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**Vol. 53, No. 5, 2018**
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Distributed worldwide by Springer. *Journal of Contemporary Mathematical Analysis* ISSN 1068-3623.

## Differential Equations

ISSN 1068-3623, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2018, Vol. 53, No. 5, pp. 247–259. © Allerton Press, Inc., 2018.
Original Russian Text © S. S. Kharibegashvili, N. N. Shavlakadze, O. M. Jokhadze, 2018, published in Izvestiya Natsional'noi Akademii Nauk Armenii, Matematika , 2018, No. 5, pp. 32-52.

DIFFERENTIAL EQUATIONS

# On the Solvability of a Mixed Problem for an One-dimensional Semilinear Wave Equation with a Nonlinear Boundary Condition

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Received July 25, 2016

**Abstract**—In this paper, for an one-dimensional semilinear wave equation we study a mixed problem with a nonlinear boundary condition. The questions of uniqueness and existence of global and blow-up solutions of this problem are investigated, depending on the nonlinearity nature appearing both in the equation and in the boundary condition.

**MSC2010 numbers :** 35L20, 35L71

**DOI:** 10.3103/S1068362318050011

**Keywords:** *Semilinear wave equation; nonlinear boundary condition; a priori estimate; local, global and blow-up solutions.*

## 1. INTRODUCTION. THE STATEMENT OF THE PROBLEM

In this paper, in the domain $D_T := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ 0 < t < T\}$ of the plane of independent variables $x$ and $t$, we consider a mixed problem of determination of a solution $u(x,t)$ of a semilinear wave equation of the form:

$$Lu = u_{tt} - u_{xx} + g(u) = f(x,t), \quad (x,t) \in D_T, \tag{1.1}$$

satisfying the initial conditions:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le l, \tag{1.2}$$

and the boundary conditions:

$$u_x(0,t) = F[u(0,t)] + \alpha(t), \quad u_x(l,t) = \beta(t)u(l,t) + \gamma(t), \quad 0 \le t \le T, \tag{1.3}$$

where $g, \ f, \ \varphi, \ \psi, \ \alpha, \ \beta, \ \gamma$ and $F$ are given functions, and $u$ is the unknown real function.

Note that for $f \in C(\overline{D}_T), g \in C(\mathbb{R}), \ F \in C^1(\mathbb{R}), \ \varphi \in C^2([0,l]), \ \psi \in C^1([0,l]), \alpha, \beta, \gamma \in C^1([0,T])$ necessary conditions of solvability of the problem (1.1)-(1.3) in the class $C^2(\overline{D}_T)$ are the following second order consistency conditions:

$$\begin{gathered} \varphi'(0) = F[\varphi(0)] + \alpha(0), \quad \psi'(0) = F'[\varphi(0)]\psi(0) + \alpha'(0), \\ \varphi'(l) = \beta(0)\varphi(l) + \gamma(0), \quad \psi'(l) = \beta'(0)\varphi(l) + \beta(0)\psi(l) + \gamma'(0). \end{gathered} \tag{1.4}$$

We set $\Gamma = \Gamma_1 \cup \omega_0 \cup \Gamma_2$, where $\Gamma_1 : x = 0, \ 0 \le t \le T; \ \omega_0 : t = 0, \ 0 \le x \le l; \ \Gamma_2 : x = l, \ 0 \le t \le T.$

**Definition 1.1.** *Let the functions*

$$f \in C(\overline{D}_T), \quad g, \ F \in C(\mathbb{R}), \quad \varphi \in C^1([0,l]), \quad \psi \in C([0,l]), \quad \alpha, \ \beta, \ \gamma \in C([0,T]) \tag{1.5}$$

*satisfy the following first order consistency conditions:*

$$\varphi'(0) = F[\varphi(0)] + \alpha(0), \quad \varphi'(l) = \beta(0)\varphi(l) + \gamma(0). \tag{1.6}$$

[2] The research was supported by Shota Rustaveli National Science Foundation grant No FR/86/5-109/14.

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*A function $u$ is said to be a strong generalized solution of the problem (1.1)-(1.3) of the class $C$ in the domain $D_T$ if $u \in C(\overline{D}_T)$, and there exists a sequence of functions $u_n \in C^2(\overline{D}_T)$ such that the following conditions are satisfied:*

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_T)} = 0, \tag{1.7}$$

$$\lim_{n\to\infty} \|u_n(\cdot,0) - \varphi\|_{C^1(\omega_0)} = 0, \quad \lim_{n\to\infty} \|u_{nt}(\cdot,0) - \psi\|_{C(\omega_0)} = 0, \tag{1.8}$$

$$\lim_{n\to\infty} \left\|u_{nx}(0,\cdot) - F[u_n(0,\cdot)] - \alpha(\cdot)\right\|_{C(\Gamma_1)} = 0, \tag{1.9}$$

$$\lim_{n\to\infty} \left\|u_{nx}(l,\cdot) - \beta(\cdot)u_n(l,\cdot) - \gamma(\cdot)\right\|_{C(\Gamma_2)} = 0. \tag{1.10}$$

**Remark 1.1.** In the case $\alpha = 0$ and $\gamma = 0$, in Definition 1.1 we assume that the sequence $u_n$ is such that $u_n \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma_1, \Gamma_2) := \left\{v \in C^2(\overline{D}_T) : (v_x - F(v))|_{\Gamma_1} = 0,\ (v_x - \beta v)|_{\Gamma_2} = 0\right\}$.

**Remark 1.2.** It is clear that the classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1)-(1.3) is a strong generalized solution of that problem of the class $C$ in the domain $D_T$.

Note that nonlinear boundary conditions of the form (1.3) arise, for instance, in the description of the process of longitudinal vibrations of a spring in the case of elastic fixing one of its endpoints, when tension is not subjected to linear Hooke's law and is a nonlinear function of blending (see [1], p. 41], as well as, in the description of processes in the distributed self-vibrating systems (see [2], p. 405 and [3]).

The problem (1.1)-(1.3) in the case of one-dimensional spatial variable, as well as, its multivariate version has been studied in a number of papers (see [4]-[8], and references therein). On the whole, in these papers the solution $u = u(x,t)$ of the problems of interest are considered in the energetic spaces, when the solution and its partial derivatives for a fixed $t$ belong to Sobolev spaces with respect to the spatial variables. In the paper [9], for equation (1.1) was investigated the mixed problem, when at the endpoint $x = l$ is imposed Dirichlet homogeneous condition. When jumping from this case to the case of Robin type boundary condition (see condition (1.3) with $x = l$), additional difficulties arise not only of technical nature, but also in obtaining a priori estimate of the solution, as well as, in construction of a representation of a solution of the corresponding linear problem, which plays an essential role in obtaining of an existence theorem.

