generalized Birkhoff theorem obtained in [120].

There also exists a left-invariant symplectic form ω on ΩG. It is described by defining it on the Lie algebra ΩA by the formula

$$\omega(X, Y) = \frac{1}{2\pi} \int_T (X'', Y)_A,$$

where X and Y are interpreted as elements of ΩA.

The form ω is nondegenerate since D has no kernel on based loops and one can verify by standard calculation that ω is smooth (see [120]). Summarizing all this, we conclude that ΩG is an infinite-dimensional Kähler manifold and ω is the Kähler form for the Kähler metric

$$(X, Y) = \frac{1}{2\pi} \int_T \left(\left| \frac{d}{d\phi} \right| X(\phi), Y(\phi) \right)_A d\phi.$$

Comparing with the above formula for the Sobolev metrics, we see that this is precisely the H_h-metric. Using its specific properties, an elegant formula for the corresponding Kähler connection was established in [57].

Denote by $\mathbb{T}$ the circle group; then $\mathbb{T}$ acts on the space LG of free loops by rotations and one can form the semidirect product EG of $\mathbb{T}$ and LG. Obviously,

$$\Omega G = EG/(\mathbb{T} \times G);$$

therefore, EG naturally acts on ΩG. By the general principles of Lie group actions, to this action corresponds an infinitesimal action which assigns a vector field ξ_Z to each element $Z \in EG$ (as is well known, this vector field is the *right-invariant* extension of Z to EG). Evaluation at the identity in ΩG gives the map which identifies $M_{\mathbb{C}} = M_+ \oplus M_-$ with the complexified tangent space to ΩG.

Let ∇ be the Kähler connection and ∇^L be the Lie derivative. Then, again by general principles, the difference $\nabla_{\xi_Z} - \nabla^L_{\xi_Z}$ is tensorial, hence it defines a linear transformation on M. Since both derivatives preserve the complex structure, after complexification this transformation separately preserves M_+ and M_-. Denoting by EA the semidirect product of $\mathbb{R}$ and LA, we obtain a map $\phi : EA \to L(M_+)$ as the $\mathbb{C}$-linear extension of the map defined by

$$Z \mapsto \nabla_{\xi_Z} - \nabla^L_{\xi_Z}|M_+.$$

Freed expressed the Kähler (H_h) connection by indicating a remarkable explicit formula for ϕ in terms of Toeplitz operators [57]. Families of such operators eventually allowed Freed to construct a Fredholm structure on ΩG whose characteristic classes coincided with those defined using the Chern–Weil theory for the Kähler H_h metric [58].

This circumstance is especially remarkable in the context of our approach because, as was explained in [81, 83], Fredholm structures defined by Riemann–Hilbert problems can also be induced from certain families of Toeplitz operators. Because of the arising technical complications, we do not present here a detailed discussion, but we emphasize that the approach of Freed can be used to establish some subtle geometric and topological properties of Birkhoff strata in the H_h-metric. In particular, one obtains a general approach to calculating their curvatures and characteristic classes, and it is interesting to verify whether this leads to the same results as were obtained in [40] for the groups of Hölder loops.

Another intriguing open problem is to investigate whether these "Freedholm" structures are equivalent (concordant in the sense of [49]) to some of those obtained from parametrizing the loop groups by families of linear conjugation problems as in [81, 83]. Actually, there is some evidence that the structures constructed by Freed correspond to the case of Riemann–Hilbert problems with respect to the adjoint representation of the group.

Since each Fredholm Grassmanian Gr^s by its very definition gives rise to a family of Fredholm operators parametrized by it, one can consider the Fredholm structure defined by this family. In the spirit of Remark 1.5, it is natural to conjecture that these *Fredholm structures on Fredholm Grassmanians* can be related with the preceding results using the Grassmanian embeddings of loop groups [120]. As will be explained in Sec. 2, the interpretation of a loop as a coefficient of a linear conjugation problem gives a natural mapping of an appropriate loop group LG into the group $GL(\pi_+, K)$. By virtue of the above discussion (cf. also [120]), it is clear that by posing proper regularity conditions on a loop f, one can achieve that the rotation of the subspace H_+ by the operator of multiplication by f gives a subspace in one of Grassmanians Gr^s. In this way, one obtains a natural mapping of LG into Gr^s, which is called the *Grassmanian model* (or *Grassmanian embedding*) of a loop group [120].

Some properties of these models follow from the preceding discussion; others were established in [120], basically for the case of the restricted Grassmanian $\mathrm{Gr}_r(H)$. In particular, it is well known that the group of continuously differentiable loops can be embedded in Gr_r [120]. However, the topology of LG and the topology induced on its image as a subset of Fredholm Grassmanian do not coincide in general. In fact, it is an interesting and difficult analytical problem to find exact regularity conditions which guarantee that the corresponding loop group can be realized in Gr^s for a concrete s (see examples in [120, Chap. 7]).

We avoid discussion of this problem by concentrating our attention on the group of smooth (infinitely differentiable) loops $L_\infty U_n$, which is the smallest of the interesting groups of that kind. Its image under the above embedding is called the smooth (Fredholm) Grassmanian Gr_∞. It is easy to verify that it lies in each Fredholm Grassmanian Gr^s. A more interesting circumstance is that it is homotopy equivalent to each of them [120], and it captures important global properties of these Grassmanians.

Now one can transplant various structures from $L_\infty U_n$ to Gr_∞. In particular, it is obvious that $L_\infty U_n$ can be endowed with Fredholm structures, which are just the restrictions of the Fredholm structures on H^1-loops provided by Proposition 1.10, and we obtain the same conclusion for the smooth Grassmanian.

Proposition 1.3. *With each linear* (*finite-dimensional*) *representation of* U_n*, one can associate a Fredholm structure on the smooth Grassmanian* $\mathrm{Gr}_\infty(H)$*.*

Of course, one may ask whether it is possible to extend these structures to ambient Grassmanians Gr^s, but this problem involves some delicate analytic issues which will be discussed elsewhere.

As an example of perspectives suggested by these results, let us formulate another natural problem. From the mentioned result of Elworthy and Tromba and Proposition 1.10, it follows that there exists a zero-index Fredholm mapping of the loop group L^1U_n into a Hilbert space. It would be interesting and instructive to find an explicit construction of such a mapping. The same problem can be formulated for all compact Lie groups. It would also be interesting to find such a mapping from the smooth Grassmanian $\mathrm{Gr}_\infty(H)$ in its model space.

Also, it is well known that for a Fredholm manifold M, one can define its characteristic classes $\mathrm{ch}_k(M) \in H^{2k}(M, \mathbb{Z})$ [58]. A natural and important problem is to identify these classes in the cohomology of M. In our setting, this problem permits a particularly nice formulation.

As was already mentioned, the smooth Grassmanian has the same homotopy type as Fredholm Grassmanians $\mathrm{Gr}^s(H)$; therefore, their cohomology rings are isomorphic and the structure of these rings is well known [48]. It is also well known (see [48,57]) that any Fredholm Hilbert manifold has well-defined Chern classes ch_j which are classes in the even-dimensional cohomology of this manifold. Combining these two observations, we conclude that a Fredholm structure on the smooth Grassmanian defines certain classes in $H^{2j}(\mathrm{Gr}^s(H))$. Thus, we come to the problem of calculation of these classes for the structures F_γ described above. Some results in this direction were obtained in [57,58]. It is remarkable that such Chern classes can be represented by some differential forms using traces of appropriate products of operators from Schatten classes [57], which indicates an intriguing analogy with the noncommutative geometry of Connes [34].

Now we recall the main result of [58] and explain its relation to the geometric models for Riemann–Hilbert problems discussed in this section. Recall that the group of units of the Calkin algebra $Q(H) = L(H)/K(H)$ can be naturally identified with the quotient group $Q^* = GL(H)/GK(H)$. As is well known, it is a Banach Lie group modeled on its Lie algebra $Q(H)$, which is actually a C^*-algebra [44].

One can similarly define quotient groups $Q^s = GL/GL^s$ for each $s \geq 1$. Since L^s is an ideal in $L(H)$, GL^s is a normal subgroup in $GL(H)$; therefore, the quotient Q^s is a group. Unfortunately, it cannot be made into a Banach Lie group because L^s is not closed in $L(H)$; this implies that the Lie algebra L/L^s is not Hausdorff in the quotient topology. Hence one has to regard Q^s as just an abstract group. Nevertheless, these are good objects because they are closely related to Q^*, which has a number of useful topological interpretations [48].

As was shown by Freed [58], the hidden “nice” structure of groups Q^s can be revealed by considering some special homomorphisms $G \to Q^s$ of a Banach Lie group G, which factor through the projection $\pi : F_0(H) \to Q^s$. This setting is actually a particular case of the notion of *Fredholm representations* considered by Mishchenko and his followers (cf. [107, 129]).

In such a context, Freed was able to prove a general theorem providing useful information on the Chern classes of the GK^s-bundle emerging on G via pull-back from $F_0(H)$ (recall that $F_0(H)$ is the classifying space for all $GK^s(H)$-bundles [48]). Denote the Chern classes of the universal $GK(H)$-bundle by ch_l.

Theorem 1.11 ([58]). *Let G be a Banach Lie group with the Lie algebra* $\mathrm{Lie}(G)$. *Assume that $T : G \to F_0(H)$ is a smooth map such that the composition $\pi \circ T : G \to Q^s$ is a homomorphism, i.e., $T(g)T(h) - T(gh) \in L^s$ for all $g, h \in G$. Further, assume that the map $(g, h) \mapsto T(g)T(h) - T(gh)$ is a smooth map into L^s. Let $T' : \mathrm{Lie}(G) \to L(H)$ be the differential of T at the identity, and define the left-invariant L^s-valued 2-form on G by setting*

$$\Omega(X, Y) = [T'(X), T'(Y)] - T'[X, Y], \quad X, Y \in \mathrm{Lie}(G).$$

Then for all $l \geq s$, the cohomology class $T^* \mathrm{ch}_l$ is represented invariantly by the form

$$\gamma_l = -(\mathbf{i}/2\pi)^l (l!)^{-1} \operatorname{tr}(\Omega^l).$$

Here the elements of Lie(G) are understood as left-invariant vector fields on G, which, in the definition of Ω, are evaluated at the identity of G. The assumptions of the theorem guarantee that the trace $\mathrm{tr}(\Omega^l)$ exists for all $l \geq s$. A detailed proof of this theorem can be found in [58].

Of course, it is not quite obvious how to construct such maps T as required in the theorem. Nevertheless, a natural source of examples is provided by the *Fredholm Hilbert Lie groups* G introduced below. Recall that a Fredholm structure on a group G defines in a standard way a map $G \to F_0(H)$. One just takes an F_0-map of $F : G \to H$ (which exists according to the criterion given in [49]) and takes as T the map defined by the family of differentials d_xF, $x \in G$. In many cases, this map satisfies the conditions of the theorem for some s and one can calculate some components of the total Chern class of the corresponding Fredholm structure.

An example of such kind emerges from the "Kählerian" Fredholm structures on loop groups constructed in [57]. This enabled Freed to calculate the total Chern class of such a Fredholm structure for $SU(n)$ [57].

Fredholm structures defined by families of linear conjugation problems in some cases also generate maps T which satisfy the conditions of the theorem [84]. Actually, in most cases, such families only satisfy a weaker condition, which amounts to saying that "long commutators" of operators $T(g)$ belong to some group GL^s. In [84], the same is expressed by saying that the algebra of operators generated by $T(g)$ satisfies a *polynomial identity modulo compact operators*. This situation is not covered by the Freed theorem but still it is very close to the notion of *Fredholm module*, which plays an important role in the noncommutative differential geometry of Connes [34]. Therefore, one may hope that the methods of calculation of Chern character developed in [34] can be applied in this situation. Unfortunately, this idea has not yet found sufficient development, and we delay discussion of the topic for the future.

It seems also worth noting that some properties of the so-called partial indices [20, 112] of linear conjugation problems can be formulated in the language of Fredholm structures. As was already mentioned, the Birkhoff strata define an interesting stratification of the loop group. Using Grassmanian models of loop groups and linear conjugation problems described above, one obtains the corresponding strata in the smooth Grassmanian Gr_∞ and restricted Grassmanian Gr_r (cf. [120]).

Using the well-known properties of partial indices [21], one can show that Birkhoff strata are complex analytic submanifolds of finite codimension in the group $LGL_n(\mathbb{C})$ of loops on the complex general linear group $GL_n(\mathbb{C})$ [40]. Similar conclusions derived for Birkhoff strata can be considered in the group $\Omega GL_n(\mathbb{C})$ of based loops [40]. Similar results in the context of Fredholm Grassmanians were obtained in [83, 120]. One can actually describe the homotopy type of a Birkhoff stratum B_K in terms of the indexing vector K [40, 120].

To be more precise, denote by G the group of L^1-loops on $GL_n(\mathbb{C})$ and denote by B_K the subset of all loops with a given collection of (left) partial indices K.

Theorem 1.12 ([40])**.** *B_K is a locally closed complex-analytic submanifold of G of finite codimension; its codimension is equal to*

$$\sum_{k_i > k_j} (k_i - k_j - 1).$$

The homotopy type of B_K in terms of K can be described as follows. Let g_K denote the diagonal matrix-valued function in (1.2) corresponding to the integer vector K. Denote by $GL(K)$ the centralizer in $GL_n(\mathbb{C})$ of the image of g_k and by $\Delta : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ the diagonal embedding. Finally, denote by X_K the quotient space (as a topological space)

$$(GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))/\Delta(GL(K)),$$

where in the denominator stays the image of $GL(K)$ under the diagonal embedding. Note that this is a finite-dimensional space of the type well suited for applying the usual methods of algebraic topology which allow one to calculate many topological invariants of this space and eventually determine its homotopy type.

Theorem 1.13 ([40]). *B_K is homotopy equivalent to X_K.*

Further results on the topological type of Birkhoff strata can be found in [40, 57]. Most of the preceding discussion is applicable to Birkhoff strata in the loop group LG of an arbitrary compact Lie group G (cf. the discussion of G-exponents in the next section).

We proceed by explaining how one can approach the study of Birkhoff strata in the context of Fredholm structures. For this, one can use a slight generalization of the notion of a *proper Fredholm submanifold* introduced in [116, 136].

The generalization we have in mind can be naturally formulated in the setting of *Riemannian Hilbert manifolds*. To be more precise, let us recall some relevant notions from the theory of infinite-dimensional manifolds.

Let M be a smooth infinite-dimensional manifold modeled on a separable Hilbert space H. As is well known, one can introduce all basic concepts of differential geometry (vector fields, bracket operation, p-forms, tensors, etc.) in the same way as for finite-dimensional manifolds [94].

A Riemannian metric on M is defined as a smooth section g of $S^2(T*M)$ such that $g(x)$ is an inner (scalar) product on T_xM equivalent to the inner product on H for all $x \in M$. If such g is given, then (M, g) is called a *Riemannian Hilbert manifold*. It is well known that there exists a unique torsion-free *connection* (sometimes spelled as *connexion*) (in the usual sense of differential geometry [94]) compatible with the metric g. Such a connection is called the *Levi-Civita connection* of g.

Let M and N be Hilbert manifolds, g be a Riemannian metric on N, and ∇ be the Levi-Civita connection of g. A smooth map is called an *immersion* if d_xf is injective and $d_xf(T_xM)$ is a closed linear subspace of $T_{f(x)}N$ for all $x \in M$. Then the restriction of $g(x)$ to $d_xf(T_xM)$ defines a Riemannian metric on M. The *normal bundle* $\nu(M)$ of $f(M)$ in N is defined as the bundle over M with fibers $\nu_x(M)$ isomorphic to the orthogonal complement of $d_xf(T_xM)$ in $T_{f(x)}N$ [94]. A globally bijective immersion is called an *embedding* and its image is called a submanifold in N. We deal only with submanifolds of *finite codimension*, i.e., $d_xf(T_xM)$ has finite codimension in $T_{f(x)}N$ for all $x \in M$.

For an embedded submanifold M, one can define in a standard way its *tubular neighborhood* $t(M)$ and the exponential mapping $\exp : t(M) \to N$ [94]. Finally, using the natural identification of $t(M)$ with a (locally trivial) fibration of open normal discs (balls) $D_r(M) = D_r(\nu(M))$ of (sufficiently small) radius r in $\nu(M)$, one can define the so-called *endpoint map* $Y_r : D_r(M) \to N$.

Note that in the particular case where N is just the model space H, our definition coincides with the definition of the endpoint map for submanifolds of the Hilbert space used in [116, 136]. Since $D_r(M)$ is also a Riemannian Hilbert manifold, one can speak of Fredholm mappings of $D_r(M)$ into any other Hilbert manifold. Thus, in particular, it make sense to speak of Fredholm maps into N.

Definition 1.7. M is called a *proper Fredholm* submanifold (*PF-submanifold*) of N if the endpoint map $Y_r : D_r(M) \to N$ is proper and Fredholm for all sufficiently small $r > 0$.

Many examples of proper Fredholm submanifolds of a Hilbert space can be found in [116, 136]. Their curvature operators have remarkable compactness properties [136] and they can be successfully studied by differential-geometric methods [136]. This concept is also well suited for the case where the ambient manifold N is endowed with a Fredholm structure. This situation repeatedly appears in the context of loop groups and Fredholm Grassmanians; we introduce some relevant concepts.

Definition 1.8. An infinite-dimensional Riemannian Hilbert manifold M is called a *Fredholm Riemannian Hilbert manifold* (*FRH-manifold*) if M is endowed with a compatible Fredholm atlas. If M is simultaneously a Lie group and the metric is invariant, then M is called a *Fredholm Hilbert Lie group* (*FHL-group*).

Now it is clear that one can also define the notion of a *Fredholm action* of an FHL-group. From the preceding discussion it follows that these concepts are "nonvacuous." In particular, the smooth Fredholm Grassmanians give examples of FRH-manifolds. Moreover, as was explained above, the groups of H^1-loops on compact Lie groups appear to have natural Fredholm structures arising from the exponential mapping [109]. Then it is easy to verify that they actually give examples of FHL-groups. Finally, an important example of a Fredholm action of a FHL-group is provided by the action of a loop group on the appropriate space of connections described in [116, 136]. Actually, some actions appearing in the context of integrable systems [111] can also be interpreted as Fredholm actions of FHL-groups.

Remark 1.6. These definitions permit a lot of variations (e.g., in the setting of more general Banach manifolds) and the range of applicability of these concepts can be substantially extended (e.g., in the framework of *Euler equations on infinite-dimensional Lie groups* [111] and *Poisson Hilbert Lie groups* [101]), but we cannot dwell upon such developments in the present paper.

By using these concepts, we can formulate some related results with a view to applications to Birkhoff strata in loop groups.

Proposition 1.4. *A PF-submanifold of finite codimension of an FRH-manifold inherits a Fredholm structure whose equivalence class is uniquely determined by the embedding.*

This follows from [136, Proposition 2.8]. Indeed, the Fredholm structure constructed there on a submanifold M of a Hilbert space was defined by the family of curvature operators of M. Since these operators can be calculated from an arbitrarily small tubular neighborhood of M, one can easily verify that the argument used in [136] can be applied in our situation.

Taking into account the general definition of a *Fredholm submanifold* of a given Fredholm manifold [49], we can reformulate this proposition by saying that a PF-submanifold is a Fredholm submanifold. In the framework of Fredholm structure theory, one can introduce and investigate various geometric properties of Fredholm submanifolds.

To apply all this to Birkhoff strata, note that the exponential mapping and curvature operators of loop groups and Birkhoff strata were calculated in [57, 109]. Translated into our language, the results of [57, 109] mean that the endpoint maps of Birkhoff strata are proper Fredholm; therefore, they are PF-submanifolds of loop groups (as was shown above, they have finite codimension). Now the preceding proposition allows us to place Birkhoff strata in the context of Fredholm structures.

Proposition 1.5. *The Birkhoff strata are Fredholm submanifolds of* $\mathrm{Gr}_\infty(H)$ *and, as follows from the general results of* [48]*, each of them has a well-defined fundamental class in the even-codimensional cohomology of* $\mathrm{Gr}_r(H)$.

Some calculations of the fundamental classes of Birkhoff strata can be found in [40]. In particular, Disney managed to calculate the fundamental classes for all collections of partial indices with not more than three different components k_i. It would be interesting to complete his results by finding a general formula for the fundamental class of B_K in $H^*(LGL_n(\mathbb{C}))$.

Remark 1.7. Taking into account the fact that the stratification of LG given by Birkhoff strata is of a complex-analytic nature [40], one can investigate its properties along the usual lines of stratification theory [67]. In particular, it is interesting to verify whether this stratification satisfies the Whitney conditions [67]. Some further results about the geometry of Birkhoff strata can be derived from the results on the structure of the so-called *isoparametric submanifolds* obtained by Palais and Terng [116, 136].

