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LOOP SPACES AND RIEMANN-HILBERT PROBLEMS

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Abstract. We present a survey of recent results concerned with generalizations of the classical Riemann-Hilbert transmission problem in the context of loop spaces. Specifically, we present a general formulation of a Riemann-Hilbert problem with values in an almost complex manifold and illustrate it by discussing two particular cases in more detail. First, using the generalized Birkhoff factorization theorem of A. Pressley and G. Segal we give a criterion of solvability for generalized Riemann-Hilbert problems with coefficients in the loop group of a compact Lie group. Next, we present a visual example of solution to a Riemann-Hilbert problem with values in the immersed loop space of three-dimensional sphere. Finally, we describe a geometric construction of Fredholm structures on loop groups and relate them to the canonical Fredholm structures on Kato Grassmannians.

Introduction. The aim of this paper is to present a survey of recent developments which emerged in the framework of a geometric approach to Riemann-Hilbert problems suggested in [3], [4] and further developed in [5], [6], [31], [21], [22], [23], [7], [8]. Most of those developments can be naturally formulated in terms of loop groups and, more generally, of certain loop spaces. It is the arising interplay between loop spaces and Riemann-Hilbert problems that we are going to describe and advocate.

It should be noted that the settings and results presented below owe much to discussions and joint investigations with B. Bojarski (cf. [7], [8]). Actually, the present paper only covers a part of the results presented in a joint talk of B. Bojarski and the present author at the "Geometry and topology of manifolds" conference in Będlewo in May 2005. The results presented here were obtained by the present author independently and announced in [23], [24], [25]. The other results mentioned in the foregoing talk were obtained jointly with B. Bojarski and will be presented in a forthcoming joint publication.

It should be added that different aspects of Riemann-Hilbert problems were discussed in joint papers of B. Bojarski and A. Weber [9], [10]. The author uses this opportunity

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to thank Professor B. Bojarski for inspiring cooperation and supporting the idea of this publication. Thanks also go to G. Misiolek for several useful discussions on the geometry of loop groups and detailed comments on his results from [28], [29] which strongly influenced our discussion of Fredholm structures on loop groups given in Section 4.

To provide some general background and motivation for our considerations, notice first that loops are actually involved in the very formulation of the Riemann-Hilbert transmission problem. Indeed, nondegenerate matrix functions on a simple closed contour can be naturally interpreted as loops in general complex linear group $GL(n,\mathbb{C})$. Thus loops in $GL(n,\mathbb{C})$ can be thought of as *coefficients* of the classical Riemann-Hilbert problems. As was shown in [21], [22], one can formulate a natural analog of Riemann-Hilbert problem where coefficients are taken from the group of regular loops in a compact Lie group. A considerable part of the classical theory can be extended to this setting and in the present paper we present a solvability criterion in terms of the so-called *generalized Birkhoff factorization* developed in [31].

Another type of generalization of Riemann-Hilbert problem arises in relation to Gromov's theory of pseudoholomorphic mappings between almost complex manifolds [19]. Along these lines, a general definition of linear conjugation problem in the context of almost complex manifolds was suggested in [24], [25] which gave a wide extension of the classical Riemann-Hilbert problem. If the almost complex manifolds in question are finite-dimensional, a version of Fredholm theory for such linear conjugation problems can be derived from Gromov's results. When the source manifold is just the Riemann sphere $\mathbb{CP}^1$ one obtains Fredholm theory for *analytic discs* in almost complex manifolds, which is a straightforward generalization of the classical Fredholm theory for Riemann-Hilbert problem.

As a natural next step, it seems reasonable to consider such problems in the case when a target manifold is infinite dimensional. As was observed in [31], [7], loop groups and restricted (Kato) Grassmannians often have natural almost complex structures so it seems natural to consider Riemann-Hilbert problems for functions with values in such spaces. This is the second type of generalized Riemann-Hilbert problems, called *loopy Riemann-Hilbert problems*, which we consider in this paper.

Recall that, as was revealed in [3], [4], many geometric aspects of classical linear conjugation problems with sufficiently regular (differentiable, Hölder) coefficients can be formulated and successfully studied in terms of restricted Grassmannians and loop groups. Thus our loopy Riemann-Hilbert problems reveal new geometric aspects of the classical Riemann-Hilbert problem. It should be added that some problems of modern mathematical physics (such as construction of instantons in Yang-Mills theory [1]) appear closely related to our Riemann-Hilbert problems with values in loop spaces. For this reason we believe that the setting suggested below may appear useful and deserves consideration by its own.

