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CONFIGURATION SPACES AND SIGNATURE FORMULAS

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ABSTRACT. We show that nontrivial topological and geometric information about configuration spaces of linkages and tensegrities can be obtained using the signature formulas for the mapping degree and Euler characteristic. In particular, we prove that the Euler characteristics of such configuration spaces can be effectively calculated using signature formulas. We also investigate the critical points of signed area function on the configuration space of a planar polygon. We show that our approach enables one to effectively count the critical points in question and discuss a few related problems. One of them is concerned with the so-called cyclic polygons and formulas of Brahmagupta type. We describe an effective method of counting cyclic configurations of a given polygon and formulate four general conjectures about the critical points of the signed area function on the configuration space of a generic planar polygon. Several concrete results for planar quadrilaterals and pentagons are also presented.

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Introduction

We describe several new applications of the signature formulas for topological invariants [15] to the topological study of configuration spaces of linkages [12] and tensegrities [19]. Our approach is based on the possibility of representing the configuration spaces in question as the fibers of proper polynomial mappings and calculating their topological invariants by the so-called signature formulas [15]. In such manner we obtain a number of general results about configuration spaces and investigate in some detail the critical points of the signed area function on the configuration space of a planar polygon.

In line with this strategy, we begin by presenting some general results about polynomial mappings and signature formulas. The formula for the Euler characteristic in the form suggested in [5] is especially relevant for our purposes and so we reproduce it as Theorem 1.1. In the second section we collect the necessary information about configuration spaces of linkages and tensegrities. We show that the configuration spaces we deal with are compact algebraic or semi-algebraic subsets so that the signature formulas enable one to calculate their Euler characteristics.

Next, we consider the signed area as a function on the configuration space of a planar polygonal linkage and study its critical points. A general method of counting its critical points is described and a few concrete results are presented. It turns out that in all those cases the critical points are given by the the so-called *cyclic* configurations (i.e., the ones which can be inscribed in a circle) of linkage. For this reason we study the cyclic configurations as well and, in particular, give a general method of counting those configurations by means of signature formulas. We also discuss some corollaries and related results concerned with the areas of cyclic polygons and generalized Heron polynomials of

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D. Robbins [20]. In conclusion, we formulate four general conjectures about the critical points of an area function which are suggested by our results and considerations.

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1. Signature Formulas for Topological Invariants

The algebraic formulas for the mapping degree [7, 13] and Euler characteristic [5, 13] have already found many applications in concrete problems of geometry, topology, and singularity theory (cf. [15]). They are often called *signature formulas* [15] because they express the topological invariants in question in terms of signatures of effectively constructible quadratic forms. As was outlined in [13] and further developed in [5, 14], those formulas, in particular, provided an effective method for calculating the Euler characteristic of an explicitly given compact algebraic or semi-algebraic subset. It should be added that thanks to the existing computer programs for calculating the mapping degree (see, e.g., [18]) such calculations can be done quite effectively.

Recall that a real polynomial mapping $F : \mathbb{R}^s \to \mathbb{R}^t$ is defined by a collection of t polynomials $F_1, \dots, F_t$ in s variables with real coefficients. We will only deal with the cases where $s \geq t$. If $s = t$, then F is called a *polynomial endomorphism* (or a *polynomial vector field* as in [2]).

For any $y \in \mathbb{R}^t$, the set $F^{-1}(y)$ is called a fiber of F over the point y. A point $x \in \mathbb{R}^s$ is called a regular point of F if the rank of Jacobi matrix $J(F)(x)$ is maximal, i.e., equal to t (for $s = t$ this is obviously equivalent to $\det JF(x) \neq 0$). In the opposite case (i.e., if this rank is less than t) point x is called a singular point of F. A fiber $F^{-1}(y)$ is called regular if it does not contain singular points of F. In this case the point y is called the regular value of F. The set of regular values of F is denoted $\operatorname{Reg} F$.