In this paper, we study the problem (1.1)-(1.3) in the class of continuous functions for sufficiently broad classes of nonlinear functions, appearing both in equation (1.1) and in boundary condition (1.3).

The paper is organized as follows. In Section 2, under some conditions imposed on functions $g, F, \alpha, \beta, \gamma$ appearing in equation (1.1), we obtain a priori estimate for a strong generalized solution $u$ of the problem (1.1)-(1.3) of the class $C$ in the domain $D_T$ in the sense of Definition 1.1. In Section 3, we reduce the problem (1.1)-(1.3) to an equivalent system of Volterra type nonlinear integral equations in the class of continuous functions. Section 4 is devoted to the proof of local solvability of the problem (1.1)-(1.3) in variable $t$. In Section 5, we prove a uniqueness theorem for a solution of the nonlinear mixed problem (1.1)-(1.3). In Section 6, we consider the question of solvability on the whole in the domain $D_T, T \le l$ of the problem (1.1)-(1.3) in the class of continuous functions, as well as, the question of existence of a global classical solution of this problem in the domain $D_\infty$. Finally, in Section 7, we consider the question of existence of a blow-up solution of the problem (1.1)-(1.3).

## 2. AN A PRIORI ESTIMATE OF A SOLUTION OF THE PROBLEM (1.1)-(1.3)

Consider the following conditions:

$$G(g;s) := \int_0^s g(s_1)ds_1 \ge -M_1 s^2 - M_2, \ \int_0^s F(s_1)ds_1 \ge -M_3 \quad \forall s \in \mathbb{R}, \tag{2.1}$$

$$\alpha = \gamma = 0, \quad \beta \in C^1([0,T]), \quad \beta(t) \le 0, \quad \beta'(t) \ge 0, \quad 0 \le t \le T, \tag{2.2}$$

where $M_i = const \ge 0,\ 1 \le i \le 3$.

**Lemma 2.1.** *Let the conditions (2.1) and (2.2) be satisfied. Then for a strong generalized solution $u$ of the problem (1.1)-(1.3) of the class $C$ in the domain $D_T$ in the sense of Definition 1.1 the following a priori estimate is fulfilled:*

$$\|u\|_{C(\overline{D}_T)} \leq c_1\|f\|_{C(\overline{D}_T)} + c_2\|\varphi\|_{C^1(\omega_0)} + c_3\|\psi\|_{C(\omega_0)} + c_4\|G(|g|;|\varphi|)\|^{\frac{1}{2}}_{C(\omega_0)} + c_5\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + c_6, \tag{2.3}$$

*where $c_i = c_i(M_1, M_2, M_3, l, T, \beta(0))$, $1 \leq i \leq 6$ are positive constants, independent of functions $u, f, \varphi$ and $\psi$.*

**Proof.** Let $u$ be a strong generalized solution $u$ of the problem (1.1)-(1.3) of the class $C$ in the domain $D_T$. Then by (2.2), Definition 1.1 and Remark 1.1, there exists a sequence of functions $u_n \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma_1, \Gamma_2)$, such that the limiting relations (1.7) and (1.8) are satisfied. Denote

$$f_n := Lu_n, \tag{2.4}$$

$$\varphi_n := u_n|_{\omega_0}, \quad \psi_n := u_{nt}|_{\omega_0}. \tag{2.5}$$

Multiplying both sides of equality (2.4) by $u_{nt}$ and integrating over the domain $D_\tau$, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int_{D_\tau}(u_{nt}^2)_t dxdt - \int_{D_\tau} u_{nxx}u_{nt}dxdt + \int_{D_\tau}\big[G(g;u_n)\big]_t dxdt = \int_{D_\tau} f_n u_{nt}dxdt. \tag{2.6}$$

We set $\omega_\tau : t = \tau,\ 0 \leq x \leq l;\ 0 \leq \tau \leq T$. Let $\nu = (\nu_x, \nu_t)$ be the unit vector of the exterior normal to $\partial D_\tau$. It is easy to see that

$$\begin{gathered} \nu_x|_{\omega_\tau} = 0, \quad 0 \leq \tau \leq T, \quad \nu_x|_{\Gamma_1} = -1, \quad \nu_x|_{\Gamma_2} = 1, \\ \nu_t|_{\Gamma_1\cup\Gamma_2} = 0, \quad \nu_t|_{\omega_0} = -1, \quad \nu_t|_{\omega_\tau} = 1, \quad 0 < \tau \leq T. \end{gathered} \tag{2.7}$$

Applying integration by parts (Green's formula), and taking into account (2.5), (2.7), and that $u_n \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma_1, \Gamma_2)$, we can write

$$\begin{aligned} &\frac{1}{2}\int_{D_\tau}(u_{nt}^2)_t dxdt + \int_{D_\tau}\big[G(g;u_n)\big]_t dxdt = \frac{1}{2}\int_{\partial D_\tau} u_{nt}^2\nu_t ds + \int_{\partial D_\tau} G(g;u_n)\nu_t ds \\ &= \frac{1}{2}\int_{\omega_\tau} u_{nt}^2 dx - \frac{1}{2}\int_{\omega_0}\psi_n^2 dx + \int_{\omega_\tau} G(g;u_n)dx - \int_{\omega_0} G(g;\varphi_n)dx, - \int_{D_\tau} u_{nxx}u_{nt}dxdt \\ &= \int_{D_\tau}[u_{nx}u_{ntx} - (u_{nx}u_{nt})_x]dxdt = \frac{1}{2}\int_{D_\tau}(u_{nx}^2)_t dxdt - \int_{\partial D_\tau} u_{nx}u_{nt}\nu_x ds \\ &= \frac{1}{2}\int_{\partial D_\tau} u_{nx}^2\nu_t ds + \int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt - \int_{\Gamma_{2,\tau}}\beta u_n u_{nt}dt = \frac{1}{2}\int_{\omega_\tau} u_{nx}^2 dx \\ &-\frac{1}{2}\int_{\omega_0}\varphi_{nx}^2 dx + \int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt - \frac{1}{2}\beta(\tau)u_n^2(l,\tau) + \frac{1}{2}\beta(0)\varphi_n^2(l) + \frac{1}{2}\int_{\Gamma_{2,\tau}}\beta' u_n^2 dt, \end{aligned} \tag{2.8}$$

where $\Gamma_{i,\tau} = \Gamma_i \cap \{t \leq \tau\}$, $i = 1, 2$. In view of (2.8), the equality (2.6) we can write in the form:

$$\begin{aligned} 2\int_{D_\tau} f_n u_{nt}dxdt &= 2\int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt - \beta(\tau)u_n^2(l,\tau) + \beta(0)\varphi_n^2(l) + \int_{\Gamma_{2,\tau}}\beta' u_n^2 dt \\ &+ \int_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)dx + 2\int_{\omega_\tau} G(g;u_n)dx - \int_{\omega_0}(\varphi_{nx}^2 + \psi_n^2)dx - 2\int_{\omega_0} G(g;\varphi_n)dx. \end{aligned} \tag{2.9}$$