Let us now say a few words about the structure of the paper. We begin by recalling necessary definitions and auxiliary results about loop spaces and Riemann-Hilbert problems. In particular, we give a general formulation of Riemann-Hilbert problems in the context of almost complex manifolds. Generalized Riemann-Hilbert problems with coefficients in loop groups are discussed in section 2. In section 3 we give some explicit examples of solutions to Riemann-Hilbert problems with values in loop spaces. In conclusion we give an explicit construction of Fredholm structures on loop groups and relate them to the canonical Fredholm structures on the Kato Grassmannians constructed in [8].

1. Preliminaries on loop spaces and Riemann-Hilbert problems. We begin with recalling a few basic concepts and definitions. Let X be a topological space and $\mathbb{T} = S^1$ be the unit circle. Recall that the *free loop space* LX of X is defined as the set of all continuous maps $\mathbb{T} \to X$ endowed with the compact-open topology [14]. If $x_0 \in X$ is a distinguished point then the *based loop space* ΩX is defined as the set of all those loops which send the number $1 \in \mathbb{T}$ to the distinguished point x_0.

We are only interested in the case when $X = M$ is a smooth (infinitely differentiable) manifold of positive dimension. Then one can also consider subspaces consisting of loops of a fixed regularity class (C^k, Hölder, Sobolev). All of them are referred to as loop spaces of M and denoted by symbols LM or ΩM decorated by appropriate indices and/or exponents.

It is well known that loop spaces of Riemannian manifold M carry a number of interesting geometric structures. In particular, they often have natural complex or almost complex structures [31], [26] and they can also be endowed with various natural metrics induced from the metric on M. Of the main interest for us is the case when $M = G$ is a compact Lie group with a left-invariant metric [14].

Then loop spaces LG and ΩG endowed with pointwise multiplication of loops become infinite-dimensional topological groups. Groups of such type are called *loop groups* [31]. One can obtain Banach Lie groups by considering only loops of appropriate regularity class (e.g., Sobolev).

The group of based loops ΩG has a natural complex structure for which the operator J is defined as the Hilbert transform on the Lie algebra of ΩG [31]. Almost complex structures on loop spaces of three-dimensional manifolds were introduced by J.-L. Brylinski [12] and L. Lempert [26]. We use those structures to formulate the Riemann-Hilbert problem for loop valued functions.

Another aim we pursue in this paper is to show that loop groups can be endowed with so-called *Fredholm structures* [15]. Such structures were first constructed using the generalized Riemann-Hilbert problems introduced in [21]. Now we wish to show that the same structures can be constructed using the riemannian exponential mapping on loop groups. This construction is presented in section 4. Fredholm structures on loop groups have already been discussed in the literature (see, e.g., [17], [21], [8], [25]). However our construction, which relies on the properties of Riemannian exponential mapping established in [28], [29], essentially differs from the approaches used in preceding papers on the same topic.

We give now a formulation of Riemann-Hilbert problem in the context of almost complex manifolds appropriate for the topics considered below. Before doing so, notice that there exist nowadays a number of commonly used concepts of Riemann-Hilbert problem. We only deal with Riemann-Hilbert problems considered as *boundary value* *problems for holomorphic functions.* A general formulation of Riemann-Hilbert problem of such type was given in [24]. Here we elaborate upon the definition from [24] so that it becomes applicable to holomorphic functions of one complex variable with values in loop spaces of certain types.

Recall that an almost complex structure J on a smooth manifold M is defined as a smooth family of linear operators $J_p = J(p)$ in tangent spaces $T_pM, p \in M$, such that $J_p^2 = -I$ (here and in the sequel I always denotes the identity mapping of the corresponding space). In particular, each complex manifold (for example, $\mathbb{C}^n$ or $\mathbb{CP}^n$) has a canonical complex structure defined by the operator of multiplication by $\imath$ in each tangent space. The concept of holomorphic mapping between complex manifolds is generalized in the context of almost complex manifolds as follows.

Consider two almost complex manifolds (M, J) and (N, J'). A differentiable mapping $F : M \to N$ is called *holomorphic* if its differential dF intertwines the given almost complex structures, namely:

$$dF(p)J_p = J'_{F(p)}dF(p), \tag{1}$$

for each $p \in M$. Sometimes such mappings are called *pseudo-holomorphic* (cf. [19]) but we prefer to omit the prefix "pseudo" since this cannot lead to a misunderstanding in the sequel. As is well known, the local description of such mappings is closely related to *Bers-Vekua equation* and *generalized analytic functions* [19], [5].

It is easy to verify that, for finite-dimensional complex manifolds, the above definition gives the usual concept of holomorphic mapping. In particular, taking a domain in the complex plane endowed with the canonical complex structure we get a concept of holomorphic function of one complex variable with values in an almost complex manifold N. If M or/and N are infinite-dimensional complex manifolds modeled on complex Banach spaces, proving equivalence of the two definitions of holomorphic map requires some care but we need not discuss here those nuances.