As is well known, each regular fiber is a smooth manifold of dimension $s - t$ (see [4]). A mapping F is called *proper* if the preimage $F^{-1}(X)$ of any compact set $X \subset \mathbb{R}^t$ is a compact set in $\mathbb{R}^s$ (here and below we only consider the topologies induced by Euclidean metric). For convenience and brevity, a *proper real polynomial mapping* F as above will be referred to as a *propomap*.

Thus the fibers of a propomap are compact algebraic varieties. Correspondingly, regular fibers of a propomap F are compact smooth manifolds. By Sard's lemma [2], the set of singular values of F has measure zero, so a "generic" fiber of F is a smooth compact manifold of dimension $s - t$.

In particular, if $s = t$, then each fiber $F^{-1}(y)$ consists of a finite amount of points and it appears reasonable to consider the algebraic number of preimages of y, which leads to the concept of the mapping degree. More precisely, one can define the *mapping degree* $\deg F$ by the well-known formula

$$\deg F = \sum_{x \in F^{-1}(y)} \operatorname{sign} \det JF(x), \tag{1.1}$$

where sign denotes the sign of a real number, and $y \in \operatorname{Reg} F$ is an arbitrary regular value of F.

The sum on the right-hand side does not depend on $y \in \operatorname{Reg} F$ so we obtain an invariant of F. Moreover, the degree is invariant under proper homotopies and enjoys some other nice properties which make it very useful in many problems of topology and nonlinear analysis. An important role is also played by a local version of this notion, the *local (topological) degree* of an endomorphism which is often called the *index of an isolated zero* of a vector field [2].

The local degree $\deg_p F$ of a given polynomial endomorphism F is defined at any point p which is isolated in $F^{-1}(F(p))$. What is especially important is that, from the results of [7, 13], it follows that $\deg_p F$ can be computed in a purely algebraic way as the signature of a certain nondegenerate quadratic form which is explicitly constructible using the coefficients of components F_i of F. The same refers to the (global) topological degree $\deg F$. This method of computing topological degree can be implemented as a computer algorithm based on the computation of the Gröbner basis of the ideal (F) generated by the components of F in the algebra of real polynomials in s indeterminates (see, e.g., [18]).

The mapping degree provides one of the main topological tools of real algebraic geometry and we essentially use the fact that the Euler characteristic of any fiber of a given propomap can be expressed through the local degree as follows. As is known, it is sufficient to deal only with the case where all components of the map are homogeneous polynomials of the same degree.

Let $F : \mathbb{R}^s \to \mathbb{R}^t$ be a propomap with the components F_i of degree d. Let $y \in \mathbb{R}^t$, $X_y = F^{-1}(y)$, and $f_i = F_i - y_i$, $i = 1, \dots, t$. Note that X_y is a compact real algebraic variety, so its Euler characteristic is well-defined [4]. Set

$$h_i(x_0, \dots, x_s) = x_0^{d+1} f_i \left(\frac{x_1}{x_0}, \dots, \frac{x_s}{x_0} \right), \quad H_y = \sum_{j=1}^{t} h_i^2 - \sum_{k=0}^{s} x_k^{2d+4}.$$

Denote by $\operatorname{grad} H_y$ the gradient of H_y considered as a polynomial endomorphism of $\mathbb{R}^{s+1}$.

Theorem 1.1 (see [5]). *With the assumptions and notation as above, polynomial H_y has an isolated critical point at the origin and the Euler characteristic of X_y is given by the formula*

$$2\chi(X_y) = (-1)^s - \deg_{\mathbb{O}} \operatorname{grad} H_y, \tag{1.2}$$

where $\deg_{\mathbb{O}} \operatorname{grad} H_y$ *denotes the local degree of* $\operatorname{grad} H_y$ *at the origin of* $\mathbb{R}^{s+1}$.

Since the local degree is equal to the signature of the so-called Gorenstein quadratic form on the local algebra of mapping [7, 13], this theorem shows that the Euler characteristic is expressible through the signature of an explicitly constructible quadratic form. For this reason, we refer to formula (1.1) as the signature formula for the Euler characteristic. Using this formula and a standard process of eliminating inequalities by considering two-fold coverings [4], [14], one can obtain a similar result for a compact *semi-algebraic* subset.