Since $u_n \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma_1, \Gamma_2)$, we have

$$\int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt = \int_0^\tau F[u_n(0,t)]du_n(0,t) = \int_{\varphi_n(0)}^{u_n(0,\tau)} F(s)ds = \int_{\varphi_n(0)}^0 F(s)ds + \int_0^{u_n(0,\tau)} F(s)ds. \tag{2.10}$$

In view of (2.1), (2.2) and (2.10), from (2.9) we obtain

$$w_n(\tau) := \int_{\omega_\tau} (u_{nx}^2 + u_{nt}^2)dx \le 2\int_{D_\tau} f_n u_{nt} dxdt - \beta(0)\varphi_n^2(l) + \int_{\omega_0} (\varphi_{nx}^2 + \psi_n^2)dx$$

$$+2\int_{\omega_0} G(g;\varphi_n)dx + 2M_1 \int_{\omega_\tau} u_n^2 dx + 2\int_0^{\varphi_n(0)} F(s)ds + 2(M_2 l + M_3). \tag{2.11}$$

Next, since by (2.5)

$$u_n(x,\tau) = \varphi_n(x) + \int_0^\tau u_{nt}(x,t)dt, \tag{2.12}$$

we have

$$|u_n(x,\tau)|^2 \le 2\varphi_n^2(x) + 2\Big(\int_0^\tau u_{nt}(x,t)dt\Big)^2 \le 2\varphi_n^2(x) + 2\tau\int_0^\tau u_{nt}^2(x,t)dt,$$

implying that

$$\int_{\omega_\tau} u_n^2 dx \le 2\|\varphi_n\|^2_{L_2(\omega_0)} + 2T\int_0^\tau w_n(t)dt, \tag{2.13}$$

where $w_n$ is as in (2.11). Taking into account (2.13) and the following inequalities

$$2f_n u_{nt} \le u_{nt}^2 + f_n^2, \quad \|f_n\|^2_{L_2(D_\tau)} \le lT\|f_n\|^2_{C(\overline{D}_T)}, \quad \int_{D_\tau} u_{nt}^2 dxdt = \int_0^\tau \Big[\int_{\omega_t} u_{nt}^2 dx\Big]dt \le \int_0^\tau w_n(t)dt,$$

$$\int_{\omega_0} (\varphi_{nx}^2 + \psi_n^2)dx + 2\int_{\omega_0} G(g;\varphi_n)dx \le l\|\varphi_n'\|^2_{C(\omega_0)} + l\|\psi_n\|^2_{C(\omega_0)} + 2l\|G(|g|;|\varphi_n|)\|_{C(\omega_0)},$$

$$2\int_0^{\varphi_n(0)} F(s)ds \le 2|\varphi_n(0)|\|F\|_{C([-|\varphi_n(0)|,|\varphi_n(0)|])} \le \varphi_n^2(0) + \|F\|^2_{C([-|\varphi_n(0)|,|\varphi_n(0)|])},$$

$$4M_1\|\varphi_n\|^2_{L_2(\omega_0)} + \varphi_n^2(0) - \beta(0)\varphi_n^2(l) + l\|\varphi_n'\|^2_{C(\omega_0)} \le (4M_1 l + 1 + |\beta(0)|)\|\varphi_n\|^2_{C(\omega_0)}$$

$$+l\|\varphi_n'\|^2_{C(\omega_0)} \le l_0(\|\varphi_n\|^2_{C(\omega_0)} + \|\varphi_n'\|^2_{C(\omega_0)}) \le l_0\|\varphi_n\|^2_{C^1(\omega_0)}, \quad l_0 := max\ (4M_1 l + 1 + |\beta(0)|, l),$$

from (2.11) we get

$$w_n(\tau) \le (4M_1 T + 1)\int_0^\tau w_n(t)dt + lT\|f_n\|^2_{C(\overline{D}_T)} + l_0\|\varphi_n\|^2_{C^1(\omega_0)} + l\|\psi_n\|^2_{C(\omega_0)}$$

$$+2l\|G(|g|;|\varphi_n|)\|_{C(\omega_0)} + \|F\|^2_{C([-|\varphi_n(0)|,|\varphi_n(0)|])} + 2(M_2 l + M_3).$$

Therefore, in view of Gronwall's lemma, we obtain

$$w_n(\tau) \le \Big[lT\|f_n\|^2_{C(\overline{D}_T n)} + l_0\|\varphi_n\|^2_{C^1(\omega_0)} + l\|\psi_n\|^2_{C(\omega_0)} + 2l\|G(|g|;|\varphi_n|)\|_{C(\omega_0)}$$

$$+\|F\|^2_{C([-|\varphi_n(0)|,|\varphi_n(0)|])} + 2(M_2 l + M_3)\Big] \exp\big[T(4M_1 T + 1)\big]. \tag{2.14}$$

For $(x,t) \in \overline{D}_T$, by integrating with respect to variable $\xi \in [0,l]$ the following obvious inequality

$$|u_n(x,t)|^2 = \Big|u_n(\xi,t) + \int_\xi^x u_{nx}(x_1,t)dx_1\Big|^2 \le 2|u_n(\xi,t)|^2 + 2l\int_0^l u_{nx}^2(x,t)dx,$$

we obtain

$$|u_n(x,t)|^2 \le \frac{2}{l}\int_0^l |u_n(\xi,t)|^2 d\xi + 2l w_n(t). \tag{2.15}$$

By similar arguments, in view of (2.12), we obtain

$$\int_0^l |u_n(x,t)|^2 dx \le 2\|\varphi_n\|^2_{L_2(\omega_0)} + 2l\int_0^l dx\int_0^t u^2_{nt}(x,\sigma)d\sigma \le 2\|\varphi_n\|^2_{L_2(\omega_0)} + 2l\int_0^t w_n(\sigma)d\sigma.$$

Hence, taking into account (2.15), we get

$$|u_n(x,t)|^2 \le \frac{4}{l}\|\varphi_n\|^2_{L_2(\omega_0)} + 4\int_0^t w_n(\sigma)d\sigma + 2lw_n(t)$$

$$\le \frac{4}{l}\|\varphi_n\|^2_{L_2(\omega_0)} + 6l\max_{\sigma\in[0,T]} w_n(\sigma) \le 4\|\varphi_n\|^2_{C(\omega_0)} + 6l\max_{\sigma\in[0,T]} w_n(\sigma). \tag{2.16}$$

