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Loewner Conjecture for Quasihomogeneous Polynomials

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ABSTRACT. We prove that the Loewner conjecture holds true for quasihomogeneous real polynomials of two variables and give an explicit bound for the topological index of the Loewner vector field in terms of the weights of variables. We also formulate a quaternionic analog of the named conjecture and discuss its connections with several topics of differential topology and singularity theory.

Key words: *Cauchy-Riemann operator, Loewner vector field, topological index, mapping degree, Euler characteristic, umbilical point.*

1. We deal with a well-known conjecture attributed to Loewner (see, e.g., [1, 2, 3]). Let P be a real polynomial of two variables. Obviously, P defines a smooth (infinitely differentiable) real-valued function on the plane which we identify with the complex plane $\mathbf{C}$. Recall that the Cauchy-Riemann operator D acts on a differentiable function f on $\mathbf{C}$ by the formula $Df = \partial f / \partial x + i\, \partial f / \partial y$. Separating the real and imaginary parts of Df one obtains a pair of real-valued functions or a vector field on $\mathbf{C}$. If f is a smooth function, then one can iterate the action of D and, for each natural n, consider the vector field $L_n f$ corresponding to the function $D^n f$. In particular, one can apply this construction to a real polynomial P as above. The resulting vector-fields $L_n P$ are called *Loewner vector fields* of P (as in [2, 3]). Thus, for each natural n, one can speak of the n-th Loewner field of P. Recall that, for a smooth vector field V, the local topological (or mapping) degree deg (V,z) is defined at any isolated zero z of V (see, e.g., [4] for a modern exposition of the theory of mapping degree).

The Loewner conjecture is concerned with the local degrees of vector fields $L_n f$. Namely, let P be a polynomial which does not contain monomials of degree less than three and such that the origin is its isolated critical point.

The *Loewner conjecture* claims that, for an arbitrary natural n, if the vector field $L_n P$ has an isolated zero at the origin then its index (local topological degree) at the origin does not exceed n. In other words:

$$\deg(L_n P, 0) \le n.$$

Thus this is in fact a whole infinite sequence of conjectures and it is convenient to denote by $L(n)$ the special case referring to the n-th Loewner field. It is well known that the validity of $L(2)$ has important consequences in the theory of umbilical points of two-dimensional surfaces, in particular, it would imply the famous Caratheodory conjecture about the existence of (at least) two umbilical points on a smooth surface homeomorphic to the sphere [1].

Despite such a simple formulation and important applications, the Loewner conjecture remains largely open. For n=1, its validity follows from the classical results of Poincare and Bendixson (see, e.g., [3]). In order to put this fact in a proper context, we will explain below how it can be derived from a general result on the local degree of gradient vector field established by the present author and proved in full detail in [4]. For clarity and convenience this observation is formulated as Theorem 1. It is easy to show that the estimate given in Theorem 1 is exact. Thus the Loewner conjecture for n=1 ($L(1)$ in our notation) is well understood.

However, the situation is drastically different already for n=2. Namely, $L(2)$ was only established for various special classes of polynomials [2,3,5]. The most general result belongs to N. Ando who proved that $L(2)$ holds true for any *homogeneous* polynomial (of two real variables) P and gave exact estimates for deg $(L_2P, 0)$ in terms of the (algebraic) degree of P [5].

The aim of this note is to show that $L(2)$ holds for each quasihomogeneous polynomial and give estimates for deg$(L_2P, 0)$ in terms of the quasihomogeneous type of P (weights of variables and quasihomogeneous degree of P). Our Theorem 2 gives a direct generalization of the aforementioned result of N. Ando. The method of proof is essentially different from the one used by N. Ando and reveals curious connections of LC with some topics from differential topology and singularity theory. Our approach also suggests further extensions of LC, some of which are mentioned in the sequel. This research was conducted in the framework of an INTAS project. The author acknowledges financial support by INTAS grant No. 05-1000008-7805.

2. In order to give precise formulations of our results let us recall a few necessary concepts and constructions. Recall that a critical point of polynomial P is called algebraically isolated if it is isolated as the critical points of P considered as function of two complex variables [6]. In such a case the Milnor number of the critical point is defined which serves as a certain complexity measure of the critical point [6]. Recall also that the corank of P is defined as the corank of its Hessian.

Theorem 1. *Let P be a real polynomial with an algebraically isolated critical point at the origin of corank 2 and let m be the Milnor number of P at the origin. Then*

$$-[\sqrt{m}]-1 \leq \deg(L_1P, 0) \leq 1.$$

As will be shown below, this follows from our previous results about the geometry of local level surfaces. Before giving the details of the argument we present our second result which may be considered as the main result of this paper. To this end recall that polynomial P is called $(w_1, w_2; d)$ quasihomogeneous if all the monomials present in P have the same degree d as the variables are considered with the weights w_1, w_2. For such a polynomial, the Milnor number at the origin can be expressed through the weights and degree by the well-known formula going back to J. Milnor [6].