If M is a one-dimensional complex manifold (Riemann surface) then the image of a holomorphic mapping $M \to N$ is called a *holomorphic curve* in N. In particular, if M is a domain in $\mathbb{C}$, such an image is called an *analytic disc.* If $M = \mathbb{CP}$ is the Riemann sphere then its holomorphic images are called *holomorphic spheres.* Obviously, a holomorphic sphere is a union of two analytic discs glued along their boundaries. In the third section we present an example of such situation in the loop space of a 3-sphere.

In order to formulate Riemann-Hilbert problem in almost complex setting, suppose moreover that M is decomposed into two (open) parts M_+, M_- by a smooth divisor (hypersurface) Γ. Introduce the function spaces as follows. For an open subset $U \subset M$, let $A(U, N)$ denote the set of all mappings defined and continuous in $\overline{U}$ taking their values in N and holomorphic in U. Fix finally a continuous mapping (current) Φ on Γ with values in a subgroup G of infinite-dimensional Lie-Frechet group Diff N consisting of smooth diffeomorphisms of N.

Then Riemann-Hilbert problem defined by quintiple (M, N, Γ, G, Φ) is formulated as the problem of describing the totality of pairs $(X_+, X_-) \in A((M_+, N) \times A(X_-, N))$

satisfying the following condition on divisor Γ:

$$X_+(p) = \Phi(p)(X_-(p)),\ p \in \Gamma, \tag{2}$$

where $\Phi(p)$ acts on $X_-(p)$ as an element of Diff$N$. Notice that by taking $M = \mathbb{C}, \Gamma = \{|z| = 1\}, N = \mathbb{C}^n, G = GL(n, \mathbb{C})$, and some $(n \times n)$-matrix-function on Γ in the role of Φ, one obtains a classical version of Riemann-Hilbert problem called the *problem of linear conjugation* (cf. [30], [4], [22]). If we take $M = \mathbb{C}, \Gamma = \{|z| = 1\}$ and N equal to a complex representation space of a compact Lie group G, then we come to the generalized linear conjugation problem considered in the next section.

Thus we see that each pair of almost complex manifolds (M, N) yields a collection of analytic problems whose nature strongly depends on the geometry and topology of the manifolds and group G considered. The classical theory of Riemann-Hilbert problem and singular integral equations appears as a particular case of this general scheme. In a short review like this one it makes no sense to discuss the general scheme in depth so we just describe some new aspects and phenomena appearing when N is taken to be a loop space with an almost complex structure. The most natural examples are given by groups of loops in a compact Lie group [31] and immersed loop spaces of 3-folds [12]. Up to our knowledge, Riemann-Hilbert problems of the above type have never been discussed in the literature even for these concrete examples so in the sequel we make an attempt to fill this gap.

It appears more convenient to use a little bit more geometric language. Namely, in complete analogy with the finite-dimensional setting, while dealing with Riemann-Hilbert problems in loop spaces one is inevitably led to considering images of holomorphic mappings of discs into a given loop space. Thus it becomes necessary to deal with analytic discs in certain loop spaces which could be called *loop-valued analytic discs* in order to distinguish them from the usual analytic discs in finite-dimensional complex manifolds. For brevity, we call them *loopy analytic discs*. Similar geometric objects in loop spaces of 3-folds were earlier considered by J.-L. Brylinski [12] and L. Lempert [26] and appeared to be related to interesting geometric constructions. We use this terminology and related constructions in section 3.

2. Riemann-Hilbert problems with coefficients in loop groups. In this section we describe a generalization of the linear conjugation problem introduced in [21] and further investigated in [22], [23]. Consider the Riemann sphere $\mathbb{P} = \overline{\mathbb{C}}$ decomposed as the union of the unit disc D_+, unit circle T, and exterior domain D_-, which contains the infinite point ∞ denoted by N ("North Pole"). The main innovation is to permit coefficients from a loop group. More precisely, we take functions on the circle with values in a compact Lie group G as coefficients and search for piecewise holomorphic mappings with values in a given representation space of G. The precise statement of the problem is given below, and the rest of the section is devoted to its investigation. It appears that in the case of a compact Lie group G one can develop a reasonable theory analogous to the classical one which relies on the recent generalization by A. Pressley and G. Segal [31] of the well-known factorization theorem due to G. Birkhoff [30]. It is easy to indicate several natural regularity classes for coefficients which guarantee that the problem is described by a Fredholm operator in corresponding functional spaces. A natural framework for our discussion is provided by a generalized Birkhoff factorization theorem and Birkhoff stratification of a loop group so we present first some auxiliary concepts and results.