Using the above approach for loop groups and Fredholm Grassmanians associated with compact Lie groups [120], one can generalize Theorem 1.10 in this context. The formulation which we present follows from the results of [83], which, in turn, are based on the Fredholm theory for linear conjugation problems developed in [80, 83]. The existence follows from the Fredholmness of the corresponding linear conjugation problem for G [81]. Recall that for any compact Lie group, one can naturally define the smooth Grassmanian Gr_∞^G lying in $\mathrm{Gr}_r^G(H)$.

Proposition 1.6. *For each linear representation γ of a compact Lie group G, the smooth Grassmanian* $\mathrm{Gr}_\infty^G(H)$ *has a canonical Fredholm structure F_γ induced by γ.*

As was already mentioned, for any Fredholm structure on a complex Banach manifold, one can define its Chern classes [48]; therefore, we can introduce some global topological invariants of such Grassmanians.

Corollary 1.5. *For each even k, there exists a canonical cohomology class in $H^{2k}(\mathrm{Gr}^G_\infty(H))$ which can be defined as the Chern class of the canonical Fredholm structure F_γ.*

Now it is obvious that one can formulate a number of natural questions related to such Fredholm structures. It is the author's hope that the approach described in this section can lead to new insights about global properties of the classical linear conjugation problems and geometric objects naturally associated with them. In the next two sections, we discuss some generalizations of classical linear conjugation problems which naturally arise in the framework of this approach.

2. Linear Conjugation Problems for Compact Lie Groups

In this section, we describe a generalization of the linear conjugation problem introduced recently by the author in the framework of loop groups [81].

We consider the Riemann sphere $\mathbb{P} = \overline{\mathbb{C}}$ decomposed into the union of the unit disc D_+, unit circle T, and exterior domain D_- containing the infinite point ∞ denoted by N ("north pole").

The main innovation is to permit more general coefficients in the transmission equation (1.1). It is natural to take as a coefficient the function on the circle with values in a compact Lie group G and for piecewise-holomorphic mappings with values in a given representation space of G. The precise statement of the problem is given below, and the rest of the section is devoted to its investigation.

It appears that in the case of a Lie group G one can develop a reasonable theory similar to the classical theory [140] which relies on the recent generalization by Pressley and Segal [120] of the well-known Birkhoff factorization theorem [112].

It is easy to indicate several natural regularity classes for a coefficient which guarantee that the problem is described by a Fredholm operator in corresponding functional spaces. One can also obtain an index formula (Theorem 2.2) in terms of so-called *partial G-indices* which is a direct generalization of the corresponding classical result [112]. A natural framework for our discussion is provided by the generalized Birkhoff factorization theorem and Birkhoff stratification of a loop group, and we have to present some auxiliary concepts and results.

Let G be a connected compact Lie group of rank p with the Lie algebra A. As is well known [120], each of such groups has a complexification $G_{\mathbb{C}}$ with the Lie algebra $A_{\mathbb{C}} = A \otimes \mathbb{C}$. This fact is very important since it provides complex structures on loop groups and this is the main reason why our discussion is restricted to compact groups.

Let LG denote the group of continuous based (i.e., sending the number 1 to the unit of G) loops on G endowed with the pointwise multiplication and usual topology [120]. We need some regularity conditions on loops and, for simplicity, we assume that all loops under consideration are (at least once) continuously differentiable.

For an open set U in $\mathbb{P}$, let $A(U, \mathbb{C}^n)$ denote the subset of $C(\widetilde{U}, \mathbb{C}^n)$ formed by vector-valued functions holomorphic in U.

Assume also that we are given a fixed linear representation r of the group G in a vector space V. For our purposes, it is natural to assume that V is a complex vector space. Note that for a compact group, one has a complete description of all complex linear representations [1].

Now we are in the position to formulate our generalization. Namely, having fixed a loop $f \in LG$, the (homogeneous) generalized linear conjugation problem (GLCP) R_f with coefficient f is formulated as a question on the existence and cardinality of pairs $(X_+, X_-) \in A(D_+, V) \times A(D_-, V)$ with $X_-(N) = 0$ satisfying the transition condition on T:

$$X_+(z) = r(f(z)) \cdot X_-(z). \tag{2.1}$$

For any loop h on V, we also obtain an inhomogeneous problem $R_{f,h}$ (with the right-hand side h) by replacing the transition equation (2.1) by the condition

$$X_+(z) - r(f(z)) \cdot X_-(z) = h(z). \tag{2.2}$$

In other words, we are interested in the kernel and cokernel of the natural linear operator T_f expressed by the left-hand side of formula (2.2) and acting from the space of piecewise holomorphic vector-functions on $\mathbb{P}$ with values in V into the loop space LV. To avoid annoying repetition, when dealing with the inhomogeneous GLCP it will always be assumed that the loop h is Hölder-continuous, which is a usual assumption in the classical theory [112].

Remark 2.1. In the particular case where $G = U(n)$ is the unitary group, we obtain that $G_{\mathbb{C}} = GL(n, \mathbb{C})$ is the general linear group. If we take r to be the standard representation on $\mathbb{C}^n$, then Eqs. (2.1) and (1.1) coincide and we obtain the classical linear conjugation problem. Note that even in this classical case, one obtains plenty of such problems by taking various representations of $U(n)$, and the result below can be best illustrated in this situation.

Needless to say, the same picture is observed for all groups, but as a matter of fact only irreducible representations of simple groups are essential. Moreover, the exceptional groups of Cartan's list will also be excluded and the remaining groups will be termed "classical simple groups."

It would not be appropriate to reproduce and discuss here all necessary concepts and constructions of Lie group theory. All necessary results on Lie groups, in a form suitable for our purposes, are contained in [1] and we repeatedly refer to this book in the sequel.

Let f be a loop on G. We associate with f some numerical invariant similar to the classical partial indices [112]. For this, let us choose a maximal torus T^p in G and a system of positive roots. Then, following [120], one can define the nilpotent subgroups $N_0^{\pm}$ of $G_{\mathbb{C}}$ whose Lie algebras are spanned by root vectors of $A_{\mathbb{C}}$ corresponding to the positive (respectively, negative) roots. We also introduce subgroups $L^{\pm}$ of $LG_{\mathbb{C}}$ formed by the loops which are the boundary values of holomorphic mappings of the domain B_+ (respectively, B_-) into the group $G_{\mathbb{C}}$, and the subgroups $N^{\pm}$ consisting of the loops from L^+ (respectively, L_-) such that $f(0)$ belongs to N_0^+ (respectively, $f(N)$ belongs to N_0^-).

The following fundamental result was proved in [120].

Decomposition Theorem. *Let G be a classical, simple, compact Lie group and $H = L^2(T, A_{\mathbb{C}})$ be the polarized Hilbert space with $H = H_+ \oplus H_-$, where H_+ is the usual Hardy space of boundary values of holomorphic loops on $A_{\mathbb{C}}$. Then we have the following decomposition of groups of based loops LG:*

(i) *LG is the union of subsets B_K indexed by the lattice of holomorphisms of T into the maximal torus T^p;*

(ii) *B_K is the orbit of $K \cdot H_+$ under N^-, where the action is defined by the usual adjoint representation of G. Every B_K is a locally closed contractible complex submanifold of finite codimension d_K in LG, and it is diffeomorphic to the intersection L_K^+ of N^- with $K \cdot L_1^- \cdot K^{-1}$, where L_1^- consists of loops equal to the unit at the infinite point N;*

(iii) *the orbit of $K \cdot H_+$ under N^+ is a complex cell C_K of dimension d_K. It is diffeomorphic to the intersection L_K^+ of N^+ with $K \cdot L_1^- \cdot K^{-1}$ and meets B_K transversally at the single point $K \cdot H_+$;*

(iv) *the orbit of $K \cdot H_+$ under $K \cdot L_1^- \cdot K^{-1}$ is an open subset U_K of LG, and the multiplication of loops gives a diffeomorphism from $B_K \times C_K$ into U_K.*

Recall that in the classical case, this result reduces essentially to the Birkhoff factorization theorem for matrix loops [140].

Let us introduce the corresponding construction in our setting. Namely, for a loop f on G, the (left) Birkhoff factorization will be called its representation in the form

$$f = f_+ \cdot H \cdot f_-\,, \tag{2.3}$$

where $f_\pm$ belongs to the corresponding group $L^{\pm}G$ and H is some homomorphism of T into T^p.

Now it is obvious that items (ii) and (iv) of the theorem imply the following existence result.

Proposition 2.1. *Every differentiable (and even Hölder class) loop in a classical simple compact group has a factorization.*

Note that we could also introduce the right factorization with the reversed order of f_+ and f_- and the result is also valid. Our choice of the factorization type is consistent with the problem under consideration.

Taking into account that any homomorphism H in (2.3) is determined by a sequence of p integer numbers $(k_1, \dots, k_p)$, we obtain that this sequence can be associated with any loop f. These natural numbers are called (left) *G-exponents* (or *partial G-indices*) of f. Their collection is denoted $K(f)$.

It is easy to prove that $K(f)$ does not actually depend on either the terms of the representation (2.3) or on the choice of the maximal torus. For a given maximal torus, the proof of this fact can be obtained as in the classical case, while the independence on the choice of a maximal torus follows from the well-known fact that any two maximal tori are conjugate [1].

The exponents provide basic analytical invariants of loops and have some topological interpretations.

Proposition 2.2. *Two loops lie in the same connected component of LG if and only if they have the same sum of exponents.*

This follows easily from the contractibility of subgroups $L^{\pm}$ and item (ii) of the theorem.

Remark 2.2. In the classical case where $G = U(n)$, we obtain the usual partial indices, and Proposition 2.2 reduces to the obvious observation that the connected components of LU_n are classified by the sum of partial indices, which is known to coincide with the increment of the determinant argument of a matrix-valued function along the unit circle [112].

Using exponents of loops, we may identify each subset B_K with the collection of loops having a given collection of G-exponents equal to K (up to the order) and use the corresponding decomposition of LG in the topological study of GLCP. Note that in the classical theory of LCP, the geometry of Birkhoff strata B_K was the subject of an intensive investigation [20, 66]. Later on, Bojarski proposed an approach in the spirit of global analysis [21], which can also be treated from the standpoint of the theory of Fredholm structures [48].

Theorem 2.1 ([57, 81])**.** *For any classical, simple, compact group G, the group of based H^1-loops ΩG possesses a Fredholm structure such that all strata B_K are contractible Fredholm submanifolds of ΩG.*

This result provides a manifestation of the close connections between the geometric theory of Riemann–Hilbert problems and global analysis discussed in [23, 24]. Its proof makes an essential use of results obtained in [80, 120]. Another proof was given by Freed [57]. We mention some corollaries which are in the spirit of our exposition.

Corollary 2.1. *The inclusion of each stratum B_K into LG defines a cohomological fundamental class $[B_K]$ in $H^*(LG)$.*

This is an immediate consequence of the cohomology theory for Fredholm manifolds [48].

Remark 2.3. In the particular case where $G = U(n)$, this fact was established by Disney [40] without referring to Fredholm structures.

This corollary leads to the purely topological problem of calculation of these fundamental classes in terms of the classical description of $H^*(LG)$ given by Bott [29]. For $G = U(n)$, some of these fundamental classes were calculated in [40]. Further progress in this topic was obtained in [57, 88]. The general problem of calculating these fundamental classes seems to have remained unsolved.

Another type of problem arises in connection with the aforementioned Grassmanian models for LG. We describe it only in the classical case where $G = U_n$ and r is its canonical representation on $\mathbb{C}^n$.

Recall that given a polarized Hilbert space $H = H_+ \oplus H_-$, one can introduce the Fredholm Grassmanian $\mathrm{Gr}_r(H)$ consisting of all subspaces in H such that the orthogonal projection on H_+ is Fredholm and the complementary projection on H_- is of Hilbert–Schmidt class [120]. For the canonical representation of $G = U_n$, we can take H_+ such that it coincides with the usual Hardy subspace in the space H of square-integrable vector functions on the unit circle [60, 120]. This is of course just a special case of Definition 1.3.

Note that any two elements of $\mathrm{Gr}_r(H)$ form a Fredholm pair in the sense of Definition 1.1 and the index of such a pair is well-defined. As was shown in [120], smooth loop groups are embedded in Gr_r so that for any two loops, the index of the pair of images of H_+ is defined, and one can try to calculate it in terms of exponents. It is sufficient to do that in the case where one of the subspaces coincides with H_+, which corresponds to the constant loop. From the calculation of the virtual dimension of an element in $\mathrm{Gr}_r(H)$ [120], it is easy to derive a formula for the index of a Fredholm pair of such subspaces which, by virtue of [21], coincides with the index of the corresponding Riemann–Hilbert problem.

Corollary 2.2. *The index under consideration is equal to the sum of the exponents of the given loop.*

For any natural n, let B_n denote the union of all Birkhoff strata with the sum of exponents equal to n. Collecting together all observations, we see that the images of the sets B_n lie in various connected components of Gr_+ and n is equal to the Fredholm index of pr_+.

There are also some interesting differential-geometric aspects of the Fredholm stratification. In particular, one can calculate the curvature of B_K in terms of the corresponding Toeplitz operators [57] and verify that they provide examples of the so-called isoparametric submanifolds of LG [136]. These strata are among a few known examples of nonlinear Fredholm submanifolds of a geometrically interesting Banach manifold.

To obtain a formula for the index of a GLCP, it is obviously necessary to take into account the influence of a given representation r; this is easy to do since for classical compact groups, all irreducible representations are determined completely by their highest weights [1]. At the same time, it is clear that the index of a GLCP behaves additively with respect to taking a direct sum of representations. Thus, it is sufficient to assume that r is irreducible with the highest weight $w(r)$.

Recall also that with any weight w of the group G, one can associate the so-called elementary symmetric sum $S(w)$ [1] and evaluate it on any integer vector with p components.

Theorem 2.2. *Let r be an irreducible representation of a classical simple compact Lie group G. Then the index of a GLCP with a differentiable loop f as a coefficient is given by the formula*

$$\operatorname{ind} P_f = S(w(r))(k_1(f), \dots, k_p(f)), \tag{2.4}$$

where $w(r)$ is the highest weight of the representation r.

Proof. Standard facts about decompositions of characters of irreducible representations imply that it is sufficient to prove (2.4) only for the so-called basic representations, i.e., such that their classes in the representation ring $R(G)$ form a set of algebraic generators of $R(G)$ [1].

According to Proposition 2.1, we can rewrite (2.3) for the coefficient loop f. Then we insert (2.3) into Eq. (2.1) and separate the terms marked by $+$ in the left-hand side; this gives us an equivalent but much more useful form of the transmission condition:

$$(r(f_+))^{-1}(X_+) = r(H)r(f_-)(X_-).$$

Now we introduce the new functions

$$Y_+ = (r(f_+))^{-1}(X_+), \quad Y_- = r(f_-)(X_-)$$

and note that they are holomorphic in the same domains as X_+ and X_- respectively. This means that we can equivalently solve the new linear conjugation problem for the pair (Y_+, Y_-) with respect to the action of T^r.

Since representations of tori are always decomposable into the direct sum of irreducible one-dimensional representations, we conclude that the above factorization allows us to reduce the given problem to a collection of problems having the same form as classical linear conjugation problems.

It remains only to determine the exponents of the one-dimensional components of the representation $r(H)$. This can be easily done by using an effective description of the basic representations available for all classical groups [1]. These descriptions are similar for all classical groups and, therefore, we consider only one case, say, for the series A_k.

Then basic representations are of the form $s^{(i)}$ with $1 \leq i \leq k$, where the superscript denotes the exterior degree of the standard representation on $\mathbb{C}^k$ [1]. The character of $s^{(i)}$ is given by the ith elementary symmetric function in k indeterminates and the Weyl group W reduces to the symmetric group S_k. It follows that this character coincides with the elementary symmetric sum of highest weight $S(x_1+\cdots+x_i)$ and the exponents of one-dimensional representations of T^p are given by the terms of the corresponding symmetric sum for exponents of the loop f.

Note that each of the arising one-dimensional problems is simply a classical linear conjugation problem for vector-valued functions vanishing at infinity. By a classical result [112], the index of such a problem is equal to the exponent of the coefficient. Collecting all these observations, we immediately obtain the desired index formula. □

Remark 2.4. The Fredholm property of the operator T_f is automatically available due to the regularity and invertibility of the coefficient f. This easily follows from the description of our problem in terms of holomorphic principal $G_{\mathbb{C}}$-bundles on P, which is presented below.

In terms of the exponents, one can also obtain a more precise description of the space of solutions to GLCP.

Corollary 2.3. *The dimension of the kernel of a GLCP R_p is equal to the sum of all positive terms in the formal symmetric sum of exponents of its coefficient f.*

This follows from the proof of the theorem, since the corresponding fact for one-dimensional reductions of our problem is well known [140].

We would also like to indicate one aspect of the loop group theory where our result seems useful. Namely, a given loop, generally speaking, can be attributed to various ambient loops by considering some natural embeddings of the groups under consideration (e.g., $U(n) \subset O(2n)$). This can change both the highest weight and exponents, and there arises an interesting problem of describing possible changes of exponents under such embeddings of coefficient groups.

This problem is not merely of theoretical interest since it is closely related with the problem of effective calculation of exponents and factorization of a given loop. A natural way is to realize the group in question as a matrix group by considering the matrix realization of the representation r involved in the definition of GLCP, and then calculate the partial indices of the corresponding matrix-valued function using the well-known results of [33,100]. Note that for wide classes of matrix-valued functions, there exist effective calculating algorithms which are easy to implement on a computer [5]. Then it would remain to take into account the changes of exponents caused by embedding G in a matrix group.

The same problems can be formulated with respect to arbitrary homomorphisms of the groups under consideration. For example, it would be interesting to investigate the role played by the *Dynkin index* of such a homomorphism. We delay a discussion of this issue to the future and pass to another geometric topic related with GRHP.

As was already mentioned, an adequate language for classical LCP is provided by holomorphic vector bundles over $\mathbb{P}$ [120]. In the case of an arbitrary compact group G, there exists a natural connection between GLCP and principal $G_{\mathbb{C}}$-bundles over the Riemann sphere [120]. This connection works in both directions. In particular, the results on the structure of solutions of a GLCP allow one to obtain some information on deformations of $G_{\mathbb{C}}$-bundles.

Theorem 2.3. *The base of the versal deformation of a holomorphic principal $G_{\mathbb{C}}$-bundle corresponding to a loop f has dimension d_K, where $K = K(f)$ is the collection of exponents of f. Moreover, it is given by the formula*

$$d_K = \sum_{k_i > k_j} (k_i - k_j - 1). \tag{2.5}$$

This can be derived from the geometric description of the Birkhoff stratification provided by the above decomposition theorem. Indeed, each stratum corresponds to a fixed isomorphism class of the bundles under consideration [120]. In fact, item (iii) of the decomposition theorem shows that such strata possess natural transversals, which, due to the smoothness of strata, yield the germs of the base of versal deformation [117]. At the same time, the desired dimension d_K can be calculated by the general technique of deformation theory in terms of the first cohomology group of $\mathbb{P}$ with coefficients in the adjoint representation of G [117]. Then formula (2.5) follows from (2.3) and the Serre duality.

Corollary 2.4. *A holomorphic principal $G_{\mathbb{C}}$-bundle is* (*holomorphically*) *trivial if and only if all exponents of the corresponding loop are equal to zero.*

Corollary 2.5. *A holomorphic principal $G_{\mathbb{C}}$-bundle is stable if and only if all pairwise differences of its exponents do not exceed* 1*.*

One can also explicitly calculate various cohomology groups associated with such $G_{\mathbb{C}}$-bundles in terms of exponents; this is useful, for example, in some applications of GRHP to nonlinear equations [41]. One can also use the exponents for investigating which principal bundles can be realized as subbundles of a given $G_{\mathbb{C}}$-bundle. Note that for U_n-bundles, a complete solution of this problem was obtained in [125].

It is worth noting that it is also possible to formulate some reasonable nonlinear versions of GRHP: one has to consider Eq. (2.1) with respect to more general actions of compact groups, e.g., on their homogeneous spaces. In this context, one can obtain some conditions in terms of the exponents which guarantee solvability of a nonlinear problem [78].

In conclusion, it seems appropriate to point out that recently there appeared a number of papers which use the generalized Birkhoff factorization for solving nonlinear equations. Some results of such kind are presented in [120].

A highly effective application of Birkhoff factorization to proving the existence of solutions of a nonlinear equation on a Lie group can be found in [41]. The literature devoted to applications of linear conjugation problems and Riemann–Hilbert problems to nonlinear equations and integrable systems is very ample, but there is no possibility of discussing this issue here. We just indicate a recent book of Deift [36], which contains a detailed exposition of several advanced applications of Riemann–Hilbert problems to analysis and differential equations.

3. Linear Conjugation Problems over C^*-Algebras

In this section, we introduce certain geometric objects over C^*-algebras which are relevant to the homotopy classification of abstract elliptic problems of linear conjugation. The abstract problem of linear conjugation was introduced by Bojarski [21] as a natural generalization of the classical Riemann–Hilbert problem for analytic vector-valued functions [112]. As was later realized by the author [79], the whole issue fits nicely into the theory of Fredholm structures [47], more precisely, into the homotopy theory of operator groups started by Palais [115] and developed by Rieffel [121] and Thomsen [137].