Theorem 1.2 (see [14, 15]). *The Euler characteristic of an explicitly given compact semi-algebraic subset can be calculated in the algorithmic way through the signatures of explicitly constructible quadratic forms.*

In the next section we apply these results to the topological study of mechanical linkages and tensegrities.

Remark 1.1. For a quadratic mapping F, another way of calculating the Euler characteristics of fibers was proposed in [1]. This is sometimes useful because, as is well known, configuration spaces of linkages can be represented as the fibres of quadratic mappings. However, for calculating the Euler characteristics of workspaces and configuration spaces of tensegrities, one needs to use Theorems 1.1 and 1.2.

2. Configuration Spaces of Linkages and Tensegrities

Mechanical linkages, in particular planar mechanical linkages, and their configuration (moduli) spaces were studied in many papers. Linkages may be thought of as mechanisms build up from rigid bars (sticks) joined at flexible links (pin-joints). Some links may be fixed in the ambient space and the rest are supposed to be movable. In many problems it is important to know the totality of possible positions of the links in the ambient space and the range of positions available for a given link (workspace of the link). A general mathematical description of mechanical linkage runs as follows (cf. [12, 21]).

Recall that a *weighted graph* is defined as a triple $\Gamma = (V, E, d)$ consisting of a set of vertices V, a set of edges $E = \{(V_{i_k}, V_{j_k})\}$, and a weight function $d : E \to \mathbb{R}_+$ which assigns to every edge (V_{i_k}, V_{j_k}) a certain length $d(V_{i_k}, V_{j_k}) \in \mathbb{R}_+$. We always assume that Γ is connected, i.e., each pair of its vertices can be connected by a sequence of elements of E. It is also assumed that the set of vertices of V is decomposed in two subsets $V = V_f \cup V_m$, with $V_f = \{V_1, \dots, V_m\}$ and $V_m = \{V_{m+1}, \dots, V_n\}$. Vertices from V_f will be considered as *fixed*, while those from V_m as *movable*. For our further considerations it is natural to assume, and we do so, that $m \geq 2$ (which basically means that we factor out the action of the group of Euclidean motions, i.e., exclude shifts and rotations of the graph as a whole).

A connected weighted graph is called N-*realizable* (or realizable in $\mathbb{R}^N$) if there exists a mapping $f : V \to \mathbb{R}^N$ such that the Euclidean distance $|f(V_i) - f(V_j)|$ is equal to $d(V_i, V_j)$ for all $(V_i, V_j) \in E$. For example, a regular tetrahedron is (by definition) 3-realizable but not 2-realizable.

A connected weighted graph Γ is called a *planar mechanical linkage* if it is realizable in $\mathbb{R}^2$. In general, an N-realization of Γ is defined as the collection $(f(V_1), \dots, f(V_n)) \in \mathbb{R}^{nN}$. In order to get the definition of the N_d-configuration space of a mechanical linkage as above, one fixes m points $p_1, \dots, p_m$ in $\mathbb{R}^N$ such that $|p_i - p_j| = d(V_i, V_j)$ for all $V_i, V_j \in V_f$ with $(V_i, V_j) \in E$. Now the N_d-configuration space of Γ is defined as

$$C_N(\Gamma) = \{f \text{ realization of } \Gamma \mid f(V_f) = (p_1, \dots, p_m)\} / \operatorname{Iso}_+(\mathbb{R}^N),$$

where $\operatorname{Iso}_+(\mathbb{R}^N)$ denotes the group of all orientation preserving isometries of $\mathbb{R}^N$.