Let G be a connected compact Lie group of the rank p with the Lie algebra A. As is well known [31], each of such groups has a complexification $G_{\mathbb{C}}$ with the Lie algebra $A_{\mathbb{C}} = A \otimes \mathbb{C}$. This fact is very important as it provides complex structures on loop groups and this is the main reason why our discussion is restricted to compact groups. Let LG denote the group of continuous loops in G endowed with the point-wise multiplication and usual topology [31]. We need some regularity conditions on loops and for the sake of simplicity let us first assume that all loops under consideration are (at least once) continuously differentiable. For an open set U in $\mathbb{P}$ let $A(U, \mathbb{C}^n)$ denote the subset of $C(\widetilde{U}, \mathbb{C}^n)$ formed by those vector-functions which are holomorphic in U. Assume also that we are given a fixed linear representation r of the group G in a vector space V. For our purposes it is natural to assume that V is a complex vector space. Notice that for a compact group one has a complete description of all complex linear representations [31].

We are now in the position to formulate the problem we are interested in. Namely, having fixed a loop $f \in LG$, the (homogeneous) generalized linear conjugation problem (GLCP) R_f with coefficient f is formulated as a question about the existence and cardinality of pairs $(X_+, X_-) \in A(D_+, V) \times A(D_-, V)$ with $X_-(N) = 0$ satisfying the transition condition on T

$$X_+(z) = r(f(z)) \cdot X_-(z). \tag{3}$$

For any loop h on V we obtain also an inhomogeneous problem $R_{f,h}$ (with the right-hand side h) by replacing the transition equation (3) by the condition

$$X_+(z) - r(f(z)) \cdot X_-(z) = h(z). \tag{4}$$

In other words, we are interested in the kernel and cokernel of the natural linear operator T_f expressed by the left-hand side of the formula (4) and acting from the space of piecewise holomorphic vector-functions on $\mathbb{P}$ with values in V into the loop space LV. To avoid annoying repetition, when dealing with the inhomogeneous GLCP it will always be assumed that the loop h is Hölder-continuous, which is a usual assumption in the classical theory [30].

Remark 1. In the particular case when $G = U(n)$ is the unitary group we get that $G_{\mathbb{C}} = GL(n, \mathbb{C})$ is the general linear group. If we take r to be the standard representation on $\mathbb{C}^n$, then we obtain the classical linear conjugation problem. Note that even in this classical case one obtains a plenty of such problems at the expense of taking various representations of $U(n)$, and the result below can be best illustrated in this situation.

Needless to say, the same picture is observed for all groups but as a matter of fact only irreducible representations of simple groups are essential. Moreover, the exceptional groups of Cartan's list will also be excluded and the remaining groups will be termed as "classical simple groups". It would not be appropriate to reproduce and discuss here all necessary concepts and constructions from the theory of Lie groups. All necessary results on Lie groups, in a form suitable for our purposes, are contained in [31] and we repeatedly refer to this book in the sequel.

Let f be a loop on G. We would like to associate with f some numerical invariant analogous to the classical partial indices [30]. To this end let us choose a maximal torus T^p in G and a system of positive roots. Then following [31] one can define the nilpotent subgroups $N_0^{\pm}$ of $G_{\mathbb{C}}$ whose Lie algebras are spanned by the root vectors of $A_{\mathbb{C}}$ corresponding to the positive (respectively negative) roots. We also introduce subgroups $L^{\pm}$ of $LG_{\mathbb{C}}$ formed by the loops which are the boundary values of holomorphic mappings of the domain B_+ (respectively B_-) into the group $G_{\mathbb{C}}$, and the subgroups $N^{\pm}$ consisting of the loops from L^+ (respectively L_-) such that $f(0)$ belongs to N_0^+ (respectively $f(N)$ belongs to N_0^-). The following fundamental result was proved in [31].

Decomposition Theorem. *Let G be a classical simple compact Lie group, and $H = L^2(T, A_{\mathbb{C}})$ be the polarized Hilbert space with $H = H_+ \oplus H_-$, where H_+ is the usual Hardy space of boundary values of holomorphic loops on $A_{\mathbb{C}}$. Then we have the following decomposition of the groups of based loops LG:*

(i) LG is the union of subsets B_K indexed by the lattice of holomorphisms of T into the maximal torus T^p.

(ii) B_K is the orbit of $K \cdot H_+$ under N^- where the action is defined by the usual adjoint representation of G. Every B_K is a locally closed contractible complex submanifold of finite codimension d_K in LG, and it is diffeomorphic to the intersection L_K^+ of N^- with $K \cdot L_1^- \cdot K^{-1}$, where L_1^- consists of loops equal to the unit at the infinite point N.