Similar geometric objects appear in loop groups theory, K-theory, and the geometric aspects of operator algebras and have recently gained considerable attention [19,28,102,120,152]. This circumstance allowed the author [79, 86] to develop a geometric approach to the abstract linear conjugation problems presented in this section.

Recall that in 1979, Bojarski formulated a topological problem which appeared important in his investigation of the so-called Riemann–Hilbert transmission problems [21]. This problem was later solved independently in [76, 144] (cf. also [120]). Moreover, these results were used in studying several related topics of global analysis and operator theory [28, 77, 79, 145].

An important advantage of the geometric formulation of elliptic transmission problems in terms of Fredholm pairs of subspaces of a Hilbert space given in [21] was that it permitted various modifications and generalizations. Thus, it became meaningful to consider similar problems in more general contexts [79]. Along these lines, the author developed some aspects of Fredholm-structure theory [47] in the context of Hilbert C^*-modules [74, 102, 108], which led to some progress in the theory of generalized transmission problems [79, 86].

Such an approach allows one, in particular, to investigate elliptic transmission problems over an arbitrary C^*-algebra. Clearly, this gives a wide generalization of the original setting used in [21,28,76,144], since the latter corresponds to the case where the algebra is taken to be the field of complex numbers $\mathbb{C}$. This also generalizes the geometric models for classical Riemann–Hilbert problems considered in the previous section. In addition, this approach allowed the author to clarify the homotopy classification of abstract singular and bisingular operators over C^*-algebras [79].

Note also that the setting of transmission problems over C^*-algebras includes the investigation of families of elliptic transmission problems parametrized by a (locally) compact topological space X. In fact, this corresponds to considering transmission problems over the algebra of continuous functions on the parameter space $C(X)$, and classification of families of elliptic problems of such kind becomes a special case of our general results.

To make the presentation concise, we freely use terms and constructions from a number papers on related topics, especially from [19, 21, 74, 106, 108, 138]. An exhaustive description of the background and necessary topological notions is contained in [19, 74, 102, 106, 108]. Actually, all necessary results can be found in the recent work [102], which contains a detailed description of the theory of Hilbert modules over C^*-algebras.

Now we pass to the precise definitions needed to formulate a generalization of a geometric approach to transmission problems suggested by Bojarski [21]. We use essentially the same concepts as in [21], but sometimes in a slightly different form adjusted to the case of Hilbert C^*-modules.

Let A be a unital C^*-algebra. Denote by H_A the standard Hilbert module over A, i.e.,

$$H_A = \left\{ \{a_i\},\ a_i \in A,\ i = 1, 2, \ldots : \sum_{i=1}^{\infty} a_i a_i^* \in A \right\}. \tag{3.1}$$

Since there exists a natural A-valued scalar product on H_A possessing the usual properties [102], one can introduce direct-sum decompositions and consider various types of bounded linear operators on H_A. Denote by $B(H_A)$ the collection of all A-bounded linear operators having A-bounded adjoints. This algebra is one of the most fundamental objects in Hilbert C^*-module theory [74, 102, 108].

As is well known, $B(H_A)$ is a Banach algebra and it is useful to consider also its group of units $GB = GB(H_A)$ and the subgroup of unitaries $U = U(H_A)$. For our purposes, it is important to have adjoints, which, as is explained, e.g., in [102], is not the case for an arbitrary bounded operator on the Hilbert A-module H_A. In particular, for this algebra, we have an analogue of the polar decomposition [102], which implies that $GB(H_A)$ is retractable to $U(H_A)$. Thus, these two operator groups are homotopy equivalent, which is important for our consideration.

Compact linear operators on H_A are defined to be A-norm limits of finite-rank linear operators [102]. Their collection is denoted by $K(H_A)$.

Recall that one of the central objects in Bojarski's approach [21] is a special group of operators associated with a fixed direct-sum decomposition of a given complex Hilbert space. Fix a direct-sum decomposition of Hilbert A-modules of the form $H_A = H_+ + H_-$, where H_+ and H_- are isomorphic to H_A as A-modules. As is well known, any operator on H_A can be written as a (2×2)-matrix of operators with respect to this decomposition. We denote by π_+ and π_- the natural orthogonal projections defined by this decomposition.

Now introduce the subgroup $GB_r = GB_r(H_A)$ of $GB(H_A)$ consisting of operators whose off-diagonal terms belong to $K(H_A)$. Let $U_r = U_r(H_A)$ denote the subgroup of its unitary elements. To relate this to transmission problems, we must have an analogue of the restricted (Fredholm) Grassmanian introduced in [120]. In fact, this is practically equivalent to working with Fredholm pairs of subspaces, which was used in [21]. To implement all this in our generalized setting, some technical preliminaries are needed.

Recall that there is a well-defined notion of a finite rank A-submodule of a Hilbert A-module [108]. This allowed Mishchenko and Fomenko to introduce the notion of a Fredholm operator in a Hilbert A-module by requiring that its kernel and image be finite-rank A-submodules [108]. It turns out that many important properties of the usual Fredholm operators also remain valid in this context. Thus, if the collection of all Fredholm operators on H_A is denoted by $F(H_A)$, then there exists a canonical homomorphism $\mathrm{ind} = \mathrm{ind}_A : F(H_A) \longrightarrow K_0(A)$, where $K_0(A)$ is the usual topological K-group of the basic algebra A [19].

This means that Fredholm operators over C^*-algebras have indices satisfying the usual additivity law. In the sequel, we will freely refer to the detailed exposition of these results in [102, 106].

Granted the above technicalities, we can now introduce a special Grassmanian $\mathrm{Gr}_+ = \mathrm{Gr}_+(H_A)$ associated with the given decomposition. It consists of all A-submodules V of H_A such that the projection π_+ restricted on V is a Fredholm operator while the projection π_- restricted on V is compact. Using the analogues of the local coordinate systems for $\mathrm{Gr}_+(H_{\mathbf{C}})$ constructed in [120], we can verify that $\mathrm{Gr}_+(H_A)$ is a Banach manifold modeled on the Banach space $K(H_A)$. For our purposes, it suffices to consider Gr_+ as a metrizable topological space with the topology induced by the standard one on the infinite Grassmanian $\mathrm{Gr}^\infty(A)$.

Now the problem that we are interested in is to investigate the topology of $\mathrm{Gr}_+(H_A)$ and $GB_r(H_A)$. Note that for $A = \mathbb{C}$, this is the problem formulated by Bojarski in [21].

The main topological results about these objects can be formulated as follows.

Theorem 3.1. *The group $GB_r(H_A)$ acts transitively on $\mathrm{Gr}_+(H_A)$ with contractible isotropy subgroups.*

Theorem 3.2. *All even-dimensional homotopy groups of $\mathrm{Gr}_+(H_A)$ are isomorphic to the index group $K_0(A)$ and its odd-dimensional homotopy groups are isomorphic to the Milnor group $K_1(A)$.*

Of course, the same statements hold for the homotopy groups of $GB_r(H_A)$, since by Theorem 3.1 these two spaces are homotopy equivalent. We formulate the result for $\mathrm{Gr}_+(H_A)$ since it is a space of interest for the theory of transmission problems.

The homotopy groups of $GB_r(H_A)$ were first calculated by the author in [79] without considering Grassmanians. Later, similar results were obtained by Zhang [152] in the framework of K-theory. The contractibility of isotropy subgroups involved in Theorem 3.1 in the case $A = \mathbb{C}$ was established in [120].

In proving Theorem 3.1, we will obtain more precise information on the structure of isotropy subgroups. It should also be noted that the contractibility of isotropy subgroups follows from a fundamental result on C^*-modules called the generalization of the Kuiper theorem for Hilbert C^*-modules, which was obtained independently by Troitsky [138] and Mingo [106]. Particular cases of Theorem 3.2 for various commutative C^*-algebras A may be useful to construct classifying spaces for K-theory.

The solution of Bojarski's original problem is now immediate (cf. [76, 120, 144]).

Corollary 3.1. *Even-dimensional homotopy groups of the collection of classical Riemann–Hilbert problems are trivial and odd-dimensional homotopy groups are isomorphic to the additive group of integers $\mathbb{Z}$.*

Note that the nontriviality of these groups can be interpreted in terms of the so-called spectral flow of zero-order pseudo-differential operators, which has recently led to some interesting developments of Booss and Wojciechowsky [28], which shed new light on the Atiyah–Singer index formulas in the odd-dimensional case.

Similar results hold for abstract singular operators over A (for the definition of abstract singular operators, see [79]).

Corollary 3.2. *Homotopy groups of invertible singular operators over a unital C^*-algebra A are expressed by the relations*

$$\begin{gathered}\pi_0 \cong K_0(A), \quad \pi_1 \cong \mathbf{Z} \oplus \mathbf{Z} \oplus K_1(A); \\ \pi_{2n} \cong K_0(A), \quad \pi_{2n+1} \cong K_1(A),\end{gathered} \tag{3.2}$$

where n is natural and arbitrary.

Specifying this result for the algebras of continuous functions, one can, in particular, calculate the homotopy classes of invertible classical singular integral operators on arbitrary regular closed curves in the complex plane $\mathbb{C}$ (see [76, 79] for the precise definitions).

Corollary 3.3. *If $K \in \mathbb{C}$ is a smooth closed curve with k components, then homotopy groups of invertible classical singular integral operators on K are expressed by the relations*

$$\pi_0 \cong \mathbb{Z}, \quad \pi_1 \cong \mathbb{Z}^{2k+1}, \quad \pi_{2n} = 0, \quad \pi_{2n+1} \cong \mathbb{Z}, \tag{3.3}$$

where n is natural and arbitrary.

As is shown in [79], this information also allows one to find homotopy classes and index formulas for the so-called bisingular operators. The latter can be defined by purely algebraic means, starting from the algebra of abstract singular operators. Thus, one comes to the notion of a bisingular operator over a C^*-algebra and to the description of homotopy classes of elliptic bisingular operators. The notion was introduced in [79] and the description of index ranges follows from the results of this paper.

Corollary 3.4. *Abstract elliptic bisingular operators over a C^*-algebra A are homotopically classified by their indices taking values in $K_0(A)$. The index homomorphism is an epimorphism onto $K_0(A)$.*

As is well known, the usual bisingular operators correspond to certain pseudo-differential operators on the two-torus $\mathbf{T}^2$ [44]. One may similarly recover some of the known results on homotopy groups of invertible pseudo-differential operators over other two-surfaces [46].

One can also obtain an index formula for abstract bisingular operators in terms of homotopy classes of their operator-valued symbols, which can be described by Theorem 3.2. For brevity, the results concerning the index formulas for bisingular operators are presented here.

Theorem 3.1 is proved below after developing the necessary geometric constructions over C^*-algebras. We also present the outlines of proofs of Theorem 3.2 and its corollaries.

It is standard in C^*-algebra theory to identify subspaces with projections. Thus, direct-sum decompositions of the type described above correspond to the so-called infinite Grassmanian over A, which can be written as

$$\mathrm{Gr}^\infty(A) = \Big\{p \in B(H_A) : p = p^2 = p^* \text{ and } p \sim \mathrm{Id} \sim \mathrm{Id} - p\Big\}, \tag{3.4}$$

where "$\sim$" denotes the Murray–von Neumann equivalence between projections [19].

Fixing such a decomposition is equivalent to fixing a projection with image and kernel being A-modules of infinite rank. Having fixed such a projection p, which plays the role of the projection π_+ introduced above, one can readily verify the useful characterization of GB_r.

Lemma 3.1. $GB_r(H_A) = \{x \in B(H_A) : xp - px \in A \otimes K(H)\}$, *where $K(H)$ is the ideal of compact operators in the usual separable complex Hilbert space H.*

The aforementioned (2×2)-matrix representation of $x \in B(H_A)$ can be rewritten in the form

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \tag{3.5}$$

where $x_{11} = pxp$, $x_{12} = px(1-p)$, $x_{21} = (1-p)xp$, and $x_{22} = (1-p)x(1-p)$.

Now it is obvious that $GB_r(H_A)$ is $*$-isomorphic to the group of units of the C^*-algebra consisting of (2×2)-matrices over $B(H_A)$ whose off-diagonal entries are elements of $A \otimes K(H)$. Further, the existence, additivity, and stability properties of the Fredholm index (see diagram (3.6) below) imply that for $x \in GB_r(H_A)$, both x_{11} and x_{22} are Fredholm operators with opposite indices; this is important for the sequel.

Using simple algebraic identities for such (2×2)-matrices (explicitly written in [120] for matrices over $B(H)$) and the fact that $K(H_A)$ is an ideal in $B(H_A)$, it is easy to verify that if such a (2×2)-matrix is applied to an element V of $\mathrm{Gr}_+(H_A)$, then the restriction to V of the first projection π_+ is transformed into $x_{11}\pi_+ + x_{12}$ and thus remains Fredholm, while the restriction to V of the second projection gives $x_{22}\pi_- + x_{21}$ and remains compact. This means that xV is again in $\mathrm{Gr}_+(H_A)$ and we have proved the following lemma.

Lemma 3.2. *The restricted linear group $GB_r(H_A)$ acts on the special Grassmanian $\mathrm{Gr}_+(H_A)$.*

Now it is obvious that to determine the isomorphy class of stability subgroups, it is sufficient to identify it for a "coordinate submodule" H_+ in $GB_r(H_A)$. It readily follows from the existence of polar decompositions that the latter subgroup is homotopy equivalent to the isotropy subgroup of H_+ in the restricted unitary group $U_r(H_A)$ (which acts on $\mathrm{Gr}_+(H_A)$ as a subgroup of $GB_r(H_A)$).

Analyzing the description of a similar isotropy subgroup in the case of the usual Hilbert space given in [120], one easily finds that in view of the above technical results for Hilbert C^*-modules, the same conclusion also holds in our case.

Lemma 3.3. *The stability subgroup of H_+ in $U_r(H_A)$ is isomorphic to $U(H_+) \times U(H_-)$.*

Recall that the latter group is contractible according to the result of Troitsky and Mingo [106,138].

To prove Theorem 3.1, it remains to verify the transitivity, which is the most delicate part of the proof. We use the method of proof from [120] adapted to our situation. Note that Fredholm operators with vanishing indices can be transformed into invertible operators by a compact perturbation. The corresponding statement for Hilbert C^*-modules is contained in the so-called *fundamental commutative diagram* of Fredholm-structure theory. In our case, it has the form

$$\begin{array}{ccccccc} GB(M_A) & \longrightarrow & F_0 & \longrightarrow & F & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & G & \longrightarrow & G & \longrightarrow & Q \end{array}, \tag{3.6}$$

where $F(H_A)$ and $F_0(H_A)$ are semigroups of all Fredholm operators and zero-index operators, respectively, G is the group of units of the quotient algebra $B(H_A)/K(H_A)$, and G_0 is its component of the identity. The right vertical arrow is the Calkin factorization and the left one is the quotient homomorphism on the quotient group below. The upper arrows are inclusions.

The commutativity of this diagram is well known to experts and follows from the facts established in [47, 48] (cf. also [19]). Also, it is a standard verification that the lower left corner horizontal arrow is a homeomorphism. In topological terms, the latter fact means that G_0 is the classifying space for the K-functor [19] and its homotopy groups are isomorphic to the corresponding K-groups of the basic algebra. This conclusion is explained in full detail in [48].

Let us now return to our situation and take an A-submodule V belonging to the special Grassmanian $\mathrm{Gr}_+(H_A)$. By definition, there exists a Fredholm operator $T \in B(H_+, H_-)$ such that V is its graph, i.e., is the set of points (x, Tx) with respect to the given decomposition of H_A.

To prove the transitivity of the action, it is sufficient to obtain a (2×2)-matrix $M \in GB_r(H_A)$ of the form described above such that $M(H_+) = V$. For this, we first consider the diagonal matrix $\mathrm{diag}(T, T^*)$, where T^* is the adjoint operator of T. The additivity of the Fredholm index implies that this matrix has zero index when considered as an element of $B(H_A)$. Considering its class in the space of (2×2)-matrices over the Calkin algebra B/K, one sees that it is invertible. Thus, the diagram shows that this matrix can be turned into an invertible matrix by a compact perturbation. In other words, there exist compact off-diagonal terms x_{21} and x_{12} such that our diagonal matrix completed with such entries becomes invertible as an operator on H_A. This already implies the existence of the desired matrix M. One could also finish the proof arguing as in [120, Chap. 6].

The simplest way of proving Theorem 3.2 is as follows. Note that Proposition 6.2.4 in [120] suggests that $GB_r(H_A)$ is homotopy equivalent to $F(H_+)$. This would already prove Theorem 3.2 since diagram (3.6) implies that homotopy groups of $F(H_A)$ are isomorphic to the K-theory of A. In fact, it can be actually proved that $GB_r(H_A)$ is homotopy equivalent to $F(H_+)$ by using a suitable modification of the argument from [120, Chap. 6]. However, to make this argument rigorous, one needs to develop a substantial portion of Hilbert-module theory. For brevity, we prefer another way, more algebraic in spirit, which closely follows the lines of [152]. In doing so, we will borrow freely the concepts and results from [106, 152].

In this section, we use the identification of direct-sum decompositions with projections and fix $p \in B(H_A)$ such that $p = p^2 = p^*$. Below, we omit some tedious details which are standard in the theory of operator algebras and K-theory.

As was explained, it suffices to calculate the homotopy groups of the restricted linear group $GB_r(H_A)$. Denote by $GB_r^0(H_A)$ its identity component. As is well known, in dealing with K-theory invariants, it is useful to consider the conjugations by unitary operators. Taking this into account, we introduce the notation

$$UpU^* = \{vpv^* : v \in U_r(H_A)\} = \{vpv^* : v \in GB_r(H_A),\ vv^* = v^*v = \mathrm{Id}\}.$$

The following simple proposition is verified by using the standard techniques of K-theory (cf. [19]).

Lemma 3.4. *$U(upu^*)U^*$ is the path component of UpU^* containing upu^*.*

One also has an equivalent description of the K_0-functor which was used in [106, 152].

Lemma 3.5. *For any such $p \in \mathrm{Gr}^\infty(A)$, the group $K_0(A)$ is isomorphic to the fundamental group $\pi_1(UpU^*)$.*

Indeed, later we will produce an explicit isomorphism between these two groups in terms of some partial isometries associated with elements of $GB_r(H_A)$, which plays an important role in the argument.

Following [152], we say that a unitary operator $x \in U_r(H_A)$ is p-adapted if both off-diagonal terms of the corresponding (2×2)-matrix (see (3.6)) are some partial isometries in $A \otimes K(H)$.

It is easy to calculate some associated projections needed in the sequel.

Lemma 3.6 ([152])**.** *If x is a p-adapted unitary, then $p - x_{11}x_{11}^*$, $p - x_{11}^*x_{11}$, $(\mathrm{Id} - p) - x_{22}x_{22}^*$, and $(\mathrm{Id} - p) - x_{22}^*x_{22}$ are projections in $A \otimes K(H)$.*

The following results from [152] amount to a partial isometry description of the K-functor. Equivalent statements can be found in [102, 106]. A similar factorization for the case $A = C$ was also used in [120].

Proposition 3.1. *Any $X \in GB_r(H_A)$ can be represented in the form*

$$x = (\mathrm{Id} + k) \cdot \mathrm{diag}(z_1, z_2) \cdot u, \tag{3.7}$$

where $k \in A \otimes K(H)$, the second factor is invertible, and u is p-adapted unitary.

Recall that, according to one of the basic constructions, any partial isometry $b \in A \otimes K(H)$ defines a class $[bb^*] \in K_0(A)$ [19]. The following proposition follows from this construction and the equivalence relation in $K_0(A)$.

Proposition 3.2. *The class*

$$[u_{12}u_{12}^*] - [u_{21}u_{21}^*] \in K_0(A) \tag{3.8}$$

is independent of a p-adapted unitary u entering into a representation of a given $x \in GB_r(H_A)$ of the form (3.7).

Now we can define the mappings which yield the desired group isomorphisms. Our strategy is to consider the group GB_r as a fibration over its homogeneous space GB_r/GK and, then, to calculate the homotopy groups of GB_r/GK, since the homotopy groups of the fiber $GK(H_A)$, being the standard participants in K-theory, are well known.

First, observe that representation (3.7) implies the equality of cosets $x \cdot GK(H_A) = u \cdot GK(H_A)$ of the elements x and u with respect to the subgroup $GK(H_A)$. By Lemma 3.5, for such u, we have the following direct sum of projections:

$$(p - u_{12}u_{12}^*) \oplus (u_{12}u_{12}^*). \tag{3.9}$$

As is well known, direct sums do not have any influence on the stable equivalence relation involved in the definition of $K_0(A)$. In other words, it is meaningful to assign to element (3.9) the class

$$[u_{12}u_{12}*] - [u_{21}u_{21}^*] \in K_0(A). \tag{3.10}$$

A connection between the considered basic topological spaces is established by the following lemma.

Lemma 3.7. *Element* (3.9) *belongs to the subset* UpU^*.