In other words, we factor out the motions of a realization as a rigid whole. There are versions of this definition where factoring is over the group of all (not necessarily orientation preserving) isometries or over the extension of $Iso_+(\mathbb{R}^N)$ by homotheties. These differences are inessential for our tasks, so we use the simplest definition presented above. Note that the same space can be defined as

$$C_N(\Gamma) = \{x = (x_j) \in \mathbb{R}^{nN} \mid x_j = p_j,\ 1 \leq j \leq m,\ |x_i - x_j| = d(V_i, V_j) \text{ if } (V_i, V_j) \in E\} / \operatorname{Iso}_+(\mathbb{R}^N).$$

Configuration spaces are also called the *moduli spaces* of mechanical linkage [12]. All configuration spaces are considered with a natural topology induced by the Euclidean distance. If the linkage is just a polygon with fixed sidelengths, one obtains configuration spaces of polygon studied in [12, 21].

From the topological point of view, $2d$-configuration spaces are especially interesting because it turned out that for any smooth *closed* (i.e., compact and without boundary) manifold M there exists a planar mechanical linkage Γ such that one of the components of its configuration space of the latter is diffeomorphic to M [12, 21]. In many cases Γ can be chosen so that $C(\Gamma)$ is connected and itself diffeomorphic to M.

Given a graph with fixed combinatorial structure, one obtains a mechanical linkage by choosing a N-realizable weight function d as above. We say that some property of a mechanical linkage is *generic* if it holds for almost all choices of the weight function. Using Sard's lemma it is easy to see that, for a generic mechanical linkage Γ, the N_d-configuration space $C(\Gamma)$ is a compact smooth manifold of the dimension $N(n-m) - (\operatorname{card} E - k)$, where n is the number of vertices of Γ, $\operatorname{card} E$ is the total number of edges, and k is the number of edges joining the vertices from V_f.

Configuration spaces of planar polygons, which gained a lot of attention in last ten years [12, 21], appear as a particular case of this general definition. Recall that Euler characteristics of configuration spaces of polygons were calculated in many cases (see, e.g., [11]). Our first observation is that, in principle, all such results could be obtained using Theorem 1.1.

Theorem 2.1. *The Euler characteristic of N_d-configuration space of any mechanical linkage $\Gamma = (V, E, d)$ as above can be calculated in an algorithmic way from its combinatorial data (V, E) and weight function d.*

The proof follows by noting that each configuration space is naturally represented as the fiber of a certain explicitly given quadratic mapping Q_Γ and then referring to Theorem 1.1. The components of the mapping Q_Γ are obtained by writing down (in the standard coordinates on $\mathbb{R}^{nN}$) the conditions that the squared distance from $f(V_i)$ to $f(V_j)$ should be equal to $[d(V_i, V_j)]^2$ for each edge $(V_i, V_j) \in E$. Obviously one has exactly $t(\Gamma) = \operatorname{card} E - k$ such conditions, where k is the number of edges joining vertices from V_f since the latter edges impose no conditions at all. Thus Q_Γ naturally appears as a quadratic mapping from $\mathbb{R}^{N(n-m)}$ to $\mathbb{R}^{t(\Gamma)}$. For evident geometric reasons, Q_Γ is proper and so one can directly apply formula (1.2) by taking $F = Q_\Gamma$. The result follows by noticing that in virtue of [7, 13] the local topological degree in the formula (1.2) can be calculated in an algorithmic way. It may be added that the Euler characteristic of any fiber of Q_Γ can also be calculated using results of [1].

This theorem has several concrete applications. As was shown in [16], it enables one to list all possible topological types of configuration spaces of planar pentagons. Using the scheme of [16] one can also describe all possible topological types of configuration spaces of planar quadrilaterals and various mechanical linkages with two-dimensional configuration space [3, 9].

Our next observation is related to the concept of work space of a given vertex. Recall that, for each movable vertex $V_j \in V_m$, its *work space* $W_\Gamma(V_j, N)$ is defined as the subset $\{f(V_j) : f \in C(\Gamma)\}$ of $\mathbb{R}^N$.

Theorem 2.2. *The Euler characteristic of the work space of any movable vertex can be calculated in an algorithmic way.*

To prove this result we notice first that, being a projection of the algebraic set $C(\Gamma)$, the work space is a semi-algebraic subset of the plane by "general principles" [4]. Moreover, it is easy to write down the inequalities which should be added to the components of the quadratic mapping Q_Γ constructed above. Thus the desired conclusion follows from Theorem 1.2.