(iii) The orbit of $K \cdot H_+$ under N^+ is a complex cell C_K of dimension d_K. It is diffeomorphic to the intersection L_K^+ of N^+ with $K \cdot L_1^- \cdot K^{-1}$, and meets B_K transversally at the single point $K \cdot H_+$.

(iv) The orbit of $K \cdot H_+$ under $K \cdot L_1^- \cdot K^{-1}$ is an open subset U_K of LG, and the multiplication of loops gives a diffeomorphism from $B_K \times C_K$ into U_K.

Recall that in the classical case this result reduces essentially to the Birkhoff factorization theorem for matrix loops [30].

Let us introduce the corresponding construction in our setting. Namely, for a loop f on G the (left) Birkhoff factorization will be called its representation in the form

$$f = f_+ \cdot H \cdot f_- , \tag{5}$$

where f_+ belongs to the corresponding group $L^{\pm}G$ and H is some homomorphism of T into T^p. Now it is evident that the points (ii) and (iv) of the theorem imply the following existence result.

Proposition 1. *Every differentiable (and even Hölder class) loop in a classical simple compact group has a factorization.*

Note that we could also introduce the right factorization with the reversed order of f_+ and f_- and the result would also be valid. Our choice of the factorization type is consistent with the problem under consideration. Taking into account that any homomorphism H from (5) is determined by a sequence of p integer numbers $(k_1, \dots, k_p)$, we get that this sequence can be associated with any loop f. These natural numbers are called (left) *G-exponents* (or *partial G-indices*) of f. Their collection will be denoted $K(f)$.

It is easy to prove that $K(f)$ does not actually depend neither on the terms of the representation (5) nor on the choice of the maximal torus. For a given maximal torus the proof of this fact can be obtained as in the classical case, while the independence on the choice of a maximal torus follows from the well-known fact that any two maximal tori are conjugate [31]. The exponents provide basic analytical invariants of loops and also permit a topological interpretation.

PROPOSITION 2. *Two loops lie in the same connected component of LG if and only if they have the same sum of exponents.*

This follows easily from the contractibility of subgroups $L^{\pm}$ and the point (ii) of the Decomposition Theorem.

REMARK 2. In the classical case when $G = U(n)$ we obtain the usual partial indices, and Proposition 2 reduces to the evident observation that the connected components of LU_n are classified by the sum of partial indices which is known to coincide with the increment of the determinant argument of a matrix function along the unit circle [30].

In these terms it appears possible to give a simple solvability criterion and find the dimension of solution space for an linear conjugation problem with coefficients in a loop group [21].

THEOREM 1. *Let G be a compact Lie group. A linear conjugation problem with coefficient $f \in LG$ is solvable if and only if there exist nonnegative G-exponents of f. The dimension of kernel is equal to the sum of all positive exponents of f.*

The index formula is also analogous to the classical case. These results enable one to develop a sufficiently complete Fredholm theory and investigate the stability properties of G-exponents. This theory has several applications discussed in [22], [23]. One of the most spectacular applications was the construction of pairwise non-isomorphic Fredholm structures on loop group LG indexed by irreducible representations of G [23]. In section 4 we show that Fredholm structures on loop groups can also be constructed in an essentially different and seemingly more direct way.

3. Explicit solution to a loopy Riemann-Hilbert problem in S^3. As was mentioned in section 1, an interesting instance of our general Riemann-Hilbert problem (2) arises if one takes target manifold N to be the immersed loop space of a 3-fold. Such loop spaces were introduced by J.-L. Brylinski [12] and have important applications in modern mathematical physics. Actually, in this case one may visualize solutions to such problems and we now wish to give an explicit example of such kind based on the famous Hopf fibration $S^3 \to S^2$.

For our purposes it is appropriate to define the Hopf fibration in complex setting. Consider the unit sphere $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ and the Riemann sphere $\mathbb{P} = \overline{\mathbb{C}} \cong S^2$. Then the Hopf fibration $H : S^3 \to S^2$ is defined by sending each point $(z_1, z_2) \in S^3$ into the ratio of its coordinates interpreted as a point of $\mathbb{P}$, i.e. $H(z_1, z_2) = z_1/z_2$. It is evident that fibres of H are the the *complex big circles*, i.e. intersections of complex lines in $\mathbb{C}^2$ with S^3, so one can consider its "inverse" as a map from S^2 into the space of smooth loops on S^3.