For the proof, it suffices to observe that this statement follows from [152, Proposition 3.1]: for any two projections $r_1, r_2 \in A \otimes K(H)$, there exists a unitary $w \in GK(H_A)$ such that $wpw^* = (p - r_1) \oplus r_2$.

By virtue of these lemmas, we obtain the basic correspondence giving the desired isomorphism at the level of fundamental groups. Below we assume that the base point of GB_r is the identity and the base point of UPU^* is p.

Proposition 3.3. *The maps defined by the relations*

$$u \cdot GB_r(H_A) \mapsto [(p - u_{12}u_{12}^*) \oplus u_{21}u_{21}^*]_{UpU^*} \mapsto [u_{21}u_{21}^*] - [u_{12}u_{12}^*] \in K_0(A) \tag{3.11}$$

are the bijections inducing the isomorphisms

$$\pi_0(GB_r)(= GB_r(H_A)/GK(H_A)) \cong \pi_0(UpU^*) \cong K_0(A). \tag{3.12}$$

The results concerning the calculation of higher homotopy groups can be formulated now as follows (cf. [152]).

Proposition 3.4. *For any natural n, one has the isomorphisms*

$$\pi_{2n+1}(GB_r(H_A)) \cong \pi_{2n+1}(UpU^*) \cong K_1(A), \tag{3.13}$$

$$\pi_{2n+2}(GB_r(H_A)) \cong \pi_{2n+2}(UpU^*) \cong K_0(A). \tag{3.14}$$

These isomorphisms can be verified by means of the long exact sequence of homotopy groups associated with a natural operator fibration over UpU^* with the contractible total space $U_\infty(A)$, which is, as above, the group of unitaries in the unitization of $A \otimes K(H)$.

For this, we consider the map defined by $u \mapsto upu^*$. Clearly, its fibers are isomorphic to the commutant of p in U, i.e., $(p')_U = \{u \in U_\infty(A) : \; up = pu\}$. It is also easy to verify that this map is a submersion and, according to an infinite-dimensional generalization of the Ehresmann theorem [47], defines a locally trivial fibration with the fiber p'.

The long exact homotopy sequence of this fibration breaks, as usual, into short exact sequences:

$$0 \to \pi_{k+1}(UpU^*) \to \pi_k(p') \to \pi_k(U_\infty(A)) \to 0. \tag{3.15}$$

Since the homotopy groups of the stabilized unitary group $U_\infty(A)$ are isomorphic to the K-groups of A, these exact sequences immediately imply that $\pi_{2n+2}(UpU^*) \cong K_0(A)$ and $\pi_{2n+1} \cong K_1(A)$. Recalling that UpU^* is weakly homotopy equivalent to $GB_r(H_A)$, we obtain the desired conclusion.

Now Theorem 3.2 becomes an immediate consequence of Propositions 3.3 and 3.4.

We make a few comments on the formulations and proofs of the corollaries.

Corollary 3.1 is simply a special case of Theorem 3.2, where $A = C(S^1)$ is the algebra of continuous functions on the unit circle, which is clear from the interpretation of Riemann–Hilbert problems given in [22]. By a similar reasoning, Corollary 3.3 follows from Corollary 3.2.

Corollary 3.2 can be derived from Theorem 3.2 by using the scheme of [79], where the same result for classical singular integral operators on closed contours was derived from the solution of Bojarski's original problem. To do this, we need to clarify which one of several possible definitions of abstract singular operators (cf. [21, 119]) is actually appropriate in our setting.

We use a modification of the approach in [119] (cf. [79]). Fix an invertible operator $U \in GB(H_A)$ with the following properties:

(1) both operators U and U^{-1} have spectral radii equal to 1;

(2) there exists a projector $p \in GB(H_A)$, $p \sim \mathrm{Id} \sim \mathrm{Id} - p$ such that

$$Up = pUp, \quad Up \neq pU, \quad pU^{-1} = pU^{-1}p; \tag{3.16}$$

(3) $\operatorname{coker}(U \mid \operatorname{im} p)$ is an A-module of finite rank.

There are many such operators. For example, one can take the right shift in a Hilbert A-module and the projector on the "positive halfspace" (these are the abstract counterparts of multiplication by the independent variable and the Hardy projector from the theory of classical singular integral operators [119]). Denote by $R(U)$ the C^*-subalgebra generated by U and U^{-1}. It is easy to verify that for any $T \in R(U)$, the commutator $[T, p] = Tp - pT$ is compact, i.e., $[T, p] \in K(H_A)$.

Moreover, information about A-Fredholm operators contained in diagram (3.6) allows one to apply the arguments from [119] and obtain a description of invertible elements in $R = R(U)$.

Proposition 3.5. *Invertible operators are dense in* $R(U)$ *and characterized by the following condition: at least one of their restrictions on* $\operatorname{im} p$ *or* $\operatorname{im}(\mathrm{Id} - p)$ *is a semi-Fredholm operator.*

Following [119], any operator of the form

$$T = Lp + Mq + C, \tag{3.17}$$

where $q = 1 - p$, L, $M \in R(U)$, and $C \in K(H_A)$, is called an abstract singular operator over A (associated with the pair (U,p)). Their collection is denoted by $S(U)$.

This is a true generalization of the usual singular operators which are obtained when $A = \mathbb{C}$, U is the unitary operator of multiplication by an independent variable in $H = L_2(S^1)$, and p is the Hardy projector (for details, see [119]).

A standard application of the Gelfand spectrum theory provides symbols of singular operators which are functions on the spectrum of U. Assuming that U is unitary, we see that with any operator T of the form (3.17), one can naturally associate a pair of continuous functions $h(T) = (h(L), h(M))$ on the unit circle. A symbol is called nondegenerate if both its components are nowhere vanishing on S^1. As usual, the index $\operatorname{ind} h(T)$ of such a nondegenerate symbol is defined as the difference of argument increments of its components along S^1. Thus, we can now formulate the key characterization of elliptic singular operators.

Proposition 3.6. *An operator* $T \in S(U)$ *of the form* (3.17) *is a Fredholm operator if and only if its symbol is nondegenerate, i.e., both its coefficients* L *and* M *are invertible operators.*

After the above preparations, the proof runs in complete analogy with that in [119]. To calculate $\pi_*(S(U))$ over A, one has only to calculate the homotopy groups of pairs of invertible operators in $GR(U)$. The latter group is homotopy equivalent to $GB_r(H_A)$ with $\pi_+ = p$; the answer is provided by Theorem 3.2. Adding the groups from the latter theorem to the homotopy groups of nondegenerate symbols calculated in [76], one obtains Corollary 3.2.

Finally, Corollary 3.4 can be obtained from Corollary 3.2 by using the scheme of [76], where this was done for the classical counterparts of our results. However, this requires a lot of technical preparation. In particular, one needs to generalize the tensor-product construction of conventional bisingular operators from the algebra of pseudodifferential operators on the unit circle (see [44]). These technicalities are rather tedious and require a separate presentation.

Note that in the geometry of Hilbert C^*-modules, there are some related topics which admit a nice presentation in terms of special Grassmanians and transmission problems. Part of these results can be found in [102, 152].

Here we discuss only one topic most closely related to the geometric study of elliptic transmission problems [21, 120]. The point is that Theorem 3.2 suggests that there should exist a finer geometric structure of the Grassmanian $\mathrm{Gr}_+(H_A)$ expressed in terms of a stratification similar to the Birkhoff stratification by partial indices of invertible matrix-valued functions on the unit circle [20, 66], which plays a prominent role in the classical theory of transmission problems [60, 140].

Such a stratification can be constructed by using the geometric language developed in this paper. For this, let us fix a path component Gr_γ of the Grassmanian $\mathrm{Gr}_+(H_A)$ corresponding to a certain element $\gamma \in K_0(A)$. By Proposition 3.3, it is clear that γ is essentially the Fredholm index of the projection π_+ restricted to any element V of this component.

Since $K_0(A)$ is a group, it is reasonable to consider all pairs $(\alpha, \beta) \in K_0 \times K_0$, where $\alpha - \beta = \gamma$. For any pair, denote by $B_{\alpha,\beta}$ the subset of all V such that the following relations hold for classes in $K_0(A)$ (recall that any projective A-module generates a class in $K_0(A)$):

$$[\ker \pi_+ \mid V] = \alpha, \quad [\operatorname{coker} \pi_+ \mid V] = \beta. \tag{3.18}$$

Obviously, such a collection is a subset of the given component and one has

$$\mathrm{Gr}_\gamma = \bigcup B_{\alpha,\beta}. \tag{3.19}$$

The path component Gr_γ is arbitrary, and we obtain a natural decomposition of the special Grassmanian Gr_+ which is similar to the classical Birkhoff stratification [20, 66] (in fact, our decomposition is cruder, which can be seen in the case of classical transmission problems with respect to the unit circle). Of course, it is tempting to verify which properties of the Birkhoff stratification are still valid in our generalized setting and to generalize some of the results on its geometric structure obtained in the classical case [22, 120], but this topic remains uninvestigated.

We conclude with a purely geometric problem suggested by our constructions. It leads to nontrivial homological calculations even in the classical case [40, 57].

Recall that a complex analytic subset of a complex Banach manifold has a well-defined cohomological fundamental class in the cohomology of the ambient manifold [48]. A discussion of the orientation classes for K-theory in [49] shows that the same is valid for extraordinary cohomological theories, e.g., K-theory. Hence, fundamental classes of $B_{\alpha,\beta}$ are well defined, and there arises a problem of calculating them in terms of K-theory. As was mentioned, some results for the classical case were obtained in [40], but our knowledge of these fundamental classes is still very poor.

An intriguing open problem is to construct a finer analytic stratification of the special Grassmanian $\mathrm{Gr}_+(H_A)$ similar to that in [120] to obtain more topological invariants for transmission problems. There is some evidence that this should be possible for commutative A.

Our constructions and results can also be interpreted in terms of Fredholm structures over A. Granted diagram (3.6), the basic notions of this theory can be introduced as in [48]. By analogy with the discussion in Sec. 1, one can obtain an A-Fredholm structure on $\mathrm{Gr}_+(H_A)$ and try to determine some invariants of this structure.

4. Riemann–Hilbert Problems in Higher Dimensions

Since holomorphic functions can be considered as solutions to the Cauchy–Riemann system on the plane, a natural way of introducing multidimensional generalizations of the classical linear conjugation problems and Riemann–Hilbert problems is related with the consideration of elliptic first-order systems of differential equations with constant coefficients defined on Euclidean spaces of higher dimension [143,143]. For brevity, such systems are called *elementary elliptic systems* (EES).

Given such a system S, one can take a smooth domain D_+ in the source space of the system, choose a matrix-valued function G (of proper size) on the boundary ∂D_+, and find couples $X_\pm$ of solutions to the system S in the domains D_+ and D_- (the complement of D_+) satisfying the linear conjugation condition of the form (1.1). This gives a natural analogue of the classical linear conjugation problem.

In many cases, it is also important to consider a more general type of (local) boundary-value problems for EES, which, similarly to the one-dimensional case, are called Riemann–Hilbert problems for EES [79, 133]. Linear conjugation problems appear to be a special case of such problems [79, 133]. In a multidimensional setting, these generalized Riemann–Hilbert problems exhibit a more complicated and more interesting behavior than linear conjugation problems. In particular, a longstanding problem was to characterize EES that possess elliptic Riemann–Hilbert problems [134], while for linear conjugation problems, this issue is substantially simpler (see Theorem 4.8 below).

For these reasons, in the sequel our main focus is on generalized Riemann–Hilbert problems. They possess a number of remarkable properties and have gained considerable attention [18,32,114]. In particular, such problems for Euclidean Dirac operators play a significant role in the Clifford analysis [28,32,61].

In many situations, it is desirable that such problems could be described by Fredholm operators in appropriate functional spaces. The general theory of elliptic boundary-value problems indicates a natural approach to this topic [28, 143]. In particular, they can be reduced to a system of integral equations on the boundary [143] and this approach appeared quite effective in the case of linear conjugation problems for EES defining quaternionic regular functions [126].

However, this approach does not automatically lead to effective conditions of Fredholmness and there does not seem to exist a version of Fredholm theory for Riemann–Hilbert problems applicable in the case of an arbitrary EES. Actually, it is well known that such systems do not always possess local elliptic boundary-value problems [18] and some natural boundary-value problems fail to be Fredholm.

In line with the discussion in previous sections, we will only deal with Fredholm boundary-value problems for a class of especially well-behaved EES called *generalized Cauchy–Riemann systems* (GCRS), which were introduced by Stein and Weiss [132]. Even for such systems, the problem of describing those of them that possess elliptic Riemann–Hilbert problems is quite nontrivial and remained unsolved for a long time (for a comprehensive review of the topic, see [133]). A good understanding of this issue would open a way toward generalizing many results of previous sections to higher dimensions, so it is of major importance for our approach and we will consider it in some detail.

More precisely, we will study the Riemann–Hilbert problems for generalized Cauchy–Riemann systems. In order to provide a visual description of the class of GCRS's, we first present some basic results about the structure of such systems. The results presented below are scattered in several sources [61, 132–134], so bringing them together seemed to be reasonable.

It should also be noted that at present, there exist two approaches to the Fredholm theory of Riemann–Hilbert problems for GCRS. The first approach is direct and uses an explicit form of the Shapiro–Lopatinski condition for GCRS obtained in [134]. The second approach used in the author's papers [86, 88] is more sophisticated. It relies on some recent results about elliptic operators and K-theory [11, 68].

The direct approach permits a self-contained exposition of many aspects of Fredholm theory for GRHP's, and we basically present the main results available within the approach of [133, 134]. However, the second approach is indispensable in order to complete the list of GCRS's possessing elliptic GRHP (see Theorems 4.6 and 4.7 below), but it uses rather complicated topological machinery in the spirit of the K-homology approach to boundary-value problems [11, 68]. It would hardly be possible to give a reasonable exposition of this topic in a paper of such length, so we simply present the main results and briefly mention the related concepts.

To begin with, consider a general *homogeneous elliptic first-order system with constant coefficients* of the form

$$\sum_{j=0}^{n} M_j \frac{\partial w}{\partial x_i} = 0, \tag{EES}$$

where M_j are constant complex $(m \times m)$-matrices and w is a differentiable mapping from $\mathbb{R}^{n+1}$ to $\mathbb{C}^m$.

Definition 4.1 ([132])**.** If every solution w to system (EES) has only harmonic components w_j, $j = 1, \dots, m$, then (EES) is called a generalized Cauchy–Riemann system.

The ellipticity of such an (EES) becomes an easy consequence of results in [132].

Theorem 4.1. *Every generalized Cauchy–Riemann system of the form* (EES) *is elliptic, i.e.,*

$$\det\left(\sum_{j=0}^{n} \lambda_j M_j\right) \neq 0$$

for all $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

Indeed, assume the contrary; then there exist two vectors $\lambda \neq 0$ and $\nu \neq 0$, $\lambda, \nu \in \mathbb{C}^m$, such that

$$\left(\sum_{j=0}^{n} \lambda_j M_j\right)\nu = 0.$$

From this, one can conclude by straightforward calculations that the vector-valued function

$$w(x) = \left(\exp \sum_{j=0}^{n} \lambda_j x_j\right)\nu$$

is a nonharmonic solution and, therefore, (EES) is not a generalized Cauchy–Riemann system.

Remark 4.1. If (EES) is elliptic, then every matrix M_j is necessarily invertible. Premultiplying system (1.1) by M_0^{-1}, we obtain

$$E \frac{\partial w}{\partial x_0} + A_1 \frac{\partial w}{\partial x_1} + \dots + A_n \frac{\partial w}{\partial x_n} = 0, \tag{4.1}$$

where $A_j = M_0^{-1} M_j$, $j = 0, \dots, n$, $A_0 = E =$ identity.

In the sequel, we always consider, without loss of generality, generalized Cauchy–Riemann systems which are already of the form (4.1).

The following two theorems characterize GCRS by the properties of their coefficient matrices.

Theorem 4.2 ([133])**.** *Let system* (4.1) *be a generalized Cauchy–Riemann system; then the coefficient matrices satisfy the relations*

$$\begin{gathered} A_j^2 = -E, \quad j = 1, \dots, n, \\ A_i A_j + A_j A_i = 0, \quad i, j = 1, \dots, n, \quad i \neq j. \end{gathered} \tag{4.2}$$

Proof. The proof is direct and instructive; we present an outline of it. Setting $A_0 = E$, consider the function

$$w(x) = 2x_i x_j b - (x_i^2 A_i^{-1} A_j + x_j^2 A_j^{-1} A_i) b, \quad i \neq j,$$

where b is an arbitrary vector in $\mathbb{C}^m$. Then w is a solution of (4.1) since we have

$$\begin{aligned} E \frac{\partial w}{\partial x_0} + A_1 \frac{\partial w}{\partial x_1} + \dots + A_n \frac{\partial w}{\partial x_n} &= A_i \frac{\partial w}{\partial x_i} + A_j \frac{\partial w}{\partial x_j} \\ &= A_i(2x_j - 2x_i A_i^{-1} A_j) b + A_j (2x_i - 2x_j A_j^{-1} A_i) b = 0. \end{aligned}$$

Furthermore, one has

$$\Delta w = -2(A_i^{-1}A_j + A_j^{-1}A_i)b.$$

Assume that (4.1) is a generalized Cauchy–Riemann system; then $\Delta w = 0$. This and the fact that b can be chosen arbitrarily yield

$$A_i^{-1}A_j + A_j^{-1}A_i = 0, \quad i, j = 0, \dots, n, \quad i \neq j. \tag{4.3}$$

Setting $i = 0$, we obtain

$$A_j^{-1} = -A_j, \quad A_j^2 = -E, \quad j = 1, \dots, n.$$

This combined with (4.3) yields

$$A_iA_j + A_jA_i = 0, \quad i, j = 1, \dots, n, \quad i \neq j.$$

The theorem is proved. □

Theorem 4.3 ([133]). *Let w be a solution to system* (4.1) *belonging to the Sobolev space $W_2^1(G)$ and let the coefficient matrices of* (4.1) *satisfy the relations*

$$A_iA_j + A_jA_i = -2\delta_{ij}E, \quad i, j = 1, \dots, n. \tag{4.4}$$

Then w is a harmonic vector in the domain G.

Proof. For a twice continuously differentiable solution w, we have

$$\left(E\,\frac{\partial}{\partial x_0} - \sum_{j=1}^{n} A_j\,\frac{\partial}{\partial x_j}\right)\left(E\,\frac{\partial}{\partial x_0} - \sum_{j=1}^{n} A_j\,\frac{\partial}{\partial x_j}\right)w(x) = 0.$$

A formal calculation using (4.4) shows that the second-order operator is just the Laplacian and, therefore, $w \in C^2(G)$ must be harmonic. Now, assuming that $w \in W_2^1(G)$, we can use the Weyl lemma [143] in a standard way to complete the proof. □

Remark 4.2. The above algebraic relations are of course the famous relations for the generators of a Clifford algebra Cl_n [61]. The generalized Cauchy–Riemann systems are often defined by postulating algebraic relations (4.4); therefore, they can be interpreted in the context of Clifford analysis [32]. Thus, the A_j's are often considered not as matrices but as so-called *hypercomplex units* (generators of a Clifford algebra). We work here exclusively with matrices since this permits us to apply the same considerations to more general systems. Hypercomplex systems can be transformed into matrix notation by writing every hypercomplex component as a single equation. The dimension of the obtained coefficient matrices always equals a power of two.

Thus, in the sequel, we deal with systems of the form

$$\begin{gathered} E\,\frac{\partial w}{\partial x_0} + A_1\,\frac{\partial w}{\partial x_1} + \dots + A_n\,\frac{\partial w}{\partial x_n} = 0, \\ A_iA_j + A_jA_j = -2\delta_{ij}E, \end{gathered} \tag{4.5}$$

where A_j are constant complex $(m \times m)$-matrices and w is a vector-valued function with values in $\mathbb{C}^m$.

In fact, by taking a closer look at the coefficient matrices, one can specify all generalized Cauchy–Riemann systems explicitly. A natural way to do this is to refer to the representation theory of Clifford algebras [61]. Indeed, it is easy to see that such matrices A_i define a representation of a Clifford algebra Cl_n of an n-dimensional vector space V [61]. This interpretation of matrices A_i leads to an explicit description of their structure since representations of Clifford algebras are completely known.

We do not reproduce here the general definition of a Clifford algebra [61]. For our purposes, it is sufficient to recall that by choosing an orthonormal basis e_i in V, one can describe Cl_n as the associative algebra generated by

$$e_0 = 1,\ e_1,\ e_2,\ \dots,\ e_n$$

with the following properties:

$$\begin{array}{ll}\text{(a)} & e_j^2=-1, \quad j=1,\dots,n, \\ \text{(b)} & e_ie_j=e_je_i=0, \quad i\neq j, \quad i,j=1,\dots,n, \\ \text{(c)} & (e_ie_j)e_k=e_i(e_je_k).\end{array} \tag{4.6}$$

Generators e_i are sometimes called hypercomplex units [61]. Obviously, each product of generators can be (up to a sign) transformed to one of the following expressions:

$$1,\ e_1,\ \dots,\ e_n, \quad e_1e_2,\ e_1e_3,\ \dots, \quad e_1e_2e_3,\ \dots, \quad e_1e_2\cdots e_n. \tag{4.7}$$

These products are linearly independent and provide a basis for Cl_n and for this reason, we sometimes say that Cl_n is generated by hypercomplex units e_i.