Recently, it turned out that similar results can be obtained for the so-called *tensegrities* [19] which were brought to our attention by O. Karpenkov during a conference in Nijmegen in February of 2008. Informally, tensegrity is a subclass of the so-called spatial (3d) *truss structures* consisting of pin-jointed bars (struts) and cables (see, e.g., [19]). Thus they can be obtained from spatial linkages by replacing some of the edges by cables instead of rigid bars (struts). Tensegrities are usually characterized by certain stability properties. We will use one of the possible definitions which is most suitable for our approach. Namely, a truss structure is called a tensegrity if it has 3-realizations and the set of those is finite. In such a case, each of the realizations is automatically stable, in a natural and physically meaningful sense [19].

Tensegrities were invented by Kenneth Snelson in 1948 and gained popularity in the 1970s, particularly due to the activity of Buckminster Fuller. Their main applications are in architecture and engineering, while the mathematical aspects of this topic seem to be less developed [19]. Our goal is to show that our approach enables one to find the number of stable configurations of a tensegrity with prescribed combinatorial structure and fixed lengths of struts and cables.

To this end, notice first that the combinatorial structure of tensegrity T can be described by a weighted graph as above. Next, the fact that tensegrities are spatial structures corresponds to considering the 3-realizations of a given graph in $\mathbb{R}^3$. Thus it is appropriate to deal with the $3d$-configuration space, i.e., take $N = 3$. Finally, we can also formalize the fact that there are edges of two kinds. Namely, divide E in two subsets E_s (struts) and E_c (cables) and modify the above definition of configuration space as follows. The conditions corresponding to edges from E_s remain unchanged, while for each pair $(V_i, V_j) \in E$ one imposes a quadratic inequality $|x_i - x_j|^2 \le [d(V_i, V_j)]^2$ instead of the corresponding equality. In this way we obtain a semi-algebraic set called the configuration space $C(T) = C_3(T)$ of tensegrity T. The above definition of tensegrity obviously implies that $C(T)$ is a finite set. Hence the number of 3-relizations (stable configurations) of T is equal to the Euler characteristic of $C(T)$. These remarks combined with Theorem 1.2 immediately yield an analog of Theorem 2.1 for tensegrities.

Theorem 2.3. *The number of distinct configurations of a tensegrity with a given combinatorial structure and prescribed lengths of elements can be computed in an algorithmic way through the signatures of explicitly constructible quadratic forms.*

Since it is recognized that the problem of counting all possible configurations of a tensegrity is far from trivial [19], this result can hopefully find useful applications to analysis of concrete tensegrities. However, it should be added that the computational complexity of the corresponding algorithm is quite high even for the simplest tensegrities, so that we were unable to obtain new results in practically interesting cases.

Similar results can be obtained for configuration spaces of various different types. For instance, one may consider linkages such that certain vertices can slide along prescribed subspaces or submanifolds of the ambient space (in other words, possible positions of those vertices should belong to prescribed submanifolds). Such situations arise in circle packing problems (the author owes this observation to E. Wegert) and in gravitation toys of "spiderman" type (those were brought to our attention by T. Lomadze and N. Khimshiashvili). In a variety of such cases one can find the Euler characteristic of configuration space by our methods and derive certain conclusions about the qualitative behavior of the objects considered. Development and exposition of these topics is left for the future and we turn now to description of another type of application of signature formulas to the study of configuration spaces, which yields rather interesting results.

3. Signed Area of Planar Polygons

We proceed by investigating the signed (oriented) area of planar polygon (see, e.g., [6]) as a function on configuration space of a planar linkage. Recall that given an ordered set of n points $p_1 = (x_1, y_1), \dots, p_n = (x_n, y_n)$ in the plane, the signed area S of the corresponding n-gon is defined by the formula

$$2S = (x_1 y_2 - y_1 x_2) + \ldots + (x_n y_1 - y_n x_1). \tag{3.1}$$

In this section, we use symbol $C(n, 2)$ to denote the configuration space of a planar n-gon with unspecified but fixed sidelength vector l. In other words, $C(n, 2)$ is the collection of all possible planar configurations of an n-gon linkage (i.e., of a polygon with fixed sidelengths). In this context "n-gon linkage" means essentially the same as "n-gon with fixed (prescribed) sidelengths."