Let us endow S^3 with the standard riemannian metric inherited from the ambient Euclidean space. The sphere S^3 endowed with this metric will be called the round 3-sphere and denoted S^3_r. We can now consider the corresponding Brylinski loop space BS^3_r [12] and get a map $H^{-1} : \mathbb{P} \to BS^3_r$. Thus it becomes possible to treat the latter map from the viewpoint developed in previous sections. A straightforward calculation shows that its differential dH^{-1} intertwines the almost complex structures on $\mathbb{P}$ and BS^3_r and so it defines a holomorphic curve in BS^3_r. Details of the argument can be found in [24].

Correspondingly, the restriction of H^{-1} on any disc in $\mathbb{P}$ defines a loopy analytic disc in BS^3_r. In particular, taking the unit disc and its complement we get a solution to loopy Riemann-Hilbert problem (2) with the constant coefficient whose value at each point $p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the identity mapping of BS^3_r. In terms of analytic discs, one can state that S^3_r is the union of images of two loopy analytic discs. Despite their simplicity, these observations lead to an instructive conclusion provided by the following theorem from [24].

THEOREM 2. *The map H^{-1} defines a holomorphic embedding of the Riemann sphere into the space of oriented immersed loops on S^3_r. In other words, H^{-1} defines a holomorphic curve in BS^3_r. In particular, a round 3-sphere S^3_r can be represented as a union of two loopy analytic discs glued along their boundaries, i.e., S^3_r foliated by the complex great circles is a solution to a loopy Riemann-Hilbert problem in BS^3.*

We can now use the above observations and stereographic projection $\Pi : S^3 \to \mathbb{R}^3$ in order to obtain a similar geometric picture in $\mathbb{R}^3$, which, in particular, enables one to visualize certain analytic discs in $\mathbb{R}^3$. It is well known that the image of the unit disc under $\Phi = \Pi \circ H^{-1}$ is a solid torus $\mathbb{T}$ bounded by a round torus (torus of revolution) $\mathbb{T} \cong T^2$ in $\mathbb{R}^3$ (see, e.g., Ch.10 in [2]). The same holds for any disc in $\mathbb{C} \subset \mathbb{P}$ centered at the origin.

It is also known (but probably not so "well-known") that the images of complex big circles under Π are genuine (metric) circles which have been discovered by I.Villarceau (nowadays they are called *Villarceau circles* [2]). They can be defined as the intersections of a round torus T^2_r with the bitangent plane passing through the center of torus T^2_r [2]. Thus the preimages $\Phi^{-1}(w)$ of points w from the unit disc are exactly the Villarceau circles.

On each round torus T^2_r, Villarceau circles come in two families each of which consists of nonintersecting circles. Two Villarceau circles belonging to the same family will be called *coherent*. Thus each of the two families of coherent Villarceau circles defines a foliation of a round torus. Any two circles in the same family on a given round torus are linked with the linking number 1 which corresponds to the well-known fact that the Hopf invariant of Hopf fibration is equal to one.

Consider now a *round solid torus* $\mathbb{T}_r$ defined as the closure of interior of a round torus T^2_r. Obviously, $\mathbb{T}_r$ is a union of continual family of coaxial round tori lying inside $\mathbb{T}_r$ and the axial circle which is equal to the intersection of their interiors. One sees now that the family of coherent Villarceau circles of all those round tori can be chosen in such a way that together with axial circle they give a foliation of $\mathbb{T}_r$ by loops (circles) which are mutually linked with the linking number 1. Taking into account the above remarks we conclude that a Villarceau round torus gives a precise picture of a loopy analytic disc in $\mathbb{R}^3$ which we call *Villarceau toroid*. Thus we have established the following final result.

PROPOSITION 3. *Each round solid torus in $\mathbb{R}^3$ foliated by Villarceau circles is the image of an analytic disc in $B\mathbb{R}^3$.*

To our mind, this beautiful geometric picture alone gives a sufficient justification for the setting and considerations presented above. Using methods of nonlinear analysis it is possible to show that one can deform a Villarceau toroid in such a way that all leaves of the foliation remain closed and it still represents a loopy analytic disc. Such deformations can be described by explicit equations using methods of deformation theory. For us the main point is that they provide examples of loopy analytic discs different from Villarceau toroids.

PROPOSITION 4. *There exist small perturbations of the Villarceau toroid which can be represented as the images of loopy analytic discs.*

This fact may be used to construct solutions to loopy Riemann-Hilbert problems with non-constant coefficients which are sufficiently close to the identity. It would be very interesting to construct similar examples with coefficients not necessarily close to identity. Clearly, similar constructions and results make sense for other 3-folds foliated by loops, for example, for tangent circle bundles of compact orientable two-dimensional surfaces without boundary. One can also consider similar problems for Seifert fibrations, which suggests a number of interesting open problems (cf. [24]).