Thus, the matrices A_j, $j=1,\dots,n$, in system (4.5) can be considered as a generating system of Cl_n since they satisfy relations (4.6). One can obtain an explicit description of matrices A_i by using the well-known results on representations of Clifford algebras [61].

First, it is known that each representation of a Clifford algebra is completely reducible [61]. As is well known, when considering the representations of Cl_n, it is reasonable to distinguish between two cases depending on whether n is even or odd.

Proposition 4.1 ([61])**.** *There exists exactly one irreducible representation of Cl_{2k} of degree 2^k. Every representation of Cl_{2k} is exact and its degree is a multiple of 2^k.*

Proposition 4.2 ([61])**.** *There exist exactly two nonequivalent representations of Cl_{2k+1} of degree 2^k. A representation of Cl_{2k+1} is exact if and only if its decomposition into the sum of irreducible representations contains both irreducible representations at least once. The degree of each representation is a multiple of 2^k.*

These results can be used to obtain an explicit description of the shape of the matrices A_k. One can also answer the following question: what is a way of finding out how many times each of the irreducible representations is contained in any given representation of the Clifford algebra Cl_{2k+1}. An answer to this question will allow us to determine the discrete invariants of a given GCRS.

A solution can be given in terms of characters of representations [1]. Recall that the character of representation D is defined as the function

$$\chi(s)=\operatorname{trace}D(s).$$

As is well known, characters allow one to distinguish between nonequivalent representations [1]. Let us use characters to determine the number of irreducible components into which a representation of Cl_{2k+1} can be split.

First, note that the base element $e_1e_2\cdots e_n$ is mapped through the first irreducible representation to the matrix $\imath^{k+1}E$ and through the second one to the matrix $-\imath^{k+1}E$.

In the first case, the trace equals $\imath^{k+1}\cdot 2^k$ and in the second case, it equals $-\imath^{k+1}\cdot 2^k$. Now let D be a representation of Cl_{2k+1} of degree $m=2^k\cdot l$. The first irreducible representation may appear in $D(s)$ l_1 times and the second one l_2 times. Then the following relations are valid for the unknown l_1 and l_2:

$$l=l_1+l_2, \quad \operatorname{trace}D(e_1e_2\cdots e_n)=(l_1-l_2)\cdot\imath^{k+1}\cdot 2^k.$$

From this, we can directly obtain l_1 and l_2. Hence, we see that it is sufficient to calculate the trace of a single matrix to identify the representation D of degree $2^k\cdot l$ of Cl_{2k+1}.

Thus, we can describe all representations of the Clifford algebra Cl_n and the way to their identification. Therefore, we know in principle the explicit shapes of all generalized Cauchy–Riemann systems. For the sake of visuality, we write down some examples of such systems in low dimensions.

For $n = 1$, we have the classical Cauchy–Riemann system for two real functions $u(x, y), v(x, y)$ of two real variables:

$$u_x - v_y = 0,$$
$$u_y + v_x = 0.$$

For $n = 2$, the corresponding irreducible systems for four real functions s, u, v, and w of three real variables is the well-known Moisil–Theodoresco system, which can be written by using standard operators on vector-valued functions in $\mathbb{R}^3$ [61]:

$$\operatorname{div}(u, v, w) = 0, \quad \operatorname{grad} s + \operatorname{rot}(u, v, w) = 0.$$

For $n = 3$, the corresponding irreducible system coincides with the so-called *Fueter system* for four real functions f_i of four real variables x_j [61]:

$$\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0,$$
$$\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0,$$
$$\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} = 0,$$
$$\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = 0.$$

As is well known, the latter system is a natural counterpart of the Cauchy–Riemann system for a function of one quaternionic variable. Its solutions, called quaternionic regular (or monogenic) functions, have many interesting properties similar to those of the usual holomorphic functions of one complex variable [61,118,126]. In particular, the main result of [118] provides remarkable insight into the structure of monogenic functions of several quaternionic variables. A brief discussion of this result is presented at the end of the section.

Actually, each GCRS gives rise to a nontrivial function theory similar to the theory of holomorphic functions, which is the sample theory associated with the classical Cauchy–Riemann system. For example, basic properties of solutions to the Moisil–Theodoresco system were established in [114]. A general approach to the function-theoretic study of solutions to GCRSs can be found in [32, 61].

Before passing to boundary-value problems, let us mention another important property of generalized Cauchy–Riemann systems. We know that the classical two-dimensional Cauchy–Riemann system is invariant with respect to rotations of the coordinate system, i.e., if $D = (d_{ij})$ is an orthogonal matrix and $w(x)$ represents a holomorphic function, then $w(Dx)$ is also holomorphic. Generalized Cauchy–Riemann systems possess a similar property. If $w(x)$ is a solution and D an orthogonal transformation, then $Mw(Dx)$ is also a solution, where M is a certain nonsingular matrix depending on D.

Remark 4.3. Stein and Weiss defined generalized Cauchy–Riemann systems by the property of rotational invariance [132]. In this way, they obtained the same systems as presented above.

The development of functional-analytic methods for elliptic first-order systems has shown that a big part of the classical two-dimensional theory can be generalized to higher-dimensional GCRS [32, 61]. Unfortunately, this is no longer the case if one tries to study boundary-value problems for such systems. In principle, the Plemelj–Sokhotsky formulas can be used to transform boundary-value problems into systems of singular integral equations, but this does not lead to any effective criteria of solvability or Fredholmness.

Recently, Stern [133] succeeded in obtaining an explicit criterion of Fredholmness by using some general results of the theory of elliptic boundary-value problems in Sobolev spaces [143]. This allowed her to obtain an extensive list of GCRS possessing elliptic boundary-value problems of Riemann–Hilbert type [134]. Recall that the problem of deciding which GCRS possess elliptic local boundary-value problems attracted considerable attention in the last three decades [10,18]. The progress achieved by Stern allowed the author to guess the topological mechanism which caused the existence of elliptic Riemann–Hilbert problems, which eventually led to further progress in this topic [85].

Below, we present the main results of [85,134], which give a nearly complete list of GCRSs possessing elliptic RHPs. In these papers, boundary-value problems were basically studied which were slightly different from the linear conjugation problems which are the focus of interest in the present paper, but they were also called Riemann–Hilbert problems. As was explained in [85], the linear conjugation problems for GCRS appear as a special case of the boundary problems studied in [85,134]. Actually, the problem of the existence of elliptic boundary-value problems is less interesting for linear conjugation problems since there always exist elliptic problems of that kind, and below we accept the setting of [85,134].

Let us now describe the boundary-value problems we wish to deal with. For this, let us consider a generalized Cauchy–Riemann system

$$E\,\frac{\partial w}{\partial x_0} + A_1\,\frac{\partial w}{\partial x_1} + \cdots + A_n\,\frac{\partial w}{\partial x_n} + Dw = f \tag{4.8}$$

in a domain G with complex $(m \times m)$-coefficient matrices A_j, $j = 1, \ldots, n$, satisfying the relations

$$A_i A_j + A_j A_i = -2\delta_{ij} E, \quad i, j = 1, \ldots, n.$$

Note that, in contrast to Definition 4.1, we have added a lower-order term with a view toward greater generality. This is reasonable since the function-analytic methods used in the sequel are related essentially with the principal part of the operator. From now on, we consider the inhomogeneous system of differential equations since it is intrinsically involved in the definition of the Fredholm property.

Together with the system of differential equations, we investigate a boundary-value problem arising by imposing boundary conditions of the form

$$(B_1 B_2) \cdot w = g \quad \text{on } \partial G, \tag{4.9}$$

where B_1 and B_2 are complex matrices of dimension $m/2$ depending on boundary points and g is a vector-valued function with values in $\mathbb{C}^{m/2}$. This always makes sense since, as we have seen, m is always even for a GCRS.

In accordance with the terminology of [85,133,134], problem (4.8), (4.9) is called a Riemann–Hilbert problem for given GCRS (4.8).

Note that an equivalent problem could be formulated in the form

$$\operatorname{Re}[Cw] = h \quad \text{on } \partial G, \tag{4.10}$$

where C is a complex $(m \times m)$-matrix and h is a vector in $\mathbb{R}^m$. As is easy to see, both definitions of boundary-value problem can be transformed to each other.

We always assume that the rows of the matrix $(B_1(x)\ (B_2(x))$ are linearly independent in every boundary point $x \in \partial G$. Then they can be orthonormalized by a formal application of the Schmidt algorithm; this does not change the regularity (continuity, smoothness) of the matrix elements. Therefore, we can always consider boundary conditions with orthonormal rows.

The following theorem gives an explicit criterion of Fredholmness for Riemann–Hilbert problems, i.e., under this condition, the kernel and cokernel of the problem have finite dimensions and the image is closed.

Theorem 4.4 ([134]). *Let $G \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain of class C^k, $k \geq 1$. Consider the Riemann–Hilbert boundary-value problem*

$$E\,\frac{\partial w}{\partial x_0} + A_1\,\frac{\partial w}{\partial x_1} + \cdots + A_n\,\frac{\partial w}{\partial x_n} + Dw = f \quad \textit{in } G, \tag{4.11}$$

$$A_i A_j + A_j A_i = -2\delta_{ij} E, \quad i, j = 1, \ldots, n,$$

$$(B_1, B_2) \cdot w = g \quad \textit{on } \partial G, \tag{4.12}$$

and assume that the $(m \times m)$*-matrices* A_j *are unitary*:

$$A_j^* = -A_j, \quad j = 1, \dots, n.$$

Let the rows of the $(m/2 \times m)$*-matrix* (B_1, B_2) *be orthonormal. Furthermore, assume that* $D \in C^{k-1}(\overline{G})$ *and* $B_1, B_2 \in C^k(\partial G)$.

Then the Riemann–Hilbert boundary-value problem (4.11), (4.12) *is a Fredholm problem for* $w \in W_2^l(G)$, $f \in W_2^{l-1}(G)$, *and* $g \in W_2^{l-1/2}(\partial G)$, $1 \le l \le k$, $l \in \mathbb{Z}$, *if and only if the following relation holds for all* $x \in \partial G$:

$$\det\left[(B_1 B_2)\left(\sum_{j=0}^{n} \alpha_j A_j^*\right)\left(\sum_{j=0}^{n} t_j A_j\right)\begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} - iE\right] \neq 0, \tag{4.13}$$

where $\alpha = (\alpha_0, \dots, \alpha_n)$ *denotes the vector of the inner normal at a point* $x \in \partial G$ *and* $t = (t_0, \dots, t_n)$ *runs over all unit vectors which are tangent to* ∂G *at the same point* x.

The above results about the structure of GCRS together with Theorem 4.4 allowed Stern to show that certain GCRS do not possess any elliptic boundary-value problems of Riemann–Hilbert type. We present the main result of [134] without proof, which is quite technical and lengthy. Actually, below we present a more precise result from [85], which is more relevant for describing the state of the art in this issue.

Note that the proof in [134] possesses an amazing remarkable feature. Using the above criterion of Fredholmness, we can obtain the desired conclusion from a famous result of Adams, Lax, and Phillips about the amount of matrices of given size such that all their real combinations are nonsingular [2].

Theorem 4.5. *If the space dimension is even* (*i.e.,* $n = 2k + 1$ *is odd*) *and the representation of* Cl_n *defined by the coefficient matrices* A_i *contains an odd number of irreducible components* (*i.e., its degree equals* $m = 2^k l$*, where* l *is odd*)*, then it does not possess any elliptic Riemann–Hilbert problems.*

Stern also proved that in $\mathbb{R}^4$, there exist no elliptic Riemann–Hilbert problems for GCRSs which contain only one of two possible irreducible representations. On the other hand, it was shown in [134] that for an odd n, elliptic Riemann–Hilbert problems exist if both irreducible representations appear in the same amount. It should be noted that these results covered all results of this type, which were previously proved for various concrete systems (a review of those concrete results is contained in [134]).

Analyzing these results of Stern, the author noted that they find a nice explanation in terms in the modern approach of K-homology developed by Baum, Douglas, and Taylor [11]. More precisely, it turned out that for GCRS, one can calculate the K-homological obstruction to the existence of elliptic boundary-value problems suggested in [11] and determine all cases where it is vanishing. Combining this fact with some recent results of Gong [68], it became possible to show that these (and only these) GCRS possess elliptic Riemann–Hilbert problems.

The main results of [85] can be formulated in terms of the discrete invariants of GCRS, which are yielded by a Clifford-algebra interpretation described above. Recall that each such system is characterized by natural numbers n and m. One can equivalently substitute m by the number l of irreducible components in the associated representation of the Clifford algebra Cl_n. For odd $n = 2k + 1$, there also appear multiplicities l_1 and l_2 of each of the two irreducible representations of Cl_n (i.e., $m = 2^k l$, $l = l_1 + l_2$). In the following formulations, we assume that the domain in which we consider RHPs is smooth, convex, and contains the origin.

Theorem 4.6 ([85])**.** *If* n *is odd and* $l_1 \neq l_2$*, then there do not exist elliptic Riemann–Hilbert problems for a given GCRS. If* $l_1 = l_2$*, then elliptic Riemann–Hilbert problems do exist.*

Theorem 4.7 ([79])**.** *If* n *is even,* $n \neq 4, 6$*, then there exist elliptic boundary-value problems with a boundary condition of the form* $(B \circ r)w = g$*, where* B *is a zero-order pseudo-differential operator between* *sections of appropriate Hermitian bundles on the boundary and r is the usual trace* (*restriction to the boundary*) *map.*

The proofs of these results use rather sophisticated concepts and techniques developed in the framework of *analytic K-homology theory* [11]. Presenting such proofs in a readable form certainly requires a separate detailed exposition, and we only outline the idea of the proofs.

First, recall a few basic notions of K-theory. Let X be a compact oriented smooth manifold with a smooth boundary $Y = \partial X$. We consider smooth complex Hermitian vector bundles E_0 and E_1 over X and an elliptic first-order differential operator $T : C^\infty(E_0) \to C^\infty(E_1)$ from smooth sections of E_0 to those of E_1. Denote by $H^s(X, E_i)$ and $H^r(Y, E_i)$ the Sobolev spaces of sections of E_i and $E_i|Y$ with respect to fixed smooth measures on X and Y. This framework is currently standard and its detailed descriptions can be found in [11, 28].

It is well known that there exist topological obstructions for the existence of classical (local) elliptic (i.e., satisfying the Shapiro–Lopatinsky condition) boundary-value problems for such T [28]. The fundamental problem of describing all such obstructions eventually found an adequate interpretation in the language of K-theory, and this led to important progress in this classical topic.

More precisely, it is the K-homology functor K_* which is relevant here. It can be introduced as an extraordinary homology theory dual to the classical (cohomological) K-functor [19] and has a natural description in terms of the Kasparov bifunctor [74]. Of the crucial importance is a geometric interpretation of relative K-homology groups $K_*(X, Y)$ developed in [11] which gives rise to K-homology exact sequences such as the following fragment:

$$K_0(Y) \to K_0(X) \to K_0(X, Y) \to K_1(Y) \to K_1(X),$$

where the third arrow represents the connecting homomorphism $\partial : K_0(X, Y) \to K_1(Y)$ constructed in [11].

The basic properties of the connecting homomorphism ∂ were established in [11]. Moreover, in the same paper, it was shown that with any elliptic first-order differential operator T as above, one can associate a relative K-homology class $[T] \in K_0(X, Y)$ such that the existence of local elliptic boundary-value problems (such as RHPs in our setting) for T implies that $[T]$ comes from $K_0(X)$, i.e., lies in the image of this group under the natural homomorphism. In particular, in this case one necessarily has $\partial[T] = 0$.

Thus, the latter relation gives a necessary condition for the existence of local elliptic boundary-value problems, which may be called the BDT-condition. Our proof of Theorem 4.6 relies on identifying these classes for GCRSs and showing that in certain cases, they do not satisfy the BDT-condition. We can calculate these classes by using the explicit description of $[D]$ for a Dirac operator obtained in [11].

It was conjectured by Baum and Douglas that the BDT-condition is actually a criterion for the existence of elliptic BVPs. This conjecture was eventually shown to be very close to the truth. Namely, Gong [68] proved that if $\partial[T] = 0$ and $\dim X \neq 4, 5, 6, 7$, then there exist a complex vector bundle E_2 over Y and a zero-order pseudo-differential operator $B : C^\infty(Y, E_0) \to C^\infty(Y, E_2)$ such that the extended operator

$$(T \oplus B \circ r) : H^1(X, E_0) \to H^0(X, E_1) \oplus H^{\frac{1}{2}}(Y, E_2)$$

is a Fredholm operator, where $r : H^1(X, E_0) \to H^{\frac{1}{2}}(Y, E_0)$ is the restriction map.

Theorem 4.7 relies on this remarkable result of Gong. Using the aforementioned description of $[D]$ for Dirac operators obtained in [11], we can show that the BDT-obstruction vanishes for GCRSs having discrete parameters as specified in Theorem 4.7. Despite the simplicity of this scheme, its realization requires dealing with a lot of auxiliary technical results which are far beyond the topics considered in this paper.

It should be added that for even n, one cannot show the existence of elliptic Riemann–Hilbert problems as they were defined above. This issue is important for our approach to *hyperholomorphic cells* discussed in the last section. In fact, this can be shown for many concrete systems (in particular, for "self-conjugate systems" appearing in the last section), and the author's feeling is that elliptic Riemann–Hilbert problems exist for all even n.

As to linear conjugation problems for GCRS, the situation is much simpler and this can be established using the same ideas and techniques.

Theorem 4.8. *Elliptic linear conjugation problems with respect to a smooth convex domain exist for all generalized Cauchy–Riemann systems.*

The proof of this result can be achieved by showing that linear conjugation problems can be reduced to GRCSs such that the obstruction for existence of elliptic BVPs vanishes. For example, in the case where the domain is the unit ball $B \subset \mathbb{R}^{n+1}$, one can easily transform a linear conjugation problem to a Riemann–Hilbert problem for a GCRS of double size. This can be done by means of substituting the outer component X_- by the vector-valued function Y_+, which is the inversion of X_- in the unit sphere ∂B.

Then it is easy to verify that the new unknown vector X_+, Y_+ satisfies a GCRS S of double size. Since the new system is of a very special kind, we can show that the Baum–Douglas obstruction vanishes for this system. Indeed, as follows from [11], an inversion changes the sign of the Baum–Douglas obstruction; therefore, the obstruction for both subsystems for the components of X_+ and Y_+ are inverse to each other in $K_1(\partial B)$ and the total obstruction vanishes. Hence the system S possesses elliptic Riemann–Hilbert problems, which can be transformed back to give elliptic linear conjugation problems for the original GCRS.

All this can be worked out in detail illustrated for a concrete GCRS. Linear conjugation problems for the Moisil–Theodoresco system were considered in [18, 114], and one can easily verify that results of [18, 114] fit the above picture. Considerable progress in understanding RHPs and LCPs for the Fueter system was achieved by Shapiro and Vasilevsky [126]. It seems worth mentioning that the results of [126] allow one to develop a sufficiently complete theory of *one-dimensional* RHPs in a *quaternionic setting* (i.e., for monogenic functions of one *quaternionic* variable), which can be considered as a direct generalization of the classical scalar RHPs for holomorphic functions of one *complex* variable.

In other words, one takes a nowhere vanishing function $g : S \to \mathbb{H}$ and looks for pairs of monogenic functions $u_\pm : B_\pm \to \mathbb{H}$, $u_-(\infty) = 0$, satisfying the (left) conjugation condition $u_+(q) = g(q)u_-(q)$ on S, where S is the three-dimensional sphere of unit quaternions and B_+ and B_- are the unit ball and its complement, respectively. Similarly, one can consider the right conjugation condition of the form $u_+(q) = u_-(q)g(q)$. Of course, both these problems appear as particular cases of the RHPs considered above.

Using results of [126], one can obtain criteria of ellipticity for quaternionic RHPs which imply that there are plenty of elliptic linear conjugation problems for the Fueter system in complete accordance with Theorem 4.8. Moreover, these results appear useful for obtaining elliptic RHPs for a double Fueter system, which is related to the construction of admissible targets for monogenic cells in Sec. 6.

These results can be developed in several directions. For example, one can try to write down an explicit formula for the Fredholm index of an elliptic RHP in terms of the boundary condition. Such formulas would be, of course, just particular cases of the Atiyah–Singer index formula, but their explicit form can be useful for further investigation of GCRSs. One can also try to explicate our results for concrete GCRSs using results of [18, 114, 126, 134], which, in particular, could lead to effective algorithms of checking ellipticity for concrete boundary-value conditions.