We wish to deal only with *generic sidelengths*, which as usual means that the vector of sidelengths l is taken from an (unspecified) open dense subset U_n of the parameter space of planar n-gons. In the case of planar polygonal linkages as above, the parameter space is an open subset P_n of the positive n-orthant $\mathbb{R}^n_+$ and one can explicitly indicate a collection of hyperplanes B_n, called the bifurcation diagram of configuration space $C(n, 2)$, such that the aforementioned open subset U_n of generic sidelengths is equal to the complement of B_n in P_n. Thus most of our considerations make sense for planar polygons with the sidelength vector belonging to U_n. The connected components of U_n, called chambers, play an essential role in the topological study of configuration spaces [12].

From the discussion in Sec. 2 it follows that, for generic sidelengths (in the above sense), the configuration space $C(n, 2)$ is a compact orientable smooth manifold of dimension $n3$. Moreover, it is known that the topological type (homeomorphism class) of $C(n, 2)$ is the same for all sidelengths belonging to the same chamber [12]. Actually, $C(n, 2)$ can be defined as a fiber of a proper quadratic mapping $Q_l : \mathbb{R}^{2n-2} \to \mathbb{R}^{n-1}$ and the bifurcation diagram B_n can be identified with the set of singular values of Q_l (see [12]).

To see this, note that the action of $Iso_+(\mathbb{R}^2)$ obviously does not change the totality of configurations of a planar linkage. Thus we can change its position in the plane using the action of this group. In this way, we can obviously place the first vertex at the origin and the first side along the Ox-axis. It is also easy to realize that a homothety transformation of polygon does not change the topological type of its configuration space. Thus we may assume that the first side is of length 1, i.e. $l_1 = 1$, without loss of generality. It follows that we may assume that the first two vertices are $v_1 = (0, 0), v_2 = (1, 0)$ and we keep this assumption from now on. This done, it becomes obvious that the configuration space can be defined as a fiber of a quadratic mapping as above.

In order to complete the precise description of our problem, notice that the signed (oriented) area of a polygon naturally defines a (infinitely) differentiable function $S : C(n, 2) \to \mathbb{R}$ on each generic configuration space. Our aim is to study the critical points of S for a generic sidelength vector belonging to U_n. For $n = 4$ (planar quadrilaterals) and partially for $n = 5$ (planar pentagons), we managed to obtain considerable information about singularities of S using the signature formulas. To describe our approach to this problem in a consistent way, let us present relevant results about the topological structure of configuration spaces $C(4,2)$. In doing so we freely use standard topological concepts and methods of singularity theory [2]. The topological structure of configuration spaces of generic quadrilaterals can be described as follows.

Proposition 3.1. *The bifurcation diagram B_4 consists of eight hyperplanes $\{c_1a_1 + c_2a_2 + c_3a_3 = 1, c_i = 1\}$ and its complement in P_4 has 26 connected components.*

Proposition 3.2. *Each generic configuration space $C(4,2)$ is homeomorphic either to circle S^1 or to the disjoint union of two circles.*

Actually, one can also describe all possible topological types of $C(4,2)$ for nongeneric sidelengths as well, and the signature formulas appear helpful to this end. For example, using the method of Section 2 one easily finds that the Euler characteristic of the planar configuration space of a square is equal to -3. With some additional work it can be shown that this space is homeomorphic to the union of three circles each pair of which has a common point, which of course gives the same value of the Euler characteristic. In fact, performing similar calculations for each component of U_4 one can show that the Euler characteristic of moduli spaces of nondegenerate planar quadrilaterals can only take four values: $0, -1, -2, -3$ (formally one could also consider degenerate quadrilaterals where the length of the longest side is equal to the sum of the other three sidelengths, in which case the configuration space is obviously one point). Moreover, one can describe the local geometric structure of possible singularities of $C(4,2)$ so that the topology of $C(4,2)$ is known in great detail, which gives a sufficient background for determining the global behavior of an area function on such configuration spaces.