Theorem 2. *Let P be a real quasihomogeneous polynomial with an algebraically isolated critical point at the origin of corank 2 and let m be the Milnor number of P at the origin. Then*

$$-2[\sqrt{m}]-2 \leq \deg(L_2P, 0) \leq 2.$$

Before giving an outline of proof we make some comments. As was mentioned above, the Milnor number m can be computed explicitly. Thus our result provides a two-sided estimate for the Loewner index in terms of weights and degree of P. For a homogeneous polynomial, both weights are equal to one and it is easy to verify that the above inequalities turn into the ones given by N. Ando in [5]. As was shown in [5], these estimates are exact (sharp) for homogeneous polynomials. We were able to verify that exactness holds for many quasihomogeneous types but the exactness in general remains unknown.

3. We are now going to explain how the above theorems can be derived from our previous results about the Euler characteristic of local level surfaces near an isolated critical point. First of all notice that, for a real polynomial P as above, the Loewner vector field L_1P differs from the gradient ∂P only a multiple. Thus the values of their local degrees coincide, so in order to prove Theorem 1 it is sufficient to prove a similar statement for the gradient of P. Recall now that there exists a remarkable relation between the local gradient degree and Euler characteristics of local level surfaces. Namely, for any real polynomial in n variables with an algebraically isolated critical point at the origin one has:

$$e(\{P=a\} \cap S) = 1-(-1)^n \deg(\partial P, 0),$$

where a is a sufficiently small real number, S is a sufficiently small sphere centered at the origin and symbol e stays for the Euler characteristic (see [4] for details). Notice further that, for n=2, the intersection of the local level surface with sphere S consists of a finite amount of points, hence the Euler characteristic in the left hand side is non-negative. This immediately gives the second inequality in Theorem 1. It remains to notice that the first inequality follows from the definition of the Milnor number and a well-known estimate for the local degree. Thus we see that Theorem 1 is merely a consequence of general results of singularity theory.

As to Theorem 2, the situation appears to be entirely similar but one should use a more complicated formula, also proved in [4], which relates the Milnor number to the sum of indices of critical points of the restriction of P to S. This in fact appears to be a form of the Euler-Poincare formula for manifolds with boundary. In this way one can show that the half-sum of the local degrees at the boundary is expressible through the Milnor number. Next, one takes into account that, according to the main result of [5], this half-sum is equal to deg $(L_2P,0)$. Combining these two observations one arrives at the inequalities given in Theorem 2. In order to make this argument rigorous one needs to make some perturbations of P so that the results of [4] become applicable, which is not necessarily the case for the original polynomial P. Thus the full proof becomes rather lengthy so, the technical details of the argument will be presented elsewhere.

The link between the Loewner conjecture and singularity theory which we used above, seems interesting by itself. In particular, it is now appropriate to wonder if a similar reasoning may be applied to the next case of the Loewner conjecture, namely, for proving the inequality deg $(L_3P, 0) \leq 3$. We succeeded to prove this inequality for several series from the Arnold list of singularities of low modality. It would be interesting to do the same by applying a proper version of Morse theory as above.

4. Our discussion also suggests that the Loewner conjecture can be generalized as follows. Consider a real polynomial P in four variables and identify $\mathbf{R}^4$ with the skew-field of quaternions $\mathbf{H}$. Notice then that there exists a natural analogue of the Cauchy-Riemann operator, usually called the *Fueter operator* [7], which acts on a quaternionic function f by the formula

$$Ff = \partial f/\partial x + i\,\partial f/\partial y + j\,\partial f/\partial u + k\,\partial f/\partial v.$$

Applying it repeatedly to a polynomial P as above, we obtain a sequence of vector fields which will be called *Fueter vector fields* and denoted F_nP. A natural problem now is to estimate the local degrees of Fueter vector fields. In this setting, already the case where n=1 is more delicate. Using the aforementioned relation between the Euler characteristic and local degree, it is easy to find out that there can be no universal upper bound for the local degree of F_1P. Namely, the degree deg $(F_1P,0)$ can take arbitrarily big positive values depending on the algebraic degree of polynomial P. To our mind, this fact emphasizes a very special flavour and delicacy of the Loewner conjecture. Its natural analogue in quaternionic setting becomes then to find the exact low and upper bounds for deg $(F_1P, 0)$ for polynomials with the Milnor number m.

This problem seems quite nontrivial even for homogeneous polynomials of degree d. In this case we can indicate reasonable bounds in terms of the Petrovsky number $P(4;d)$. We refer to [4, 6] for the definition and general discussion of Petrovsky numbers.

Theorem 3. *Let P be a real homogeneous polynomial of four variables of degree $d \geq 3$ with an isolated critical point at the origin. Then*

$$-P(4;d) \leq \deg (F_1P,0) \leq P(4;d).$$

This can be proved in the same way as Theorem 1 using a natural linear transformation which connects the Fueter field with the gradient vector field of P and the estimate for the gradient degree given in [4, 6]. It remains unclear if those bounds are exact. Actually, this is a particular case of a general problem of obtaining exact estimates for gradient vector fields which remains unsolved for polynomials of degree bigger than three [4]. Anyway, for n=1 one has a reasonable conjecture but it remains unclear what are the exact estimates for deg $(F_nP,0)$ for $n \geq 2$. It would be interesting to find out if such estimates in quaternionic case have geometric applications in the spirit of the Caratheodory conjecture.

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