Next, taking into account (2.14), (2.16) and the obvious inequality $\left(\sum_{i=1}^n a_i^2\right)^{1/2} \le \sum_{i=1}^n |a_i|$, we obtain

$$\|u_n\|_{C(\overline{D}_T)} \le 2\|\varphi_n\|_{C(\omega_0)} + \Big[l\sqrt{6T}\|f_n\|_{C(\overline{D}_T)} + \sqrt{6ll_0}\|\varphi_n\|_{C^1(\omega_0)}$$

$$+l\sqrt{6}\|\psi_n\|_{C(\omega_0)} + 2l\sqrt{3}\|G(|g|;|\varphi_n|)\|^{\frac12}_{C(\omega_0)} + \sqrt{6l}\|F\|_{C([-|\varphi_n(0)|,|\varphi_n(0)|])}$$

$$+2\sqrt{3l(M_2l+M_3)}\Big]\exp\left[2^{-1}T(4M_1T+1)\right].$$

Finally, by (1.7), (1.8) and (2.5), passing to the limit (as $n\to\infty$) in the last inequality we get

$$\|u\|_{C(\overline{D}_T)} \le 2\|\varphi\|_{C(\omega_0)} + \Big[l\sqrt{6T}\|f\|_{C(\overline{D}_T)} + \sqrt{6ll_0}\|\varphi\|_{C^1(\omega_0)} + l\sqrt{6}\|\psi\|_{C(\omega_0)} \tag{2.17}$$

$$+2l\sqrt{3}\|G(|g|;|\varphi|)\|^{\frac12}_{C(\omega_0)} + \sqrt{6l}\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + 2\sqrt{3l(M_2l+M_3)}\Big]\exp\left[2^{-1}T(4M_1T+1)\right].$$

Lemma 2.1 is proved.

**Remark 2.1.** It follows from (2.17) that the constants $c_i,\ 1\le i\le 6$, in the estimate (2.3) are given by

$$\begin{gathered} c_1 = l\sqrt{6T}c_0,\quad c_2 = 2+\sqrt{6ll_0}c_0,\quad c_3 = l\sqrt{6}c_0,\quad c_4 = 2l\sqrt{3}c_0,\quad c_5=\sqrt{6l}c_0,\\ c_6 = 2\sqrt{3l(M_2l+M_3)}c_0,\quad \text{where}\ \ c_0 := \exp\left[2^{-1}T(4M_1T+1)\right]. \end{gathered} \tag{2.18}$$

**Remark 2.2.** We give examples of classes of functions, which appears frequently in applications and for which the conditions in (2.1) are fulfilled:

1. $g(s) = g_0(s)sgns + as + b$, where $g_0\in C(\mathbb{R}),\ g_0\ge 0;\ a,b,s\in\mathbb{R}$;
2. $F(s) = F_0(s)sgns + as + b$, where $F_0\in C(\mathbb{R}),\ F_0\ge 0;\ a,b,s\in\mathbb{R},\ a>0$;
3. $g\in C(\mathbb{R}),\ g\big|_{(-\infty,0)}\in L_1(-\infty,0);\ g\big|_{(0,+\infty)}\ge 0$ (for instance, $g(s)=\exp s,\ s\in\mathbb{R}$).

## 3. REDUCTION OF PROBLEM (1.1)-(1.3) TO A SYSTEM OF VOLTERRA TYPE NONLINEAR INTEGRAL EQUATIONS

We first represent the solution in the domain $D_l$ of the following mixed linear problem

$$\Box w = w_{tt} - w_{xx} = \widetilde{f}(x,t),\quad (x,t)\in D_l, \tag{3.1}$$

$$w(x,0)=\varphi(x),\quad w_t(x,0)=\psi(x),\quad 0\le x\le l, \tag{3.2}$$

$$w_x(0,t)=\widetilde{\alpha}(t),\quad w_x(l,t)=\widetilde{\gamma}(t),\quad 0\le t\le l, \tag{3.3}$$

in quadratures in a convenient form, where

$$\widetilde{f}\in C^1(\overline{D}_l),\quad \varphi\in C^2([0,l]),\quad \psi\in C^1([0,l]),\quad \widetilde{\alpha},\ \widetilde{\gamma}\in C^1([0,l]) \tag{3.4}$$

are given functions satisfying the following second order consistency conditions:

$$\varphi'(0) = \widetilde{\alpha}(0), \quad \psi'(0) = \widetilde{\alpha}'(0), \quad \varphi'(l) = \widetilde{\gamma}(0), \; \psi'(l) = \widetilde{\gamma}'(0), \tag{3.5}$$

and $w \in C^2(\overline{D}_l)$ is the unknown function.

Below the solution of the problem (3.1)-(3.3) we represent in the form:

$$w(x,t) = A_1(\widetilde{f}, \widetilde{\alpha}, \widetilde{\gamma})(x,t) + B_1(\varphi, \psi)(x,t), \quad (x,t) \in \overline{D}_l, \tag{3.6}$$

with operators $A_1$ and $B_1$, which will be constructed in explicit form.

To this end, the domain $D_l$, being a square with vertices at the points $O(0,0)$, $A(0,l)$, $B(l,l)$ and $C(l,0)$, we split into four right triangles $\Delta_1 := \Delta OO_1C$, $\Delta_2 := \Delta OO_1A$, $\Delta_3 := \Delta CO_1B$ and $\Delta_4 := \Delta O_1AB$, where the point $O_1(\frac{l}{2}, \frac{l}{2})$ is the center of the square $D_l$. It is known that the solution of the problem (3.1)-(3.3) in the triangle $\Delta_1$ is given by the following formula (see [1], p. 59):

$$w(x,t) = \frac{1}{2}\big[\varphi(x-t) + \varphi(x+t)\big] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\tau)d\tau + \frac{1}{2}\int_{\Omega^1_{x,t}} \widetilde{f}(\xi,\tau)d\xi d\tau, \quad (x,t) \in \Delta_1, \tag{3.7}$$

where $\Omega^1_{x,t}$ denotes the triangle with vertices at the points $(x,t)$, $(t-x,0)$ and $(t+x,0)$.

To obtain the solution of the problem (3.1)-(3.3) in the other triangles $\Delta_2$, $\Delta_3$ and $\Delta_4$, we use the following equality (see [10], p. 173):

$$w(P) = w(P_1) + w(P_2) - w(P_3) + \frac{1}{2}\int_{PP_1P_2P_3} \widetilde{f}(\xi,\tau)d\xi d\tau, \tag{3.8}$$

which is true for any characteristic (for equation (3.1)) rectangle $PP_1P_2P_3 \subset \overline{D}_l$, where $P$ and $P_3$, as well as, $P_1$ and $P_2$ are the opposite vertices of that rectangle, and the ordinate of the point $P$ is greater than the ordinates of the other points.