Let us now explain how one can count the critical points of S. Assume as above that $l_1 = 1$, $v_1 = (0,0)$, and $v_2 = (1,0)$. Then $C(4,2)$ is defined by three obvious quadratic equations with four unknown coordinates of movable vertices v_3, v_4. The area function in this case is given by $2S = x_3y_4x_4y_3 + y_3$. One can now introduce Lagrange multipliers c_1, c_2, c_3 and obtain a (7×7)-system of quadratic equations (first four coordinates of) the solutions to which give the critical points of S. It is not difficult to compute the Jacobian of these equations and see that it is not identically equal to zero. Hence the equations are algebraically independent and the set of solutions is finite. It follows that the number of critical points of S is also finite and equal to the number of real solutions to this system. Since the Euler characteristic of the solution set can be effectively computed using the signature formulas, we conclude that the number of critical points of S can be found for each concrete sidelength vector $l \in U_4$. Using a standard topological argument, it is easy to show that the qualitative behavior of S remains the same for each component of U_4. Thus, for a complete investigation of the area function it is sufficient to choose a vector of sidelengths in each component of U_n and find the number of critical points for those concrete sidelengths using Theorem 1.1.

Realization of this program yields the following results. If a generic configuration space has one component (homeomorphic to a circle), then the number of critical points is equal to 2 (one maximum and one minimum). If a generic configuration space has two components, then the number of critical points is equal to 4 (one maximum and one minimum on each component). It can be verified that all these critical points are nondegenerate, and so S is a Morse function on $C(4,2)$.

One can further explain this conclusion as follows. Recall that a polygon is called cyclic [20] if it can be inscribed in a circle (i.e., if it has a circumscribed circle). According to a classical result of J.Steiner, each polygonal linkage L has a convex cyclic configuration and S attains its maximum, say M, precisely at this configuration (see, e.g., [8]). Due to the skew-symmetry of the oriented area, the minimum value of S is $-M$ and it is attained at the configuration obtained from the preceding one by reflection in the first side of linkage L, which in our setting obviously reduces to changing the signs of ordinates of all vertices (since the first side is placed on the Ox-axis). One can explicitly compute M by the Brahmagupta formula:

$$M^2 = \frac{1}{16}(2a^2b^2 + \ldots + 2c^2d^2 - a^4 - b^4 - c^4 - d^4 + 8abcd) = (p-a)(p-b)(p-c)(p-d), \qquad (3.2)$$

where a, b, c, and d are the sidelengths and p is the half-perimeter of L [20] (thus this is a direct generalization of the classical Heron formula for the area of a triangle).

If the configuration space has two components, then S is positive on one of them and negative on the other. On the positive component, the maximum is again realized by the convex cyclic configuration. In this case there also exists a self-intersecting cyclic configuration of L which corresponds to the minimum of S on this component. This is the first manifestation of the curious fact that cyclic configurations provide critical points of the area function. According to D. Robbins [20], the area of a self-intersecting cyclic configuration is given by the following analog of the Brahmagupta formula:

$$m^2 = \frac{1}{16}(2a^2b^2 + \ldots + 2c^2d^2 - a^4 - b^4 - c^4 - d^4 - 8abcd). \qquad (3.3)$$

Thus, the minimum of S on the positive component can also be computed explicitly. Of course, the same picture, modulo the sign, is observed for the "negative" component. With a little more work one can verify that the critical points of S are nondegenerate, i.e., S is a Morse function on $C(4, 2)$. Moreover, the amount of critical points is the minimal one permitted by the topology of the configuration space, hence it is determined by the topology of the configuration space and remains the same for each component of U_4.

These observations can be briefly summarized as follows: the signed area defines an exact Morse function S on the configuration space of a generic quadrilateral linkage and the critical points of S are given by the cyclic configurations of the quadrilateral considered. This is in fact a quite instructive conclusion because it remains meaningful and illuminating for planar polygons with arbitrary number of sides.