Another interesting possibility is related with generalizing the so-called *Riemann–Hilbert problems for holomorphic functions of several complex variables* considered by Begehr and his coauthors [12–14]. More precisely, by a straightforward generalization of the formulations used in [13], one obtains a formulation of the Riemann–Hilbert problem for polymonogenic (separately monogenic) functions of several quaternionic variables in product domains analogous to the polydiscs used in [13].

Obviously, one needs a good understanding of the structure of polymonogenic functions to investigate such problems. Palamodov recently obtained an important result for the Fueter system [118]. With a view toward possible applications to RHPs, we now describe the main result of [118].

It turns out that it is more convenient to consider the Fueter system or functions with values in *biquaternions* (complexified quaternions) [61]. Denote by $\mathbb{B} = \mathbb{H}_{\mathbb{C}}$ the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, where the algebras $\mathbb{R}$ and $\mathbb{H}$ are considered as extensions of the field $\mathbb{R}$. Note that the Fueter system $\partial u/\partial q = 0$ can also be written for $\mathbb{B}$-valued functions. Moreover, if u is a function in an open set $U \subset \mathbb{H}^n \cong \mathbb{R}^{4n}$ with values in $\mathbb{B}$, then one can write the following system of equations with respect to quaternionic variables $q_1, \dots, q_n$:

$$\frac{\partial u}{\partial q_1} = 0, \ \dots, \ \frac{\partial u}{\partial q_n} = 0.$$

Solutions of this system are called *polymonogenic functions* [118]. It turns out that they can be described in terms of a certain integration process over the variety of proper right ideals of $\mathbb{B}$ [118].

Recall that a right ideal R in the algebra $\mathbb{B}$ is called proper if $\{0\} \neq R \neq \mathbb{B}$. Denote by $R(B)$ the collection of all proper right ideals in $\mathbb{B}$. Considered as an algebraic variety, it becomes a 2-sphere with a canonical complex algebraic structure. Take the trivial bundle $\pi : \mathbb{B} \times R(B) \to R(B)$ and consider a subbundle $r : E \to r(B)$ of π such that the fiber of r over $R \in R(B)$ is equal to the ideal R ("tautological bundle").

For an open set $U \subset \mathbb{H}^n$, denote by $Q(U)$ the space of all polymonogenic functions $u : \mathbb{H}^n \times R(B) \to \mathbb{B}$ which are polymonogenic in fibers of the bundle π and by S the sheaf of germs of generalized functions $u : \mathbb{H}^n \times R(B) \to \mathbb{B}$ which are polymonogenic in fibers of the bundle π.

For such an open set U, consider the restriction of this sheaf to $U \times R(\mathbb{B})$. By setting $S(U) = \pi_*(S|U \times R(B))$, we obtain the sheaf of germs on $R(B)$ of generalized functions with values in $Q(U)$. Set $S(U)^{*,*} = S(U) \otimes_E E^{*,*}$, where $E^{p,q}$ denotes the sheaf of germs of smooth (p,q)-forms on $R(B) \cong \mathbb{C}P^1$, $p, q = 0, 1$, with values in $\mathbb{C}$, $E = E^{0,0}$.

From the trivial connection ∇ on π and the $\overline{\partial}$-operator in $E^{*,*}$, one obtains a sheaf morphism

$$\delta = \nabla(\overline{\partial}) : S(U)^{1,0} \to S(U)^{1,1}.$$

For $R \in R(B)$, denote by $Q(U,R)$ the space of R-valued polymonogenic functions in U. It has a natural algebra structure with respect to pointwise multiplication and the family of algebras $Q(U,R)$ defines a fiber bundle on $R(B)$. Denote by $S(U, R(B))$ the subsheaf of $S(U)$ of germs of generalized sections of this bundle of algebras and consider the sheaf of $Q(U, R(B))$-valued differential forms $S(U, R(B))^{*,*}$. This subsheaf is invariant under the operator δ since $R(B)$ is an analytic fiber bundle. The section space $\Gamma(R(B), S(U, R(B))^{*,*})$ is the space of forms on $R(B)$ with values in the bundle with fibers $Q(U,R)$.

After these preparations, it becomes clear that we have the sequence of linear spaces and mappings

$$0 \to \Gamma(R(B), S(U, R(B))^{1,0}) \to \Gamma(R(B), S(U,R)^{1,1}) \to Q(U) \to 0,$$

where the second arrow represents δ and the third arrow is defined by the integration operator

$$\omega = \int_{R(B)} \omega^{1,1}.$$

We can see that this sequence is a complex, and the main result of [118] yields that its homology is trivial.

Theorem 4.9 ([118])**.** *The above sequence is exact for any convex open $U \subset \mathbb{H}^n$.*

In particular, this sequence is exact for domain products of the form $B_1 \times \ldots \times B_n$, where B_i is a ball in $\mathbb{H}$. For each such domain, one can introduce Riemann–Hilbert problems of the type considered in [13] and investigate their solvability using Theorems 4.7 and 4.9. In particular, for $n = 2$, we can easily construct elliptic Riemann–Hilbert problems of such type. We will not discuss this topic since it deserves a special presentation. Our aim was simply to indicate another way of constructing multidimensional elliptic problems of Riemann–Hilbert type.

5. Nonlinear Transmission Problems for Analytic Functions

In course of a long historical development, the linear conjugation problem and Riemann–Hilbert problem became an "organizing center" for a number of important topics of complex analysis, differential equations, topology, operator theory, and nonlinear analysis. Most of these topics are developing quite actively and continue to suggest interesting new problems and interrelations. In particular, there exists a vast amount of literature devoted to nonlinear versions of the Riemann–Hilbert problem (see, e.g., [141]).

In this section, we discuss some nonlinear problems of such kind closely related to the concept of *analytic disc*, which plays a significant role in various modern topics of complex analysis and symplectic geometry [17, 53, 54, 69].

The problem we wish to study can be considered as a direct generalization of the transmission problem (1.1). One searches for two functions Φ_- and Φ_+, which are holomorphic in an interior and an exterior domain, respectively, but now they should satisfy a *nonlinear* coupling condition on the common boundary of the two domains. More precisely, we admit nonlinear conditions of the form

$$\Phi_+(t) = G(t, \Phi_-(t)). \tag{5.1}$$

From the operator-theoretic standpoint, the linear transmission problem is related to Toeplitz operators, i.e., to the interaction of *multiplication* with the Riesz projection, while the nonlinear problem concerns interaction of the Riesz projection with *superposition.* The operator-theory approach appears helpful also for dealing with nonlinear transmission problems.

Following [142], we introduce a special class of *nonlinear transmission problems* and obtain a rather complete description of their solutions. Moreover, we discuss some relations between nonlinear transmission problems and the existence problem for so-called *attached analytic discs* [17]. Since this subject is relatively recent, we only present some sample results without trying to reach the maximal possible generality.

More precisely, for a given continuously differentiable function $G : \mathbb{T}\times\mathbb{C} \to \mathbb{C}$, we consider a *nonlinear transmission problem*

$$\Phi_+(t) = G(t, \Phi_-(t)), \qquad \forall t \in \mathbb{T}. \tag{5.2}$$

Assume that the unknown functions Φ_+ and Φ_- extend holomorphically from the complex unit circle $\mathbb{T}$ into its interior $\mathbb{D}$ and its exterior $\mathbb{E}$, respectively, and that Φ_- vanishes at infinity.

If G is linear in z and $\overline{z}$, $G(\cdot, z) = g_0 + g_1 z + g_2\overline{z}$, then we obtain a linear transmission problem with conjugation [60, 99].

The "holomorphic case" $\overline{\partial}_z G \equiv 0$ was studied in [146].

Nonlinear problem (5.2) is said to be *elliptic* if

$$|\overline{\partial}_z G(t, z)| < |\partial_z G(t, z)| \quad \forall (t, z) \in \mathbb{T} \times \mathbb{C}. \tag{5.3}$$

Another case of particular interest corresponds to real-valued G pertaining to the "parabolic case," where $|\partial_z G| = |\overline{\partial}_z G|$. In this situation, Φ_+ must be holomorphic in $\mathbb{D}$ and real-valued on $\mathbb{T}$ and hence (5.2) is equivalent to a scalar Riemann–Hilbert problem of the form

$$G(t, \Phi_-(t)) = \text{const}, \tag{5.4}$$

which is discussed in many papers (see, e.g., [60, 99, 141]). Note that, in contrast to the general nonlinear transmission problem (5.2), there exists a rather complete geometric theory of the Riemann–Hilbert problem (5.4) [141].

We say that problem (5.2) has a solution in W^1_r if the functions Φ_- and Φ_+ have boundary functions in the Sobolev space $W^1_r(\mathbb{T})$. The following existence theorem established in [142] also covers the linear elliptic case with continuously differentiable coefficients and index zero.

Theorem 5.1. *Let $G : \mathbb{T} \times \mathbb{C} \to \mathbb{C}$ be a continuously differentiable function with uniformly bounded first derivatives.*

(i) *If there exist a positive constant δ and a smooth unimodular function $g : \mathbb{T} \to \mathbb{T}$ with winding number zero, $\operatorname{wind} g = 0$, such that*

$$\big|\partial_z G(t,z)\big| - |\overline{\partial}_z G(t,z)| \ge \delta > 0 \quad \forall (t,z) \in \mathbb{T} \times \mathbb{C}, \tag{5.5}$$

$$\operatorname{Re}\big(g(t)\,\partial_z G(t,z)\big) \ge \delta > 0 \quad \forall (t,z) \in \mathbb{T} \times \mathbb{C}, \tag{5.6}$$

then the transmission problem (5.2) *has a solution in W^1_r for each $r \in (1,\infty)$.*

(ii) *The solution is unique if, in addition to the above assumptions,*

$$\operatorname{Re}\big(g(t)\,\partial_z G(t,z)\big) - |\overline{\partial}_z G(t,z)| \ge \delta > 0 \quad \forall (t,z) \in \mathbb{T} \times \mathbb{C}. \tag{5.7}$$

We reproduce the proof from [142]. It is based on several observations. First, we note that the function g in condition (5.6) admits a factorization $g = g_H/g_R$, where g_R and g_H are smooth functions on $\mathbb{T}$, g_R is real and strictly positive and g_H extends to a holomorphic function in $\mathbb{D}$ without zeros. This allows us to rewrite the boundary relation in the form

$$\widetilde{\Phi}_+ := g_H \cdot \Phi_+ = g_R \cdot g \cdot G(\cdot, \Phi_-) =: \widetilde{G}(\cdot, \Phi_-).$$

If G satisfies (5.5)–(5.7), then $\widetilde{G}$ satisfies the same conditions with $g \equiv 1$. Consequently, we can assume that $g \equiv 1$.

The following constructions serve to transform the transmission problem (5.2) into a fixed-point equation for a compact operator K. The idea is to differentiate the boundary relation along $\mathbb{T}$ (cf. [147]), which gives rise to a quasi-linear transmission problem with conjugation. The main ingredient of the operator K is a primitive of an appropriate solution of this auxiliary problem.

Fix $s \in (1,\infty)$. For a given scalar complex-valued function $\varphi \in W^1_s(\mathbb{T})$, we define

$$a(t) := \partial_z G\big(t, \varphi(t)\big), \quad b(t) := \overline{\partial}_z G\big(t, \varphi(t)\big), \quad c(t) := i\,t\,\partial_t G\big(t, \varphi(t)\big), \tag{5.8}$$

where $i\,t\,\partial_t \equiv \partial_\tau$ denotes the derivative with respect to the polar angle τ of $t \equiv e^{i\tau} \in \mathbb{T}$. Note that a, b, and c are continuous functions.

We denote by H^r_+ (respectively, H^r_-) the Hardy spaces of functions φ which extend holomorphically into $\mathbb{D}$ (respectively, in $\mathbb{E}$ with $\varphi(\infty) = 0$), and let $H^r_\pm := H^r_+ \times H^r_-$.

Lemma 5.1. *Let G be the same as in Theorem* 5.1 *with $g \equiv 1$. Let $r, s \in (1,\infty)$, $\varphi \in W^1_s$, and a, b, and c be given by* (5.8).

(i) *For each $\varphi \in W^1_s(\mathbb{T})$, the linear transmission problem*

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c \tag{5.9}$$

has a unique solution $\widetilde{\Phi} := (\widetilde{\Phi}_+, \widetilde{\Phi}_-) \in H^r_\pm$.

(ii) *For each value of the constant δ in Theorem* 5.1*, there exists $r > 1$ such that the $H^r_\pm$-norm of the solution $\widetilde{\Phi} \equiv (\widetilde{\Phi}_+, \widetilde{\Phi}_-)$ to problem* (5.9) *is bounded by a constant independent of the choice of φ.*

Proof. 1. The existence and uniqueness of the solution follow, e.g., from [141].

2. To prove (ii), we derive a representation of the solutions which involves the inverses of a certain Toeplitz operator.

The function w defined on $\mathbb{T}$ by $w(t) := \big(\overline{\Phi_-(t)}/t, \Phi_+(t)\big)$ extends holomorphically into $\mathbb{D}$. With the definitions $f := -(\operatorname{Re} c, \operatorname{Im} c)$ and

$$A := \begin{bmatrix} \overline{a} + b & -1 \\ i(\overline{a} - b) & i \end{bmatrix} \cdot \begin{bmatrix} t & 0 \\ 0 & 1, \end{bmatrix}, \tag{5.10}$$

problem (5.9) is equivalent to

$$Rw := \operatorname{Re} A w = f. \tag{5.11}$$

Let $P : L^r \to H_+^r$ denote the Riesz projection of $L^r(\mathbb{T})$ onto the Hardy space H_+^r along H_-^r. We introduce the "adjoint Riemann–Hilbert operator"

$$S : L^r \to H_+^r, \quad x \mapsto P\,\bar{t}\,\overline{A}^{-1}\operatorname{Re} x. \tag{5.12}$$

A straightforward verification shows that SR is a Toeplitz operator, $2SR = T := P\widetilde{B}P$. The symbol $\widetilde{B} := \bar{t}\,\overline{A}^{-1}A$ of T has the representation $\widetilde{B} = \frac{1}{a}\,J\,B$, where

$$J := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} |a|^2 - |b|^2 & \overline{bt} \\ -bt & 1 \end{bmatrix}. \tag{5.13}$$

Since $|a| > |b|$ and $\operatorname{wind} a = 0$, the Toeplitz operator T is invertible; this implies that the solution of problem (5.11) admits the representation

$$w = 2T^{-1}Sf. \tag{5.14}$$

3. Recall that S and T depend on the choice of φ in (5.8). It is obvious that the norm of S is bounded by a constant independent of the choice of φ in (5.8). In what follows, we prove that the norms of the inverse $T^{-1} \in \mathcal{L}(H^{1/(r-1)}, H^r)$ are also uniformly bounded with respect to φ, provided that $r > 1$ is sufficiently small. Since J is constant, we can replace T by $\widehat{T} := P(\frac{1}{a})BP$.

4. Since $\operatorname{Re}(Bz, z) = \big(|a|^2 - |b|^2\big)\,|z_1|^2 + |z_2|^2 \ge m\,\|z\|^2$ for some positive number $m = m(\delta)$, Lemma 1 of [11] shows that the inverses of the Toeplitz operators $T_B := PBP : H_+^2 \to H_+^2$ are uniformly bounded, $\|T_B^{-1}\| \le 1/m$. The invertibility of T_B implies that the (continuous) symbol B admits a generalized Wiener–Hopf factorization (canonical factorization) $B = B_-B_+$, where B_-, B_+, B_-^{-1}, $B_+^{-1} \in L^p$ for each $p < \infty$ [1, Sec. 5.5] (see also [3, 7]). Since $T_B^{-1} = B_+^{-1}PB_-^{-1}P$ and $B_+^{-1}PB_-^{-1} = B_+^{-1}PB_-^{-1}P$ on L^2, the multiplication operator

$$f \mapsto B_+^{-1}PB_-^{-1}f \tag{5.15}$$

is bounded on L^2 (uniformly with respect to $\varphi \in W_s^1$).

5. The function a is continuous and its range lies in a compact subset of the right complex halfplane (independent of φ). Consequently, a admits a Wiener–Hopf factorization $a = a_+ \cdot a_-$, where $a_+ = \exp(P \log a)$ and $a_- = \exp((I - P)\log a)$. Since $|\operatorname{Im} \log a| \le \gamma(\delta) < \pi/2$, Zygmund's lemma applies to estimate the norms of the factors a_+ and a_- in $L^{2+\varepsilon}$ (recall that $P = \frac{1}{2}(-iH + I + P_0)$, where H denotes the conjugation operator). The result is

$$\|a_+\|_{2+\varepsilon} \le C(\delta), \qquad \|a_-\|_{2+\varepsilon} \le C(\delta) \tag{5.16}$$

for some sufficiently small positive $\varepsilon = \varepsilon(\delta)$.

6. We have the factorization $(1/a)\,B = a_-^{-1}B_-B_+a_+^{-1}$ almost everywhere on $\mathbb{T}$. Using (5.15) and (5.16), we obtain that the operator

$$H_+^{1/(r-1)} \to H_+^r : w \mapsto a_+B_+^{-1}PB_-^{-1}a_-w \tag{5.17}$$

is bounded (uniformly with respect to φ).

To prove that (5.17) is the inverse of $\widehat{T}$, we note that $\widehat{T}w \equiv Pa_-^{-1}B_-B_+a_+^{-1}w = f$ is equivalent to $B_+a_+^{-1}w = PB_-^{-1}a_-w$ (note that $B_+a_+^{-1}w \in H_+^{2-\varepsilon}$ with $\varepsilon > 0$); this implies that $w = a_+B_+^{-1}PB_-^{-1}a_-f$ almost everywhere on $\mathbb{T}$. ∎

After establishing these technical facts, we continue the construction of the fixed-point equation. For any scalar complex-valued function $\varphi \in W_s^1$, we denote by $\widetilde{\Phi}_+, \widetilde{\Phi}_-$ the solution of the associated transmission problem

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c \tag{5.18}$$

with a, b, and c from (5.8). If

$$\widehat{\Phi}_-(\mathrm{e}^{i\tau}) := \int_0^\tau \widetilde{\Phi}_-(\mathrm{e}^{i\sigma})\,\mathrm{d}\,\sigma, \qquad P_0\widehat{\Phi}_- := \frac{1}{2\pi}\int_0^{2\pi} \widehat{\Phi}_-(\mathrm{e}^{i\sigma})\,\mathrm{d}\,\sigma,$$

then the operator $K : W_s^1 \to W_r^1$ is given by $K\varphi := \widehat{\Phi}_- - P_0\widehat{\Phi}_-$. The definition of $\widehat{\Phi}_-$ makes sense since $P_0\widetilde{\Phi}_- = 0$.

Lemma 5.2.

(i) *The operator $K : W_s^1 \to W_r^1$ is compact for any $r, s \in (1,\infty)$.*
(ii) *The image of $K : W_r^1 \to W_r^1$ is bounded if $r > 1$ is sufficiently small.*
(iii) *The pair $(\Phi_+, \Phi_-) \in W_r^1$ is a solution of the transmission problem* (5.2) *if and only if $K\Phi_- = \Phi_-$ and $\Phi_+ = G(.,\Phi_-)$.*

Proof. 1. The embedding $W_s^1(\mathbb{T}) \to C(\mathbb{T})$ is compact, and hence (i) follows once it is shown that $K : C(\mathbb{T}) \to W_r^1(\mathbb{T})$ is continuous. The superposition operators $\varphi \mapsto a := \partial_z G(\cdot,\varphi)$, $\varphi \mapsto b := \overline{\partial}_z G(\cdot,\varphi)$, and $\varphi \mapsto f := i\,t \cdot \partial_t G(.,\varphi)$ are continuous in $C(\mathbb{T})$, and thus the associated Toeplitz operators $T := P\widetilde{B}P$, where $\widetilde{B} := \frac{1}{a}JB$ and J and B are the same as in (5.13), and the "adjoint Riemann–Hilbert operators" S in (5.12) depend continuously on φ. Since all these operators are invertible, the solutions to the transmission problems (5.9) in $H_\pm^r$ also depend continuously on φ (cf. (5.14)). Integrating these solutions along $\mathbb{T}$ proves the continuity of $K : C(\mathbb{T}) \to W_r^1(\mathbb{T})$.

2. If $r > 1$ is sufficiently small, then, according to Lemma 5.1, the solutions $\widetilde{\Phi}_\pm$ of (5.9) are bounded in $H_\pm^r$ uniformly with respect to the choice of $\varphi \in W_r^1$, and hence Kw are uniformly bounded in W_r^1.

3. Let $\Phi = (\Phi_+, \Phi_-) \in W_r^1$ be a solution of $\Phi_+ = G(.,\Phi_-)$. Differentiating this boundary relation with respect to the polar angle τ, we obtain that $\widetilde{\Phi} := \partial_\tau\Phi \equiv it\partial_t\Phi$ is a (unique) solution of the auxiliary transmission problem (5.9). Consequently,

$$K\Phi_-(\mathrm{e}^{i\tau}) = \mathrm{const} + \int_0^\tau \widetilde{\Phi}_-(\mathrm{e}^{i\sigma})\,\mathrm{d}\,\sigma = \mathrm{const} + \int_0^\tau \partial_\tau\Phi_-(\mathrm{e}^{i\sigma})\,\mathrm{d}\,\sigma = \mathrm{const} + \Phi_-(\mathrm{e}^{i\tau}).$$

The constant on the right-hand side vanishes since $P_0\,K\Phi_- = 0$ and $P_0\Phi_- = 0$.