4. Fredholm structures on loop spaces. We pass now to Fredholm structures on loop spaces and begin with necessary definitions from functional analysis. For a Banach space E, let $L(E)$ denote the algebra of bounded linear operators in E endowed with the norm topology. Let $F(E)(F_k(E))$ denote the subset of Fredholm operators (of index k). Let also $GL(E)$ stand for the group of units and $L(E)$ and denote by $GC(E)$ the so-called *Fredholm group* of E defined as the set of all invertible operators from $L(E)$ having the form "identity plus compact".

Recall that a Fredholm structure on a smooth manifold M modeled on (infinite dimensional) Banach space E is defined as a reduction of the structural group $GL(E)$ of tangent bundle TM to subgroup $GC(E)$ [15]. In the sequel we only deal with the case when $E = H$ is a separable Hilbert space and M is taken to be the group of Sobolev H^1-loops in a compact Lie group G.

Since $GL(H)$ is contractible $F_0(H)$ is the classifying space for $GC(H)$ bundles [15]. For a Hilbert manifold M, defining a Fredholm structure on M is equivalent to constructing an index zero Fredholm map $M \to H$ [16]. It was also shown in [16] that a Fredholm structure on M can be constructed from a smooth map $\Phi : M \to F_0(H)$, i.e., from a smooth family of index zero Fredholm operators parameterized by points of M. This is actually the most effective way of constructing Fredholm structures which has already been used in [17], [22].

We are now going to describe an explicit construction of such families on appropriate loop groups using the Riemannian exponential mapping described in the first section. In the sequel we freely use its properties established in [28], [29]. Actually, the very idea of constructing Fredholm structures using exponential mapping appeared in relation with results of [28], [29] where it was proven that, for a compact Lie group G, the exponential map $\exp = \exp_e : LA \to LG$ is a Fredholm map of index zero. This fact is crucial for the construction presented in the next section. Moreover, in [28], [29] one finds a more general argument which derives the fredholmness of exponential mapping from the compactness of curvature operators and permits further generalizations to more general classes of loop spaces.

Fredholm structures on loop groups already gained some attention in [17], [18], [23]. We present now a very explicit construction which may have certain advantages from the point of view of further generalizations. A closely related construction of Fredholm structure on the so-called restricted infinite Grassmannian was given in [8].

We achieve our goal by indicating an explicitly given family of index zero Fredholms on $LG = H^1(\mathbb{T}, G)$. Discussions with G. Misiolek were crucial for finding an appropriate explicit construction. Recall that by LA we denote the loop algebra consisting of H^1-loops in Lie algebra A and there is defined the exponential map $\exp_e : LA \to LG$. For $g \in LG$, let $\exp_g : T_gLG \to LG$ be the exponential map at point g. Let $v \in LA$ and γ_v be the corresponding geodesic through e in the direction of v, i.e., $\gamma_v(t) = \exp_e(tv)$. Let further J be the Jacobi vector field along γ_v with $J(0) = 0, \nabla_{\gamma_v} J(0) = w$, where $w \in LA$. In other words, $J(t) = d\exp_e(tv)(tw)$.

Put $u(t) = \tau^v_{0,t}(J(t)) \in LA$ then u is a solution to the initial problem

$$\partial_t^2 u + \tau^v_{0,t} \circ R(\tau^v_{t,0} u, \dot\gamma_v)\dot\gamma_v = 0, u(0) = 0, \partial_t u(0) = w. \tag{6}$$

Then we can define a linear transformation $\Psi(e) : LA \to LA$ by putting $\Psi(e) = E_{t,e}(v)$, where $E_{t,e}(v)w = u(t)$. Notice that this operator is of the form $tI + K_t$, where K_t is a compact operator smoothly depending on t. Let moreover $v', w' \in T_gLG$ and consider the geodesic $\gamma_{v'}(t) = \exp_g(tv')$, where $\exp_g : T_gLG \to LG$ is the exponential map at point g. As above, let J_g be a Jacobi vector field on γ' with $J_g(0) = 0$, $\nabla_{\dot\gamma_g} J_g(0) = w'$, and let $\tau^{v'}_{t,0} : T_gLG \to T_{\gamma_g(t)}LG$ be parallel translation. Put now $u_g(t) = \tau^{v'}_{t,0} \circ J_g(t)$ and notice that $u_g(t0$ is a solution to

$$\partial_t^2 u_g + \tau^{v'}_{0,t} \circ R(\tau^{v'} t, 0 u_g, \dot\gamma_g)\dot\gamma_g = 0, u_g(0) = 0, \partial_t u_g(0) = w_g. \tag{7}$$