It is now natural and intriguing to investigate this issue for pentagon linkages. We were able to show that similar phenomena are indeed observed for certain pentagons. Consider, for example, a pentagon with $l_1 = 1$, $l_2 = l_5 = 2$, and $l_3 = l_4 = 1/3$. It is easy to see that its configuration space $C(5, 2)$ is homeomorphic to a disjoint union of two tori T^2. On the "positive" component of configuration space, function S has one maximum attained at the convex cyclic configuration and three other critical points (one minimum and two saddle-points) attained at self-intersecting cyclic configurations. In this case one can also compute the critical values of S using the generalized Heron polynomials introduced by D. Robbins [20] (see also [22]). Moreover, one can prove that S is a Morse function on $C(5, 2)$. So in this case we also obtain quite detailed information about the critical points of S.

As we have seen, cyclic configurations naturally appear in the singularity analysis of S. In view of the above, it is natural to consider them in parallel with the "critical" configurations of L corresponding to the critical points of S. A natural strategy of further research is then to count configurations of both types for all chambers (components) of U_5 and verify if the foregoing conclusion holds for generic pentagon linkages.

In general, using the same approach as for planar quadrilaterals, it is easy to see that the critical configurations in a generic configuration space $C(n, 2)$ can be counted as the real solutions to a certain $(3n - 5) \times (3n - 5)$-system of quadratic equations. It is also easy to see that the cyclic configurations correspond to the roots of a $(2n - 1) \times (2n - 1)$-system of quadratic equations. One just adds three unknowns defined as the radius and coordinates of the center C of the circumscribed circle and adds n quadratic equations expressing the fact that squared distances from all vertices to C are equal to the radius. Thus the exceptional configurations of both types can be effectively counted using the signature formula for Euler characteristic indicated in Theorem 1.1.

This approach enabled us to obtain considerable information about the critical and cyclic configurations of planar pentagons using the computer programs for calculating the degree of mapping. In all the cases considered, the results obtained on this way are similar to those for planar quadrilaterals. There is good evidence that the methods described above will allow us to completely investigate this problem for generic pentagon linkages. At the same time, our discussion suggests a sort of research program concerned with the critical points of a signed area function, which is outlined in the last section.

4. Conjectures and Remarks

The results and considerations presented above suggest four general conjectures on the critical points of a signed area function on the configuration space of a generic planar polygon.

Conjecture 1. *The critical points of signed area function are given by the cyclic configurations of a generic polygon.*

If this is really so, one can visualize the critical points and critical values of the area function in a purely geometric way by merely constructing all cyclic configurations. The problem of counting and constructing cyclic polygons was considered in [20, 22]. The geometric results contained in those papers agree with the ones obtained by our methods, which provides additional evidence in favor of this conjecture.

Conjecture 2. *All critical points of a signed area function on the configuration space of a generic polygon are nondegenerate.*

This conjecture is highly plausible for certain general reasons and should follow by a proper application of transversality theorems [2].

Conjecture 3. *The number of critical points and their types* (*Morse indices*) *are determined by the topology of a generic configuration space.*

There also exists a stronger version of this conjecture stating that the area defines an exact Morse function on generic configuration space. Since the number and indices of critical points of an exact Morse function are determined by the topology of the underlying manifold, the latter conjecture is indeed stronger. A would-be corollary is that the number and indices of critical points remain the same over each component of U_n. There is good evidence that this fact can be proved by a "wall-crossing" argument described in [12].

Conjecture 4. *The critical values of an area function on the configuration space of a generic polygon can be explicitly calculated in terms of sidelengths.*

The results of [20, 22] suggest also a stronger version of this conjecture which says that the set of critical values coincides with the set of real roots of the generalized Heron polynomial introduced by D. Robbins [20]. We believe that these conjectures are plausible and instructive. If they hold true, one obtains a conceptual generalization of previously known results on areas of cyclic polygons [8, 20, 22].

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