Now let $(x,t) \in \Delta_2$. Then setting

$$\widetilde{\mu}_1 := w|_{\Gamma_1}, \tag{3.9}$$

and applying the equality (3.8) for characteristic rectangle with vertices at the points $P(x,t)$, $P_1(0, t-x)$, $P_2(t,x)$ and $P_3(t-x,0)$, the formula (3.7) for point $P_2(t,x) \in \Delta_1$, and using (3.9), we can write

$$w(x,t) = w(P_1) + w(P_2) - w(P_3) + \frac{1}{2}\int_{PP_1P_2P_3} \widetilde{f}(\xi,\tau)d\xi d\tau = \widetilde{\mu}_1(t-x) - \varphi(t-x)$$

$$+\frac{1}{2}\big[\varphi(t-x) + \varphi(t+x)\big] + \frac{1}{2}\int_{t-x}^{t+x} \psi(\tau)d\tau + \frac{1}{2}\int_{\Omega^1_{t,x}} \widetilde{f}(\xi,\tau)d\xi d\tau + \frac{1}{2}\int_{PP_1P_2P_3} \widetilde{f}(\xi,\tau)d\xi d\tau$$

$$= \widetilde{\mu}_1(t-x) + \frac{1}{2}\big[\varphi(t+x) - \varphi(t-x)\big] + \frac{1}{2}\int_{t-x}^{t+x} \psi(\tau)d\tau + \frac{1}{2}\int_{\Omega^2_{x,t}} \widetilde{f}(\xi,\tau)d\xi d\tau, \quad (x,t) \in \Delta_2. \tag{3.10}$$

Here $\Omega^2_{x,t}$ denotes the quadrangle $P\widetilde{P}_2P_3P_1$, where $\widetilde{P}_2 = \widetilde{P}_2(t+x, 0)$.

Taking into account that for $(x,t) \in \Delta_2$

$$\int_{\Omega^2_{x,t}} \widetilde{f}(\xi,\tau)d\xi d\tau = \int_0^{t-x} d\tau \int_{-x+t-\tau}^{x+t-\tau} \widetilde{f}(\xi,\tau)d\xi + \int_{t-x}^{t} d\tau \int_{x-t+\tau}^{x+t-\tau} \widetilde{f}(\xi,\tau)d\xi,$$

in view of (3.10) we obtain

$$w_x(x,t) = -\widetilde{\mu}_1'(t-x) + \frac{1}{2}\big[\varphi'(t+x) + \varphi'(t-x) + \psi(t+x) + \psi(t-x)\big] \tag{3.11}$$

$$+\frac{1}{2}\int_0^{t-x} \big[\widetilde{f}(x+t-\tau,\tau) + \widetilde{f}(-x+t-\tau,\tau)\big]d\tau + \frac{1}{2}\int_{t-x}^{t} \big[\widetilde{f}(x+t-\tau,\tau) - \widetilde{f}(x-t+\tau,\tau)\big]d\tau.$$

Similarly, for $(x,t)\in\Delta_2$ we get

$$w_t(x,t)=\widetilde{\mu}_1'(t-x)+\frac{1}{2}\big[\varphi'(t+x)-\varphi'(t-x)+\psi(t+x)-\psi(t-x)\big]$$

$$+\frac{1}{2}\int\limits_0^{t-x}\big[\widetilde{f}(x+t-\tau,\tau)-\widetilde{f}(-x+t-\tau,\tau)\big]d\tau+\frac{1}{2}\int\limits_{t-x}^{t}\big[\widetilde{f}(x+t-\tau,\tau)+\widetilde{f}(x-t+\tau,\tau)\big]d\tau. \quad (3.12)$$

Setting $x=0$ in the equality (3.11), and taking into account the first boundary condition in (3.3), for unknown function $\widetilde{\mu}_1$ we obtain the equality:

$$-\widetilde{\mu}_1'(t)+\varphi'(t)+\psi(t)+\int_0^t\widetilde{f}(t-\tau,\tau)d\tau=\widetilde{\alpha}(t),\quad 0\le t\le l.$$

Integrating the last equality and taking into account the initial condition $\widetilde{\mu}_1(0)=\varphi(0)$, we get

$$\widetilde{\mu}_1(t)=A_2(\widetilde{f},\widetilde{\alpha},\widetilde{\gamma})(t)+B_2(\varphi,\psi)(t):=\varphi(t)-\int_0^t\widetilde{\alpha}(\tau)d\tau+\int_0^t\psi(\tau)d\tau$$

$$+\int_0^t d\tau_1\int_0^{\tau_1}\widetilde{f}(\tau_1-\tau,\tau)d\tau,\quad 0\le t\le l. \quad (3.13)$$

Now, in view of (3.10) and (3.13), the solution of the problem (3.1)-(3.3) in the domain $\Delta_2$ can be represented in the form:

$$w(x,t)=-\int_0^{t-x}\widetilde{\alpha}(\tau)d\tau+\int_0^{t-x}\psi(\tau)d\tau+\int_0^{t-x}d\tau_1\int_0^{\tau_1}\widetilde{f}(\tau_1-\tau,\tau)d\tau$$

$$+\frac{1}{2}\big[\varphi(t+x)+\varphi(t-x)\big]+\frac{1}{2}\int_{t-x}^{t+x}\psi(\tau)d\tau+\frac{1}{2}\int_{\Omega^2_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau,\quad (x,t)\in\Delta_2. \quad (3.14)$$

Next, to obtain representations for the solution of problem (3.1)-(3.3) in the domains $\Delta_3$ and $\Delta_4$, we set

$$\widetilde{\mu}_2:=w|_{\Gamma_2} \quad (3.15)$$

and use the above arguments, applied to obtain the equality (3.10), to conclude that

$$w(x,t)=\widetilde{\mu}_2(x+t-l)+\frac{1}{2}\big[\varphi(x-t)-\varphi(2l-x-t)\big]+\frac{1}{2}\int\limits_{x-t}^{2l-x-t}\psi(\tau)d\tau+\frac{1}{2}\int\limits_{\Omega^3_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau, \quad (3.16)$$

for $(x,t)\in\Delta_3$, and

$$w(x,t)=\widetilde{\mu}_1(t-x)+\widetilde{\mu}_2(x+t-l)-\frac{1}{2}\big[\varphi(t-x)+\varphi(2l-t-x)\big]$$

$$+\frac{1}{2}\int_{t-x}^{2l-t-x}\psi(\tau)d\tau+\frac{1}{2}\int_{\Omega^4_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau,\quad (x,t)\in\Delta_4. \quad (3.17)$$