4. Conversely, let $\Phi_- \in W_r^1$, $K\Phi_- = \Phi_-$, and $\Phi_+ := G(\cdot,\Phi_-)$. We prove that Φ_+ and Φ_- are holomorphic in $\mathbb{D}$ and $\mathbb{E}$, respectively, and $P_0\Phi_- = 0$.

First, we see that $\partial_\tau\Phi_- = \widetilde{\Phi}_-$. Since $\widetilde{\Phi}_-$ is holomorphic in $\mathbb{E}$, Φ_- is also holomorphic in $\mathbb{E}$. Further, $P_0\Phi_- = P_0\,K\Phi_- = 0$. Substituting $\widetilde{\Phi}_- = \partial_\tau\Phi_-$ into (5.9), we obtain

$$\widetilde{\Phi}_+ = a\,\widetilde{\Phi}_- + b\,\overline{\widetilde{\Phi}_-} + c = \frac{\mathrm{d}}{\mathrm{d}\,\tau}\,G(\cdot,\Phi_-) = \partial_\tau\Phi_+.$$

Consequently, Φ_+ is holomorphic in $\mathbb{D}$ and $\Phi := (\Phi_+,\Phi_-) \in W_r^1$ solves (5.2); this completes the proof of the lemma. □

By virtue of Lemma 5.2, the existence result (i) of Theorem 5.1 becomes a consequence of Schauder's fixed-point principle.

It remains to prove that the solution of (5.2) is unique under assumption (5.7). Let $\Phi^{(1)}, \Phi^{(2)} \in H_\pm^\infty \cap W_1^r$ be two solutions of (5.2). The difference $\Delta\Phi \equiv (\Delta\Phi_+, \Delta\Phi_-) := \Phi^{(2)} - \Phi^{(1)}$ solves the homogeneous linear transmission problem

$$\Delta\Phi_+ = a \cdot \Delta\Phi_- + b \cdot \overline{\Delta\Phi_-}, \tag{5.19}$$

where

$$a := \int_0^1 \partial_z G\big(\cdot, \lambda\Phi_-^{(1)} + (1-\lambda)\Phi_-^{(2)}\big)\,\mathrm{d}\,\lambda, \quad b := \int_0^1 \overline{\partial}_z G\big(\cdot, \lambda\Phi_-^{(1)} + (1-\lambda)\Phi_-^{(2)}\big)\,\mathrm{d}\,\lambda,$$

and $\Delta\Phi_-(\infty) = 0$. Assumption (5.7) on $\partial_z G$ and $\overline{\partial}_z G$ (with $g \equiv 0$) ensures that

$$\operatorname{Re} a - |b| \geq \delta > 0 \quad \text{on } \mathbb{T}, \tag{5.20}$$

and hence (5.19) has only the trivial solution. Thus the proof of uniqueness is also complete.

Remark 5.1. In [142], it was conjectured that the solution is unique even without a strengthened assumption (5.7). As we will see below, this can be proved by using a result of Globevnik on the local sructure of attached analytic discs.

It is instructive to compare our formulation of the nonlinear transmission problem with the so-called *Bishop problem* [17], which is related with many deep aspects of multidimensional complex analysis [53,54]. As is well known, the Bishop problem can be reformulated in terms of *analytic discs*.

Recall that an analytic disc in $\mathbb{C}^n$ is defined as a continuous (or smooth) mapping $\phi : D = \{|z| \leq 1\} \to \mathbb{C}^n$ which is holomorphic inside the unit disc D, i.e., it is defined by n functions $\phi_n \in A(D)$. If $M \subset \mathbb{C}^n$ is a submanifold and ϕ is an analytic disc with $\phi(\partial D) \subset M$, then ϕ is called an *analytic disc attached to* M.

In various problems of complex analysis, it is important to construct analytic discs attached to so-called *totally real submanifolds* and there exist many deep results on the existence and structure of such analytic discs [17, 53, 54, 69]. Recall that a linear subspace of a complex vector space is called *totally real* if it does not contain any complex line, and a submanifold M of a complex vector space is called totally real if, for each $x \in M$, the tangent plane $T_x M$ is totally real.

This topic is closely related to Riemann–Hilbert problems since structural properties (dimension, stability) of the family of analytic discs attached to M close to given properties ϕ_0 can be expressed in terms of the partial indices of matrix-valued functions defining the pullback bundle $\phi_0^*(TM)$ [54, 64, 65].

These results on attached analytic discs are very important in complex analysis and exhibit a spectacular type of application of Riemann–Hilbert problems and the Birkhoff factorization. Moreover, they can be visually interpreted in the case of a nonlinear Riemann–Hilbert problem considered above, and it seems appropriate to recall some related constructions and concepts.

Let M be a *maximal real* (i.e. totally real of maximal possible dimension n) submanifold of $\mathbb{C}^n$ and f be an analytic disc attached to M. Assume that in a neighborhood V of $f(\partial D)$, we have $M \cap V = \{x \in V : g(x) = 0\}$, where g is a smooth (it is sufficient to require $g \in C^2$) function on V having no critical points on V. We investigate the existence of nearby analytic discs attached to M. It turns out that this problem can be studied in terms of certain analytic invariants of $f(\partial D)$ in M.

For each $z \in T = \partial D$, let $T(z)$ be the tangent plane of M at $f(z)$. Since f is regular on $\overline{D}$, there is a smooth map $A : T \to GL(n, \mathbb{C})$ such that for each $z \in T$, the columns of $A(z)$ form a basis of $T(z)$. Let $B(z) = A(z)\overline{A(z)}^{-1}$, where the bar denotes complex conjugation.

Since the spaces $T(z)$ are maximal real, the map B has some remarkable properties which justify the constructions to follow. We now establish these properties of B.

Let L be a maximal real subspace of $\mathbb{C}^n$. Then, of course, $L \oplus iL = \mathbb{C}^n$. To any such subspace L, one can associate an $\mathbb{R}$-linear map R_L on $\mathbb{C}^n$ called the *reflection* in L and defined by the formula $z = x + iy \mapsto x - iy$, where we have used the above decomposition into a direct sum.

The mapping R_L is also $\mathbb{C}$-antilinear, i.e., $R_L(iv) = -iR_L(v)$ for each $v \in \mathbb{C}^n$. Let us denote by R_0 the reflection with respect to the maximal real subspace $\mathbb{R}^n \subset \mathbb{C}^n$. In the standard notation, R_0 is just the ordinary conjugation on $\mathbb{C}^n$ and for any $n \times n$ complex matrix A, one has the identity $\overline{A} = R_0 A R_0$. Now it is easy to establish the following well-known algebraic lemma, which is crucial for subsequent considerations.

Lemma 5.3. *Let L be a maximal real subspace of $\mathbb{C}^n$, $x_1, \dots, x_n$ be any basis in L, A be the matrix whose columns are the given vectors $x_j, j = 1, \dots, n$, and $B = A\overline{A^{-1}}$. Then $B = R_L R_0$; in particular, the matrix B is independent of the basis in L. Moreover, $\overline{B} = B^{-1}$ and $|\det B| = 1$.*

Proof. Obviously, A is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$, which maps $\mathbb{R}^n$ onto L. Now consider the composition $S = R_0 A^{-1} R_L A$ of automorphisms of $\mathbb{C}^n$. Then S is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$, which coincides with the identity on $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a maximal real subspace of $\mathbb{C}^n$, S is the identity on $\mathbb{C}^n$. Hence $R_0 A^{-1} = A^{-1} R_L$. Since $B = A\overline{A^{-1}} = AR_0 A^{-1} R_0$, we obtain $B = AA^{-1} R_L R_0 = R_L R_0$, as claimed, and the rest becomes obvious. □

Thus, the matrix B is uniquely determined by the bundle $T(z)$ and this allows one to extract from it the crucial invariants of this bundle by using the Birkhoff factorization theorem. Indeed, one can write

$$B(z) = B_+(z)\,\mathrm{diag}(z^k) B_-(z),$$

where the factors have the same meaning as in Sec. 1. It follows that the (left) partial indices of k_i are defined by the bundle $T(z)$. They are called *partial indices of M along $f|T$*, and their sum K is called the *total index of M along $f|T$* (or the *Maslov index of M along $f|T$*; cf. [53, 141]).

This construction becomes quite transparent for $n = 2$, and using a well-known description of nearby analytic discs in terms of the *Bishop equation* [17] or nonlinear Riemann–Hilbert problems [141], one can obtain some conclusions about the structure of attached analytic discs. For example, in the situation of Theorem 5.1, it is easy to see that all partial indices of M along the boundary of an attached analytic disc are equal to zero. At the same time, the boundaries of these discs give a foliation of M, and the family of such analytic discs is locally two-dimensional.

Also, it was shown in [53] (cf. [141]) that if both partial indices of M along $f|T$ are nonnegative, then the family of nearby analytic discs depends on $K + n$ real parameters and the same holds for small perturbations of M (or, equivalently, of its defining function g).

A general result on the structure of attached analytic discs was proved by Globevnik [65]. To give a precise formulation of this result, we need to introduce some functional spaces adapted to the situation.

Denote by $C^2(V)$ the Banach space of all real-valued C^2-functions with the standard sup-norm. Let $0 < s < 1$. Denote by H^s the Banach algebra of all real-valued functions on T with finite Lipschitz norm of exponent s, i.e.,

$$\|f\|_s = \sup |f| + \sup \frac{|f(x) - f(y)|}{|x - y|^s}, \quad x, y \in T,$$

by H^s_C the algebra of all complex-valued functions on T with finite H^s-norm, and by A_s the closed subalgebra of all $\phi \in H^s_C$ which extend holomorphically to D. The Banach space A^n_s is the space of analytic discs convenient to work with. For each $r \in (C^2(V))^n$ sufficiently close to g, we set $M_r = \{x \in V : r(x) = 0\}$.

Now assume that $f \in A^n_s$ satisfies $f(T) \subset M$, that is, $g(f(t)) = 0$ for each $t \in T$. If $U \subset A^n_s$ is a neighborhood of f so small that $h(T) \subset V$ for each $h \in U$, then the analytic disc ϕ is attached to M if and only if $g(\phi(t)) = 0$ for each $t \in T$. Consider the map Q which sends $\phi \in U$ to the map $t \mapsto g(\phi(t))$. It is easy to see that Q is (at least) a C^1-map from U into $(H^s)^n$ and $X = \{\phi \in U : Q(\phi) = 0\}$ is precisely the set of all analytic discs in U attached to M. Now our aim is to understand its structure, and the first step is to find conditions when it is a finite-dimensional smooth manifold, since then one can hope to find a reasonable parametrization of nearby attached analytic discs.

Taking into account the implicit-function theorem, we conclude that X is a C^1-manifold if $DQ(f)$ maps A^n_s onto $(H^s)^n$ and if the kernel $\ker DQ(f)$ is complemented in A^n_s. As was established by Globevnik, this is equivalent to requiring that all partial indices of M along $f|T$ are not less than -1. Moreover, one can explicitly calculate its dimension.

Theorem 5.2 ([65]). *Let M, f, and Q be as above. Let k_i be the partial indices of M along $f|T$ and K be the total index. Then $DQ(f)$ is surjective if and only if $k_i \geq -1$, $i = 1, \dots, n$. Then there exist a neighborhood $P \subset (H^s)^n$ of g and a neighborhood $W \subset A_s^n$ of f such that*

(1) *$\{(r, \phi) \in P \times W : \phi(T) \subset M_r\}$ is a C^1 submanifold of $P \times W$;*
(2) *for each $r \in P$, the set $\{\phi \in W : \phi(T) \subset M_r\}$ is a C^1 submanifold of W of dimension $K + n$.*

It is remarkable that this important result can be proved by using only the Birkhoff factorization and the implicit-function theorem in functional spaces. It seems also worth noting that in the situation of Theorem 5.1, condition (i) implies that all partial indices of emerging attached analytic discs are equal to zero, and Theorem 5.2 guarantees that family of such discs is smooth and their boundaries cover the target manifold in the "Schlicht" manner (cf. [141]). In other words, the target manifold is foliated by the boundaries of attached analytic discs even without assumption (ii) of Theorem 5.1; this answers the question posed in [142] (see the remark after the proof of Theorem 5.1).

The aforementioned Bishop problem is concerned with constructing analytic discs attached to explicitly given submanifolds of complex Euclidean spaces. In the original formulation of [17], the target manifold is just a graph of a function $\mathbb{C} \to \mathbb{C}$. For simplicity, we discuss only the case where $n = 2$; this corresponds to the original Bishop problem and to the transmission problem studied above.

Recall that Bishop considered a differentiable function $f : \mathbb{C} \to \mathbb{C}$ and its graph $\Gamma_f \subset \mathbb{C}^2$ and looked for analytic discs in $\mathbb{C}^2$ attached to Γ_f. A minute thought shows that Theorem 5.1 gives a solution to the Bishop problem for a class of functions f specified in its formulation. Note that we have actually obtained a complete description of all attached analytic discs in this case.

Nowadays, there exist a lot of papers devoted to solving the Bishop problem for various classes of functions [55]. For example, Bishop himself established the existence of such analytic discs in a number of cases and described their structure in neighborhoods of so-called *elliptic complex points* (we present the definition below) of the graph [17]. Since then, the problem of proving existence and counting complex points of various types has attracted considerable attention (for an updated review of the topic, see [55]).

The problem has a special flavor in the case where components of f are polynomials since one would like then to have some effective methods for counting complex points and establishing the existence of attached analytic discs. An approach to this problem based on the so-called *signature technique for counting real roots* [89] was recently suggested in a joint paper of Wegert and the author [91]. Below, we describe some essential ingredients of this approach, since to our mind it suggests some interesting viewpoints and perspectives.

Following [91], we treat the problem in terms of smooth transformations (self-mappings) of the plane $\mathbb{C} \cong \mathbb{R}^2$. We consider transformations with polynomial components and call them *planar endomorphisms* (*plends*).

We are interested in counting the so-called complex points of such maps; in particular, we find the number of *elliptic complex points* [17].

In many situations, it is sufficient to consider only *generic* planar endomorphisms, which are proper and satisfy some additional transversality condition (see [73, 90]). More precisely, the Gauss map of the graph of such an endomorphism should be transversal to the subset G_C of all complex lines in $\mathrm{Gr}_{\mathbf{R}}(2, 4)$. This condition holds on a dense open subset of plends and implies that the graph has only isolated complex points and a finite amount of those. For brevity, such plends are called excellent (by analogy with the "excellent maps" of Whitney [67]).

It can be shown that an arbitrary plend admits excellent deformations and a well-known paradigm of singularity theory [67] suggests that it is reasonable to count complex points in nearby excellent deformations of a given plend. Thus, one comes to the problem of counting complex points of an excellent plend, which is our main concern in the sequel. A local version of such strategy was applied in [73, 90] and it turned out that the so-called *signature formulas for topological invariants* [89] are quite effective in this context. We outline below how one can apply signature formulas in a global setting.

Denote by $\mathbf{R}_2$ the algebra of real polynomials in two variables and by P_d the subset of polynomials of algebraic degree not exceeding d. Let $F = (f, g)$, $f, g \in \mathbf{R}_2$, be a planar endomorphism (plend) with components f and g.

Identify $\mathbb{R}^2 \times \mathbb{R}^2$ with $\mathbb{C}^2$ in the usual way; then the graph Γ_F of F becomes a smooth (C^∞) two-dimensional surface in $\mathbb{C}^2$. This allows us to study Γ_F using some concepts of complex geometry. A natural step in this direction is to look at tangent planes to the graph.

Consider the Grassmanian $G = \mathrm{Gr}_{\mathbf{R}}(2,4)$ of oriented two-planes in $\mathbb{R}^4$ identified with $\mathbb{C}^2$. As usual, one can distinguish between *complex lines* and *totally real planes* [17].

The Gauss map $T : \Gamma_F \to G$ sends each point $q = (p, F(p)) \in \Gamma_F$ to the tangent plane $T_q\Gamma_F$. A well-known strategy of singularity theory suggests considering objects satisfying appropriate transversality conditions.

Recall that a point p is called a complex point of F (and its graph Γ_F) if the tangent plane $T_p\Gamma_F$ at this point is a complex line [17]. Generically, for the excellent plends defined above, a complex point can be either elliptic or hyperbolic depending on the geometric properties of Γ_F at this point [17].

Denote by

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

the usual ∂-bar operator on the plane and write $F_{\overline{z}}$ for the plend obtained by applying $\overline{\partial}$ to both components of F. Then one can give a simple characterization of complex points in terms of $\overline{\partial}$.

Lemma 5.4. *A point p is a complex point of F if and only if $F_{\overline{z}}(p) = 0$.*

Now we follow the pattern suggested by Bishop [17]. Namely, we first establish the existence of elliptic complex points and then apply the fundamental Bishop result on the existence of attached analytic discs in a vicinity of an elliptic complex point [17]. As is well known, this approach yields a lot of results about the existence of complex points on immersed compact surfaces which were basically established by topological methods [55]. In our setting, considerable information about the existence and amount of complex points can be obtained using the aforementioned signature formulas for topological invariants [89].

Note that applying the linking-number construction from [73] (cf. [90]) to sufficiently large circles on the plane, one obtains a natural integer-valued invariant of a proper plend, which can be called its (global) Maslov index $M(F)$. According to formulas in [73, 90], one can calculate $M(F)$ by properly counting complex points of nearby perfect deformations of F. It turns out that the same can be done without examination of nearby deformations.

Theorem 5.3. *The Maslov index is algorithmically computable from the coefficients of a given planar endomorphism.*

This follows from an explicit algebraic formula for the topological degree [89], which implies effective computability of mapping degrees of explicitly given polynomial mappings. Note that, by virtue of the above lemma, the Maslov index is just the topological degree of the endomorphism defined by the partial derivative $\partial F/\partial \overline{z}$.

The Maslov index alone is not sufficient for our purposes. Therefore, we consider also the numbers $C(F)$ and $C_e(F)$ of complex points and elliptic complex points, respectively. The total amount $C(F)$ of complex points can also be expressed in terms of the mapping degree.

Theorem 5.4. *The total amount of complex points of a perfect planar endomorphism is algorithmically computable from the coefficients of its components.*

This follows from a general signature formula for the Euler characteristic of a compact algebraic subset [89, Theorem 8.2]. For a perfect plend F, the set of complex points is a finite algebraic subset of $\mathbb{R}^2$ defined as the zero-set of the polynomial system $\dfrac{\partial F}{\partial \overline{z}}(x, y) = 0$. Thus, this number can be calculated as the local topological degree of an auxiliary endomorphism of $\mathbb{R}^3$, which is given by simple explicit formulas [89].

Now let us explain how one can calculate the exact amount of elliptic complex points. The trick is to represent the subset $C_e(F)$ as a semi-algebraic subset of the plane. Then it can be effectively calculated using the results of [89, Chap. 9].

In other words, we only need to indicate explicit algebraic conditions which characterize elliptic complex points. For this, we use the geometric interpretation in terms of Gaussian curvature $K_p(\Gamma_F)$ [55]. Namely, from the normal form of a generic complex point [17], it follows that the elliptic points are exactly those complex points p where $K_p(\Gamma_F)$ is positive. Now it is easy to show that this condition can be expressed as a polynomial inequality.

Indeed, one can use an explicit expression for the Gaussian curvature of a parametrized 2-surface in $\mathbb{R}^4$. It comes as no surprise that such an expression can be derived from general formulas for the first fundamental form and curvature tensor of a parameterized submanifold, which directly leads to the desired conclusion [91].

Proposition 5.1. *The set of elliptic complex points of a perfect plend F coincides with the semi-algebraic subset*

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial g}{\partial \bar{z}} = 0, \quad K(x,y) > 0,$$

where $K(x,y)$ is the Gaussian curvature of Γ_F at the point $(x, y, f(x,y), g(x,y))$.

Theorem 5.5. *The number of elliptic complex points of a perfect plend can be effectively calculated by a finite number of algebraic and logical operations over its coefficients.*

This follows from Theorem 9.1 of [89], which establishes the effective computability of the cardinality of a finite semi-algebraic subset.

Note that by using a computer program for calculating topological degree developed by Lecki and Szafraniec [95], one can easily count complex points for plends of not very high degree (everything depends just on the capacity of the computer). After determining the bifurcation diagram along the standard lines of singularity theory [67], this allows one to find all possible values of the above invariants for plends of fixed bi-degree, say, for elements of $(P_d)^2$. Some concrete results of this kind will be presented in our forthcoming publications.

6. Hyperholomorphic Cells and Riemann–Hilbert Problems

The classical boundary-value problems (BVPs) for holomorphic functions—the linear conjugation problem and the Riemann–Hilbert problem—can be described by linear operators in appropriate function spaces (see [21, 112, 119]). At present, there also exist several interesting nonlinear versions of these classical problems (see, e.g., [127, 141]).