Thus putting $E_{t,g}(v')(w') = u_g(t)$ we obtain a linear endomorphism of T_gLG. Consider now the map

$$g \mapsto \Psi(g)(\cdot) = L_{g^{-1}*g} \circ E_{t,g}(v') \circ L_{g*e} : LA \to LA. \tag{8}$$

We claim that it actually defines a smooth family of index zero Fredholms. Indeed, let us rewrite the above formula as follows:

$$\begin{aligned}\Psi(g)(\cdot) &= L_{g^{-1}*g}(\tau^{v'}_{0,t}(L_{g*\exp_e(tv)} L_{g^{-1}*g}(tv') \circ d\exp_e(L_{g^{-1}*g(tv')}) \circ L_{g^{-1}*g} \circ L_{g*e}(t(\cdot))) \\ &= L_{g^1*g} \circ \tau^{v'}_{0,t} \circ L_{g*\exp_e(tv)} \circ d\exp_e(tv)(t(\cdot)) = L_{g^1*g} \circ \tau^{v'}_{0,t} \circ L_{g*\exp_e(tv)} \circ \tau^v_{t,0}(E_{t,e}(t\cdot)).\end{aligned}$$

Notice now that $d\exp_e(tv)$ is an operator of the form "invertible plus compact" while all other operators in the last expression are bounded invertible operators. This implies that the composition is still an operator of the form "invertible plus compact" hence it is a Fredholm operator of index zero. It is easy to see that the above family of index zero Fredholms is smoothly depends on point $g \in LG$.

Collecting these observations together and taking into account the main result of [16] we can end up with a smooth Fredholm structure on LG. It is now easy to see that the exponential map becomes a Fredholm map of index zero with respect to the canonical Fredholm structure on LA and the Fredholm structure on LG provided by our construction. In this way we arrive to the following result.

THEOREM 3. *The group LG of free Sobolev H^1-loops in a compact Lie group G endowed with H^1-metric has a smooth Fredholm structure such that the riemannian exponential mapping* $\exp : T_eLG \to LG$ *becomes a Fredholm map of index zero.*

REMARK 3. It can be actually shown that the Fredholm structure provided by the theorem is uniquely defined up to the concordance by the requirement that the exponential map is Fredholm of index zero. Thus we obtain a canonical concordance class of Fredholm structures on LG.

REMARK 4. As was proven in [16], each Fredholm structure on manifold M induces a zero index Fredholm map of M in its model. It is now natural to conjecture that such a map of LG into T_eLG can be obtained by constructing a sort of "pseudo-inverse" to exponential map exp. It would be instructive to find an explicit description of such a pseudo-embedding. It would be also interesting to define the same structure by an explicitly given atlas on LG.

Using the general techniques of Fredholm structures theory, one can derive some immediate consequences of the results presented above.

COROLLARY 1. *The Fredholm structure induced by exponential mapping exists on a based loop group.*

COROLLARY 2. *The Fredholm structures on based loop groups are compatible with the Fredholm structures on the restricted Grassmannians constructed in [8].*

Furthermore, existence of Fredholm structures on loop groups enables one to study functorial properties of those groups in the framework of global analysis. For example, it is easy to verify that each homomorphism of Lie groups $\phi : G \to H$ induces an index zero Fredholm map $L\phi : LG \to LH$. Since an integer-valued mapping degree is well-defined for index zero Fredholm maps, one gets an integer $\deg L\phi$ and it becomes tempting to calculate it in terms of algebraic properties of homomorphism ϕ. Analyzing the above construction one finds out that Fredholm structures can be constructed on more general loop spaces when the ambient manifold M need not be a Lie group.

In fact, in order to perform the key construction of index zero Fredholms one just needs to have a canonical way of identifying an arbitrary tangent space with the tangent space at reference point. This can be achieved, for example, for a parallelizable manifold M and for some classes of homogeneous spaces of not necessarily compact Lie groups. The fact that this family consists of index zero Fredholms would follow from the fact that the exponential map is Fredholm. Thus our construction is applicable for loop spaces satisfying these two conditions. In such way we obtain the following generalization of Theorem 3.

THEOREM 4. *Let M be a parallelizable compact smooth riemannian manifold. Then the space of free H^1-loops LM can be endowed with a natural Fredholm structure such that the riemannian exponential map is a Fredholm map of index zero.*

In particular, immersed loop spaces of three-dimensional manifolds can be endowed with Fredholm structures and one may wish to compare them with the structures coming from the loopy Riemann-Hilbert problems. Summing up, we believe that the results presented in this paper confirm that the interplay between loop spaces and Riemann-Hilbert problems leads to interesting problems and deserves further investigation.

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