Here $\Omega^3_{x,t}$ denotes the quadrangle with vertices $P^3(x,t)$, $P^3_1(l,x+t-l)$, $P^3_2(x-t,0)$, $P^3_3(2l-x-t,0)$, and $\Omega^4_{x,t}$ denotes the pentagon with vertices $P^4(x,t)$, $P^4_1(0,t-x)$, $P^4_2(t-x,0)$, $P^4_3(2l-x-t,0)$ and $P^4_4(l,x+t-l)$. Taking into account that for $(x,t)\in\Delta_3$

$$\int_{\Omega^3_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau=\int_0^{x+t-l}d\tau\int_{x-t+\tau}^{2l-x-t+\tau}\widetilde{f}(\xi,\tau)d\xi+\int_{x+t-l}^{t}d\tau\int_{x-t+\tau}^{x+t-\tau}\widetilde{f}(\xi,\tau)d\xi,$$

and differentiating the equality (3.16) by $x$, we obtain

$$w_x(x,t) = \widetilde{\mu}_2'(x+t-l) + \frac{1}{2}\big[\varphi'(x-t) + \varphi'(2l-x-t)\big]$$

$$-\frac{1}{2}[\psi(2l-x-t) + \psi(x-t)] - \frac{1}{2}\int_0^{x+t-l} \big[\widetilde{f}(2l-x-t+\tau,\tau) + \widetilde{f}(x-t+\tau,\tau)\big]d\tau$$

$$+\frac{1}{2}\int_{x+t-l}^{t} \big[\widetilde{f}(x+t-\tau,\tau) - \widetilde{f}(x-t+\tau,\tau)\big]d\tau, \quad (x,t) \in \Delta_3. \tag{3.18}$$

Substituting the expression (3.18) with $x = l$ into the second boundary condition in (3.3), for unknown function $\widetilde{\mu}_2$ we obtain

$$\widetilde{\mu}_2'(t) - \psi(l-t) + \varphi'(l-t) - \int_0^t \widetilde{f}(l-t+\tau,\tau)d\tau = \widetilde{\gamma}(t), \quad 0 \le t \le l. \tag{3.19}$$

And, in view of (3.2) and (3.15), we have

$$\widetilde{\mu}_2(0) = \varphi(l). \tag{3.20}$$

Finally, from (3.19) and (3.20) we obtain

$$\widetilde{\mu}_2(t) = A_3(\widetilde{f},\widetilde{\alpha},\widetilde{\gamma})(t) + B_3(\varphi,\psi)(t) := \varphi(l-t) + \int_0^t \widetilde{\gamma}(\tau)d\tau + \int_{l-t}^{l} \psi(\tau)d\tau$$

$$+\int_0^t d\tau_1 \int_0^{\tau_1} \widetilde{f}(l-\tau_1+\tau,\tau)d\tau, \quad 0 \le t \le l. \tag{3.21}$$

**Remark 3.1.** If $w$ is a solution of the problem (3.1)-(3.3), then in view of equalities (3.6), (3.13) and (3.21), for the triple of functions $(w, \widetilde{\mu}_i := w|_{\Gamma_i}, i = 1, 2)$ the following integral representation holds:

$$(w, \widetilde{\mu}_1, \widetilde{\mu}_2) = A(\widetilde{f},\widetilde{\alpha},\widetilde{\gamma}) + B(\varphi,\psi), \tag{3.22}$$

where the actions of operators $A := (A_1, A_2, A_3)$, $B := (B_1, B_2, B_3)$ are specified by formulas (3.6), (3.7), (3.14), (3.16), (3.17), (3.13) and (3.21).

**Remark 3.2.** It is easy to check that in the case $\widetilde{f} \in C(\overline{D}_l)$, $\varphi \in C^1([0,l])$, $\psi \in C([0,l])$, $\widetilde{\alpha}$, $\widetilde{\gamma} \in C([0,l])$, if the first order consistency conditions $\varphi'(0) = \widetilde{\alpha}(0)$, $\varphi'(l) = \widetilde{\gamma}(0)$ are satisfied, then in view of formulas (3.11) and (3.12) for every $w_x$, $w_t$ in the domain $\Delta_2$, and also in the other domains $\Delta_1$, $\Delta_3$ and $\Delta_4$, the triple of functions $(w, \widetilde{\mu}_1, \widetilde{\mu}_2)$, defined by equality (3.22), belongs to the class $C^1(\overline{D}_l) \times C^1([0,l]) \times C^1([0,l])$. Moreover, the linear operator

$$A : C(\overline{D}_l) \times C([0,l]) \times C([0,l]) \to C^1(\overline{D}_l) \times C^1([0,l]) \times C^1([0,l]) \tag{3.23}$$

in (3.22) is continuous. A similar remark holds also for operator $B$ in the corresponding spaces of functions.

**Remark 3.3.** Similar to Remark 3.2, it can be shown that if the smoothness condition (3.4) and the second order consistency condition (3.5) are satisfied, then according to (3.6), the function $w$, constructed by means of equalities (3.7), (3.14), (3.16), (3.17), (3.13), (3.21), belongs to the class $C^2(\overline{D}_l)$, and is the classical solution of the problem (3.1)-(3.3).

**Remark 3.4.** Note that in the case where the problem (3.1)-(3.3) is considered in domain $D_T$ for $T \le l$, then for the triple of functions $(w, \widetilde{\mu}_i := w|_{\Gamma_i}, i = 1, 2)$, the integral representation (3.22) remains valid.

Now we proceed to reduce the problem (1.1)-(1.3) to a system of Volterra type nonlinear integral equations. Let $u$ be a strong generalized solution of this problem of the class $C$ in the domain $D_T$, $T \le l$, that is, $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_T)$, such that the equalities (1.7)-(1.10) are satisfied. Consider the function $u_n$ as a classical solution of the problem (3.1)-(3.3) for

$$\widetilde{f} = -g(u_n) + f_n,\ \varphi = \varphi_n,\ \psi = \psi_n,\ \widetilde{\alpha} = F(\mu_{1n}) + \alpha_n,\ \widetilde{\gamma} = \beta\mu_{2n} + \gamma_n,$$

where

$$f_n = Lu_n,\ \ \varphi_n = u_n|_{\omega_0},\ \ \psi_n = u_{nt}|_{\omega_0},\ \mu_{1n} = u_n|_{\Gamma_1},\ \ \alpha_n = u_{nx}|_{\Gamma_1} - F(\mu_{1n}),\ \ \gamma_n = u_{nx}|_{\Gamma_2} - \beta\mu_{2n}.$$