One of the most spectacular generalizations of this kind was developed in the seminal paper of Gromov concerned with the pseudo-holomorphic curves [69]. Gromov's approach involves, in particular, two important new aspects: generalizing the equation satisfied by functions (which is in some sense equivalent to working with solutions of the Bers–Vekua equation [15, 139]) and consideration of nonlinear boundary conditions in the spirit of "holomorphic discs attached to a totally real submanifold" [17].

All these results are concerned with functions which locally depend on two real variables, and one can wonder if similar results can be obtained for functions of several real variables satisfying some elliptic system of equations. Such generalizations do not seem to have attracted much attention up to now, but the existing results about linear BVPs for elliptic systems (see [28, 81, 133]) suggested that some results of this type should be available for systems of Dirac type [61].

In this section, we describe some steps in this direction. To be more precise, we discuss some geometric properties of families of solutions to certain elliptic first-order systems of linear partial differential equations with constant coefficients [143] (cf. EES of Sec. 4; such systems are also called "canonical first-order systems" (CFOS) [10]). These systems were studied in many papers and they remain an object of constant interest (see [134] for a recent review of the topic).

An especially important class of such systems is provided by the so-called "generalized Cauchy–Riemann systems" (GCRS) [132]. Solutions to generalized Cauchy–Riemann systems are often called *hyperholomorphic mappings* [126], and in many problems it is necessary to consider images of some standard domains (e.g., balls) under such mappings. Standard examples of such systems in low dimensions are the classical Cauchy–Riemann system on the plane, the Moisil–Theodoresco system in $\mathbb{R}^3$ [18], and Fueter system for functions of one quaternionic variable [32] mentioned in Sec. 4. There exists a vast amount of literature devoted to such equations (see the references in [18, 32]). In particular, some important results about the so-called *generalized analytic vectors* were obtained by Georgian mathematicians [18, 114]. Similar systems emerged in the theory of para-analytic functions developed by Frechet [56].

The main paradigm that we follow in the sequel has its origin in the theory of analytic (holomorphic) discs attached to a totally real submanifold [17], which was already discussed above. One takes a smooth bounded domain homeomorphic to a ball of appropriate dimension and considers its images under solutions to a given CFOS. Such images (we call them *elliptic cells*) are our main concern in this paper.

More precisely, inspired by the theory of attached analytic discs [17, 55] we consider elliptic cells with boundaries in a fixed submanifold M of the target space of the elliptic system in question. They are called *elliptic cells attached to M*. For our purposes, it is useful to consider them as solutions of nonlinear boundary-value problems of Riemann–Hilbert type [141]. Accepting the terminology from [141], M is called a *target manifold* (for attached elliptic cells).

Actually, we consider hyperholomorphic cells, i.e., cells which are defined by solutions to a given GCRS. Note that, except for the aforementioned theory of attached analytic discs [17], there also exist important generalizations of this classical example in the framework of symplectic geometry [69].

Recently, similar situations were discussed in mathematical physics in the relation with so-called *D-branes*. D-branes have interesting applications in topological field theory and string theory [16, 59]. It is worth noting that in these physical contexts, there also appear manifolds with boundaries attached to certain submanifolds. This confirms our trust that such objects deserve some attention on their own.

We investigate some situations where families of attached hyperholomorphic cells can be locally described as kernels of some (nonlinear) Fredholm operators. Such a phenomenon is well known in the case of analytic discs [6] and it plays an important role in Gromov's studies on pseudoholomorphic curves [69]. In particular, one can use the well-known concepts and techniques of Fredholm theory, which reveals some important topological aspects of the situation. We closely mimic Gromov's approach and establish some properties of emerging nonlinear operators using the Fredholm theory of elliptic Riemann–Hilbert problems (RHPs) for GCRS discussed in the preceding section (cf. also [79, 133]).

In particular, we show that for certain values of dimensions n, k, and m, there exist noncompact k-submanifolds in an affine m-space such that families of hyperholomorphic n-cells attached to such submanifolds are locally described by Fredholm operators. Borrowing terminology from [141], such submanifolds are called *admissible targets* (for a given GCRS). The existence of admissible targets and the Fredholmness of arising nonlinear operators are derived from the recent results on the existence of elliptic boundary-value problems for GCRS [81, 134] (cf. also [126]).

Such aspects of generalized Cauchy–Riemann systems seem to have never been discussed in the literature, and we pursue but a modest goal of describing and illustrating the framework which naturally stems from our previous results on generalized Cauchy–Riemann systems. We proceed by presenting some notions from [132, 143] in a form adjusted to our purposes.

Definition 6.1 ([132], cf. [61])**.** An elliptic system of first order with constant coefficients is called a generalized Cauchy–Riemann system (GCRS) if it is invariant under the natural action of the orthogonal group on the source space and all components of its differentiable solutions are harmonic functions. Solutions of such systems are called hyperholomorphic (hh) mappings. For a given such system S, its solutions will also be called S-mappings.

As was explained above (cf. [133]), without loss of generality, one can assume that such a system in $\mathbb{R}^{n+1}$ can be written in the canonical form described in Sec. 4:

$$E\frac{\partial w}{\partial x_0} + A_1\frac{\partial w}{\partial x_1} + \ldots + A_n\frac{\partial w}{\partial x_n} + Dw = f, \tag{6.1}$$

where A_j and D are constant complex $(m \times m)$-matrices, $A_0 = E$ is the identity matrix, and for all $i, j = 1, \ldots, n$, one has

$$A_iA_j + A_jA_i = -2\delta_{ij}E. \tag{6.2}$$

We consider such a system in a smooth domain $U \subset \mathbb{R}^{n+1}$ and assume that the unknown vector-valued function w belongs to the class $C^1(U, \mathbb{C}^m)$.

As was also shown, system (6.1) is elliptic in the usual sense [143], i.e.,

$$\det(t_0E + t_1A_1 + \ldots + t_nA_n) \neq 0$$

for all $(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} - \{0\}$.

As we have already seen in Sec. 4, such a system defines a representation of the Clifford algebra Cl_n on $\mathbb{C}^m$ [61]. Therefore, the (complex) target dimension m, being the sum of dimensions of irreducible representations of Cl_n, is an integer multiple of $2^{[n/2]}$ [28]. If, for a given system S, this dimension is the minimal possible, $m(n) = 2^{[n/2]}$, we say that system S is irreducible. In many situations, it is sufficient to consider only irreducible GCRSs.

For the sake of clarity, we identify these parameters for small n. For $n = 1$, we have $m(1) = 1$ and the corresponding irreducible system is just the classical Cauchy–Riemann system for two real functions $u(x, y)$ and $v(x, y)$ of two real variables.

For $n = 2$, we have $m(2) = 2$ and the corresponding irreducible systems for four real functions s, u, v, and w of three real variables is the Moisil–Theodoresco system in $\mathbb{R}^3$, which was already discussed in Sec. 4:

$$\mathrm{div}(u, v, w) = 0, \qquad \mathrm{grad}\, s + \mathrm{rot}(u, v, w) = 0.$$

For $n = 3$, we have $m(3) = 2$ and the corresponding irreducible system is the so-called Fueter system for four real functions f_i of four real variables x_j [61]:

$$\begin{aligned}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0,\\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} &= 0,\\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} &= 0,\\
\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} &= 0.
\end{aligned}$$

As was already mentioned, its solutions are called monogenic functions of a quaternionic variable. Such functions are at present well understood [61, 118, 126].

The general theory of PDE suggests that one can formulate various reasonable boundary-value problems (BVPs) for such systems in bounded domains [135, 143]. For our purposes, the most relevant are the classical local boundary-value problems of Riemann–Hilbert type introduced in the preceding section. In other words, one searches for solutions of (6.1) satisfying a boundary condition of the form

$$(B_1B_2) \cdot w = g, \tag{6.3}$$

where B_1 and B_2 are continuous complex $(\frac{m}{2} \times \frac{m}{2})$-matrix-valued functions on ∂U such that the rows of $(\frac{m}{2} \times m)$-matrix-valued function (B_1, B_2) are linearly independent at every boundary point and g is a continuous vector-valued function with values in $\mathbb{C}^m$.

For our purposes, Riemann–Hilbert problems elliptic in the usual sense (i.e., satisfying the Shapiro-Lopatinsky condition [143]) are especially appropriate since in this case problem (6.1), (6.3) is described by a Fredholm operator in appropriate function spaces [143]. It is well known that not all systems of type (6.1) admit elliptic boundary conditions (6.3) [18, 143]; therefore, the first natural problem is to investigate which GCRSs possess elliptic RHPs. The answer to this question given in Sec. 4 allows us to indicate cases where hyperholomorphic cells are described by Fredholm operators.

Remark 6.1. An important class of GCRS is associated with the Euclidean Dirac operators [28]. The corresponding first-order systems are called *systems of Dirac type* and their solutions are called *monogenic mappings* [61].

For notational convenience, in the sequel we denote by V the target space $\mathbb{C}^m$ of system (6.1).

We fix a GCRS of the form (6.1) and denote by B an $(n+1)$-ball in its source space. We also take some submanifold M in V and refer to it as a target.

Definition 6.2. A hyperholomorphic (hh) cell attached to M is defined as (the image of) a hyperholomorphic mapping $u : \overline{B} \to V$ such that $u(bB) \in M$. For a fixed GCRS S, we speak of S-cells attached to M.

The usual way of dealing with hh-cells attached to a given submanifold is to consider families of cells attached at a given point. Such families can be described by certain nonlinear operators in appropriate function spaces and if these operators appear to be Fredholm, then one can obtain a reasonable structural theory of such cells, as happens, for example, for pseudo-holomorphic discs and curves [6, 69]. Hence, it is natural to begin by searching for situations where hh-cells can be related to Fredholm operators. To make this idea precise, we need some constructions and definitions.

For this, consider an irreducible GCRS S in $\mathbb{R}^{n+1}$ with values in V. Consider also some smooth (C^∞) submanifold M of V of real dimension equal to the complex dimension $m(n)$ of V (in this case, we speak of a *submanifold of middle dimension*). Let B denote an $(n+1)$-ball centered at the origin of the source space of S and let q be some fixed point on its boundary n-sphere ∂B. Furthermore, we fix a point $p \in M$ and a noninteger number $r > 1$. Let H^r denote the usual Hölder class.

Let F be the space of H^{r+1}-maps $f : (B, bB, q) \to (V, M, p)$ which are homotopic to the constant map $f_p = p$ in $\pi_{n+1}(V, M, p)$ (such maps are called *homotopically trivial*). In a standard way, one verifies that F is a smooth, complex Banach manifold (cf. [6]). Let G be the complex Banach space of all H^r-maps $g : B \to V$. Define also a submanifold in $F \times G$ by setting

$$H = \{(f, g) \in F \times G : D(f) = g\}, \tag{6.4}$$

where D denotes the matricial partial differential operator defined by the left-hand side of (6.1).

Then it is easy to see that H is a connected submanifold of $F \times G$ and one can define the projection map $L_p : H \to G$ given by $L_p(f, g) = g$. It is also easy to verify that L_p is a smooth map of H into the Banach space G.

Definition 6.3. The mapping L_p is called the Gromov operator of the pair (S, M) at a point p. The manifold M is called an S-admissible target if the Gromov operator $L_p(S, M)$ is a Fredholm operator (mapping) for every $p \in M$.

Similar operators were introduced by Gromov for analytic discs [69] (cf. also [6]). General techniques of functional analysis (Fredholm theory, Sard–Smale theorem, theory of Fredholm structures) suggest that if this operator appears to be Fredholm, one can count on a reasonable structural theory for attached elliptic cells. In some sense, this is the most natural way of formulating a version of Fredholm theory for elliptic cells.

Now we present a typical result of this type available in our context. For us, of special importance are those targets M for which L_p is a Fredholm mapping at any point $p \in M$, and we introduce short-hand *admissible targets* for the targets possessing this property.

Recall that we are given an irreducible GCRS S in $\mathbb{R}^{n+1}$ with values in V. Construct another GCRS $D(S)$ with values in $W = V \times V$, which is a sort of "double" of S. If n is even, then $D(S)$ simply consists of two identical copies of S. If n is odd, then one adds to S the canonical GCRS corresponding to the second irreducible representation of Cl_n. Thus, the complex dimension of the target space of $D(S)$ is $2m(n)$.

Remark 6.2. Consideration of such "doubles" is suggested by the results of Sec. 4. From the standpoint of K-theory, this can be considered as a sort of "stabilization" and it is quite natural that this operation improves certain properties of the system (see [28]).

It turns out that noncompact admissible targets M in W can be obtained as images of appropriate embeddings of $\mathbb{R}^{2m(n)}$. We assume that all spaces of smooth mappings are endowed with the Whitney topology [70].

Theorem 6.1. *There exists an open set of embeddings $f : \mathbb{R}^{2m(n)} \to W$ such that for every point p of $M = f(\mathbb{R}^{2m(n)})$, the Gromov operator of the pair $(D(S), M)$ at point p is a (nonlinear) Fredholm operator of zero index.*

In other words, noncompact admissible targets exist for systems of the form $D(S)$. At the moment, we do not have any general existence results for compact admissible targets. Note that for every (compact or noncompact) admissible target, the Fredholmness of the Gromov operators combined with a standard application of the implicit-function theorem for Banach spaces in the spirit of [70] implies that the homotopically trivial families of attached elliptic cells are locally finite-dimensional. Note that we need not restrict ourselves to systems of the form $D(S)$.

Corollary 6.1. *If M is an S-admissible target, then the family of homotopically trivial S-hyperholomorphic cells attached to M at p is finite-dimensional.*

This result can be considered as a description of the subset of all hh-cells attached to M, which are close to a "degenerate" cell $f_p = p$. One obtains its natural generalization by considering the subset of all hh-cells attached to M which are sufficiently close to an arbitrary given cell g attached to M. One need not even assume vanishing of the class of g in $\pi_{n+1}(V, M)$.

Corollary 6.2. *For a given S-cell g attached to an S-admissible target M, the set of all S-cells attached to M near g is finite-dimensional.*

In both these cases, one encounters a natural problem of calculating the "virtual dimension of nearby attached hh cells" (see [70]) in terms of S, M, and the given cell g. Such formulas are available for pseudoanalytic discs (or Cauchy–Riemann cells, in our terminology) and they involve the notion of the Maslov index of a curve along a totally real submanifold M [55, 69].

To the author's knowledge, in the general case, this is an unsolved problem. As one can see from the discussion presented in the next section, progress in this problem depends on the availability of explicit index formulas for elliptic linear Riemann–Hilbert problems for GCRSs. Obviously, in particular cases, one can try to apply the analytic formulas for indices of elliptic boundary-value problems in Euclidean space obtained by Dynin [46] and Fedosov [51].

We omit the proof of Theorem 6.1, which relies on general results on the existence of elliptic boundary-value problems for GCRSs which were presented in Sec. 4. Examining the proof, one can conclude that for GCRSs in spaces of odd dimension (in our notation, this means that n is even), the same result can be obtained without passing to doubles; this yields the second main result of this section.

Theorem 6.2. *For every irreducible GCRS on a space of odd dimension distinct from 5 and 7, there exists an open set of embeddings of $\mathbb{R}^{m(n)}$ into $\mathbb{C}^{m(n)}$ such that their images are admissible targets for attached S-cells.*

Remark 6.3. The situation with compact admissible targets remains unclear. It is well known that there might be topological obstructions to the existence of such targets, which happens already for the classical Cauchy–Riemann system [55]. To clarify this, it is necessary to achieve better understanding of geometric conditions on admissible targets, which can hopefully be done in terms of transversality to a certain subspace of the Grassmanian $\mathrm{Gr}(2m(n), 4m(n))$. This can be done in some simple cases, for example, the necessary "algebraic analysis" of the Moisil–Theodoresco system is presented in [124]. In the general case, this seems to be quite difficult, and it is even unclear what is the dimension of the subset of "prohibited" $2m(n)$-subspaces. This point of view is related to some other approaches to the construction of admissible targets, which will be briefly discussed below.

We would like to emphasize that despite certain analogies with analytic discs, the situation with hh-cells is much more subtle. In particular, the restriction to systems of type $D(S)$ cannot be omitted.

For example, the most straightforward generalization of analytic discs attached to totally real surfaces [55] would be to consider the Fueter system in $\mathbb{R}^4 \cong \mathbb{H}$ (quaternionic regular functions [28]) and try to construct Fueter cells attached to 4-dimensional submanifolds in $\mathbb{R}^8$. However, it turns out that in this way, one cannot obtain a reasonable Fredholm theory for such cells, since in this situation, the Gromov operator is never Fredholm. The latter fact follows directly from Theorem 6.1 since the resulting system has $2 = l_1 \neq l_2 = 0$.

We conclude the section by mentioning some geometric problems suggested by our approach.

A natural problem raised by our results is to understand how can one characterize admissible targets geometrically. Gromov's general approach to solving underdeterminate systems [70] suggests that this issue should be related to certain special subsets of appropriate Grassmanians. Indeed, some initial steps in this direction can be made in a quite natural way and we proceed with a brief discussion of these matters.

Actually, a more comprehensive investigation of these connections shows that they can be conveniently described in terms of so-called Grassmanian geometries and calibrations, in the sense of [71]. We do not discuss relations to calibrated geometries, but some of those ideas are implicitly present in the comments to follow.

For a given GCRS, it is also interesting to investigate what can be the minimal possible dimensions k of target manifolds for which one can prove the Fredholmness of the Gromov operators. The Gromov h-principle suggests that admissible targets should satisfy some transversality condition with respect to certain special subset of Grassmanian $\mathrm{Gr}(k, 2m)$ defined by the characteristic matrix of the system in question.

To make this idea more precise, we first reexamine the classical case of analytic discs. Results of Gromov [70] and Alexander [6] translated to our language mean that admissible targets for analytic discs are precisely totally real submanifolds of $\mathbb{C}^k$. For $k = 2$, the condition of total reality means that the image of Gauss mapping Γ_M of a submanifold M does not intersect the subset of complex lines in $\mathrm{Gr}_{\mathbb{R}}(2, 4)$. Since in this case the target M is two-dimensional, this suggests considering generic targets M such that Γ_M is transversal to the two-dimensional subset of complex lines $\mathrm{Gr}_{\mathbb{C}}(1, 2)$ in the four-dimensional real Grassmanian $\mathrm{Gr}_{\mathbb{R}}(2, 4)$.

For such generic targets, their tangent planes can coincide with complex lines only at isolated points and one can eliminate these points by deforming M. For compact M, it is well known [17] that the only obstruction for eliminating points with complex tangents is given by the Euler characteristic $\chi(M)$. It can also be shown that for a noncompact contractible manifold M homeomorphic to $\mathbb{R}^2$, the set of embeddings into $\mathbb{R}^4$ without complex tangents is open and dense in the set of all such embeddings. The latter statement is exactly the special case of Theorem 6.1 for the classical Cauchy–Riemann system in $\mathbb{R}^2$.

Thus, it becomes clear that admissible targets can admit characterization by some genericity conditions like transversality, and to find such conditions, one should try to describe the subset of n-planes in $\mathbb{C}^k$ which can be represented as images of differentials of solutions to system S. Note that this is exactly what happens in the classical case, since for the usual Cauchy–Riemann systems, these images are complex lines.

In this case, admissible targets coincide with totally real ($2k$-dimensional) submanifolds. Note that they are not generic $2k$-dimensional manifolds since they can have complex tangents, and, actually, homological properties of the set of complex tangents are closely related to the topology of the submanifold considered [55].

As was already noted, for $n = 2$ and 3, one can indicate sufficient conditions on T_pM for a target manifold M to be admissible. This follows from the explicit criteria of the Fredholmness for RHPs for the Moisil–Theodoresco and Fueter systems obtained in [126]. It is interesting to obtain similar results for general GCRSs. It also seems tempting to relate these geometric aspects of GCRSs with the comprehensive algebraic analysis of the Moisil–Theodoresco and Fueter system developed in [3,124] by using the computer program CoCoA.

We would like to mention also the general problem of calculating the index of an elliptic RHP for GCRS. In principle, this is possible by using general results of Atiyah and Bott, which should lead to explicit formulas of Dynin–Fedosov type [46, 51], but it does not seem that anyone has ever written down these explicit formulas in terms of the characteristic matrix and boundary condition. Thus, it would be illuminating to obtain an exact recipe, or even an algorithm applicable in concrete situations. In low dimensions, some useful preparatory work for developing such an algorithm was done in [126].

We conclude by mentioning another promising perspective which emerges from the aforementioned connection between special Grassmanians and calibrated geometries. It is concerned with finding a proper calibration for a given GCRS. For the classical Cauchy–Riemann system, this can be worked out in full detail, and it turns out that the desired calibration is provided by the properly normalized Kähler form on the target space [71]. In fact, many properties of families of analytic discs can be derived directly from this interpretation, and one may hope that finding a proper calibration will be useful for achieving further progress in the theory of hyperholomorphic cells.

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