Then, by equality (3.22), for function $u_n$ and its truncations $\mu_{in} := u_n|_{\Gamma_i}, i = 1, 2$, the following equalities hold:

$$\begin{aligned} u_n &= A_1\big(-g(u_n) + f_n, F(\mu_{1n}) + \alpha_n, \beta\mu_{2n} + \gamma_n\big) + B_1(\varphi_n, \psi_n),\\ \mu_{in} &= A_{i+1}\big(-g(u_n) + f_n, F(\mu_{1n}) + \alpha_n, \beta\mu_{2n} + \gamma_n\big) + B_{i+1}(\varphi_n, \psi_n), \quad i = 1, 2. \end{aligned} \tag{3.24}$$

Taking into account Remark 3.2, the equalities (1.7)-(1.10) and (3.22), and passing to the limit in equations (3.24) as $n \to \infty$, we conclude that the triple of functions $(u, \mu_i := u|_{\Gamma_i}, i = 1, 2)$ satisfies the nonlinear operator equation:

$$(u, \mu_1, \mu_2) = A_0(u, \mu_1, \mu_2), \tag{3.25}$$

where

$$A_0(u, \mu_1, \mu_2) = A(-g(u) + f, F(\mu_1) + \alpha, \beta\mu_2 + \gamma) + B(\varphi, \psi). \tag{3.26}$$

**Remark 3.5.** In view of Remark 3.2, the operator $A_0$ defined in (3.26) acts continuously from the space $C(\overline{D}_T) \times C([0,T]) \times C([0,T])$ to the space $C^1(\overline{D}_T) \times C^1([0,T]) \times C^1([0,T]),\ T \leq l$. Hence, taking into account that the space $C^1(\overline{D}_T) \times C^1([0,T]) \times C^1([0,T])$ is compactly embedded into the space $C(\overline{D}_T) \times C([0,T]) \times C([0,T])$ (see [11], p. 135)], we conclude that the operator

$$A_0 :\ C(\overline{D}_T) \times C([0,T]) \times C([0,T]) \to C(\overline{D}_T) \times C([0,T]) \times C([0,T]) \tag{3.27}$$

is compact.

**Remark 3.6.** It is easy to see that if $(\xi, \tau) \in \Omega^i_{x,t},\ 1 \leq i \leq 4$, then $\tau \leq t$, which in view of formulas (3.7), (3.14), (3.16), (3.17), (3.13), (3.21), permits to consider (3.25) as a system of Volterra type nonlinear integral equations with respect to variable $t$. Notice that in the linear case, for this system can be applied a converging method of Picard's successive approximations in the corresponding spaces of functions.

**Remark 3.7.** Similar to Remark 3.3, in view of (3.25) we can conclude that if $u$ is a strong generalized solution of the problem (1.1)-(1.3) of the class $C$ in the domain $D_T, T \leq l$, and the following smoothness conditions

$$f \in C^1(\overline{D}_T), \quad g,\ F \in C^1(\mathbb{R}), \quad \varphi \in C^2([0,l]), \quad \psi \in C^1([0,l]), \quad \alpha,\ \beta,\ \gamma \in C^1([0,T]) \tag{3.28}$$

and the second order consistency condition (1.4) are satisfied, then $u$ will be the classical solution of this problem from the space $C^2(\overline{D}_T)$.

**Remark 3.8.** From the above presented arguments it follows that if the smoothness condition (1.5) and the first order consistency condition (1.6) are satisfied, and if a function $u$ is a strong generalized solution of the problem (1.1)-(1.3) of the class $C$ in the domain $D_T$ in the sense of Definition 1.1, then the triple of functions $(u, \mu_i := u|_{\Gamma_i}, i = 1, 2)$ is a continuous solution of the system of Volterra type nonlinear integral equations (3.25). Using arguments similar to those applied in [9], it can easily be shown that the converse assertion also holds.

## 4. LOCAL SOLVABILITY IN $t$ OF PROBLEM (1.1)-(1.3)

**Theorem 4.1.** *Let the functions $f \in C(\overline{D}_l),\ g, F \in C(\mathbb{R}),\ \varphi \in C^1([0,l]), \psi,\ \alpha,\ \beta, \gamma \in C([0,l])$ satisfy the consistency condition (1.6). Then a positive number $T_0 = T_0(f, g, F, \varphi, \psi, \alpha, \beta, \gamma) \leq l$ can be found such that for $T \leq T_0$ the problem (1.1)-(1.3) in the domain $D_T$ will have at least one strong generalized solution $u$ of the class $C$.*

**Proof.** In Section 3, the problem (1.1)-(1.3) in the space $C(\overline{D}_T) \times C([0,T]) \times C([0,T]),\ T \leq l$, was reduced to the equivalent equation (3.25), where by Remark 3.5 the operator $A_0$ is continuous and compact, acting in the space $C(\overline{D}_T) \times C([0,T]) \times C([0,T])$. Hence, according to Schauder theorem, for solvability of equation (3.25) it is enough to show that the operator $A_0$ transfers some ball $B_{R_0}(u^0, \mu_1^0, \mu_2^0)$ with center at point $(u^0, \mu_1^0, \mu_2^0)$ and of radius $R_0 > 0$ of the Banach space $C(\overline{D}_T) \times$

$C([0,T]) \times C([0,T])$ to itself. We show that this is the case for small enough $T \le l$. Indeed, in view of Remark 3.1 and equality (3.22), the operator equation (3.25) can be written in the form:

$$(u,\mu_1,\mu_2) = A_0(u,\mu_1,\mu_2) = (u^0,\mu_1^0,\mu_2^0) + A\big(-g(u), F(\mu_1), \beta\mu_2\big), \tag{4.1}$$

where

$$u^0 = A_1(f,\alpha,\gamma) + B_1(\varphi,\psi), \quad \mu_i^0 = A_{i+1}(f,\alpha,\gamma) + B_{i+1}(\varphi,\psi), \quad i=1,2.$$

It is easy to see that if $(\widetilde{u},\widetilde{\mu}_1,\widetilde{\mu}_2)$ belongs to the ball $B_{R_0}(u^0,\mu_1^0,\mu_2^0)$ and, according to Remark 3.6, the linear operator $A$ from (3.23) is a Volterra type integral operator by the variable $t \le T$, then

$$\|A\big(-g(u), F(\mu_1), \beta\mu_2\big)\|_{C(\overline{D}_T)\times C([0,T])\times C([0,T])} \le TM, \tag{4.2}$$

where

$$0 < M := M(\|g\|_{C([-R,R])}, \|F\|_{C([-R,R])}, \|\beta\|_{C([0,l])}R) < \infty,$$
$$R = \|(u^0,\mu_1^0,\mu_2^0)\|_{C(\overline{D}_l)\times C([0,l])\times C([0,l])} + R_0,$$

and $R_0$ is an arbitrary fixed positive number, and the function $M = M(s_1,s_2,s_3)$ is continuous and nondecreasing by each of the argument $s_i \ge 0$, $i=1,2,3$. Taking $T \le \frac{R_0}{M}$, from (4.1) and (4.2) for $(\widetilde{u},\widetilde{\mu}_1,\widetilde{\mu}_2) \in B_{R_0}(u^0,\mu_1^0,\mu_2^0)$, we obtain

$$\|A_0(\widetilde{u},\widetilde{\mu}_1,\widetilde{\mu}_2) - (u^0,\mu_1^0,\mu_2^0)\|_{C(\overline{D}_T)\times C([0,T])\times C([0,T])} \le R_0,$$

implying that $A_0:\ B_{R_0}(u^0,\mu_1^0,\mu_2^0) \to B_{R_0}(u^0,\mu_1^0,\mu_2^0)$, and the result follows. Theorem 4.1 is proved.

## 5. UNIQUENESS OF A SOLUTION OF PROBLEM (1.1)-(1.3)

**Theorem 5.1.** *The problem (1.1)-(1.3) cannot have more than one strong generalized solution of the class $C$ in the domain $D_T, T \le l$ in the sense of Definition 1.1, if in (1.5) it is assumed additionally that $g, F \in C^1(\mathbb{R})$.*

**Proof.** Assume that the problem (1.1)-(1.3) has two distinct strong generalized solutions $u^1$ and $u^2$ of the class $C$ in the domain $D_T, T \le l$. Then, according to Remark 3.8, the triples of functions $(u^1, \mu_1^1 := u^1|_{\Gamma_1}, \mu_2^1 := u^1|_{\Gamma_2})$ and $(u^2, \mu_1^2 := u^2|_{\Gamma_1}, \mu_2^2 := u^2|_{\Gamma_2})$ are continuous solutions of the system of nonlinear integral equations (3.25). Setting $u^0 := u^2 - u^1$, $\mu_i^0 := \mu_i^2 - \mu_i^1$, $i=1,2$, and taking into account (3.13), (3.14) and Remark 3.4, we can write

$$\mu_1^0(t) = -\int_0^t [F(\mu_1^2) - F(\mu_1^1)](\tau)d\tau - \int_0^t d\tau_1 \int_0^{\tau_1} [g(u^2) - g(u^1)](\tau_1-\tau,\tau)d\tau,\ 0 \le t \le T,$$
$$u^0(x,t) = -\int_0^{t-x} [F(\mu_1^2) - F(\mu_1^1)](\tau)d\tau - \int_0^{t-x} d\tau_1 \int_0^{\tau_1} [g(u^2) - g(u^1)](\tau_1-\tau,\tau)d\tau \tag{5.1}$$
$$-\frac{1}{2}\int_{\Omega^2_{x,t}} [g(u^2) - g(u^1)](\xi,\tau)d\xi d\tau,\ (x,t) \in \Delta_2 \cap \{t < T\}.$$

Next, since

$$\begin{aligned} F(\mu_1^2) - F(\mu_1^1) &= \left[\int_0^1 F'[\mu_1^1 + (\mu_1^2 - \mu_1^1)s]ds\right]\mu_1^0, \\ g(u^2) - g(u^1) &= \left[\int_0^1 g'[u^1 + (u^2 - u^1)s]ds\right]u^0, \end{aligned} \tag{5.2}$$

then assuming $u^i, \mu_1^i, i = 1, 2$ to be fixed functions and setting $\overline{u}(t) = \max_{0\le x\le l} |u^0(x,t)|$, $0 \le t \le T$, by (5.1) and (5.2), we obtain

$$|u^0(x,t)| \le M_0 \int_0^t \big[|\mu_1^0(\tau)| + \overline{u}(\tau)\big] d\tau$$

$$\le M_0 \int_0^t \big[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\big] d\tau, \quad (x,t) \in \Delta_2 \cap \{t < T\}, \tag{5.3}$$

$$|\mu_1^0(t)| \le M_0 \int_0^t \big[|\mu_1^0(\tau)| + \overline{u}(\tau)\big] d\tau \le M_0 \int_0^t \big[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\big] d\tau, \quad 0 \le t \le T,$$

where $M_0$ is a positive constant depending on $g, F$ and on fixed functions $u^i, \mu_j^i, i, j = 1, 2$. Similar arguments, carried out in the other domains $\Delta_j \cap \{t < T\}$, and possibly, by enlarging $M_0$, allow to obtain the following inequalities:

$$\begin{aligned} |u^0(x,t)| &\le M_0 \int_0^t \big[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\big] d\tau, \quad (x,t) \in \Delta_j \cap \{t < T\},\ j = 1, 3, 4, \\ |\mu_2^0(t)| &\le M_0 \int_0^t \big[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\big] d\tau, \quad 0 \le t \le T. \end{aligned} \tag{5.4}$$

It follows from (5.3) and (5.4) that

$$|\mu_1^0(t)| + |\mu_2^0(t)| + \overline{u}(t) \le 2M_0 \int_0^t \big[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\big] d\tau, \quad 0 \le t \le T.$$

Therefore, in view of Gronwall's lemma, we conclude that $\overline{u}(t) = 0,\ 0 \le t \le T$, that is, $u^1 = u^2$. The obtained contradiction completes the proof of the theorem. Theorem 5.1 is proved.

## 6. THE SOLVABILITY OF PROBLEM (1.1)-(1.3) IN DOMAIN $D_T$ FOR ANY $T \le l$ IN THE CASE $\alpha = \gamma = 0$

Let $\tau \in [0, 1]$, and let $u = u_\tau$ be a strong generalized solution of the class $C$ in the domain $D_T, T \le l$ of the following problem

$$\begin{gathered} u_{tt} - u_{xx} = \tau[-g(u) + f(x,t)], \quad (x,t) \in D_T, \\ u(x,0) = \tau\varphi(x), \quad u_t(x,0) = \tau\psi(x), \quad 0 \le x \le l, \\ u_x(0,t) = \tau F[u(0,t)], \quad u_x(l,t) = \tau\beta(t)u(l,t), \quad 0 \le t \le T, \end{gathered} \tag{6.1}$$

provided that the smoothness condition (1.5) and the following consistency condition (an analog of condition (1.6)): $\varphi'(0) = F[\tau\varphi(0)], \varphi'(l) = \tau\beta(0)\varphi(l)$ are satisfied. It is easy to see that these conditions will be satisfied for any $\tau \in [0, 1]$ if, for instance,

$$\varphi(0) = 0,\ \varphi'(0) = F(0),\ \varphi(l) = 0,\ \varphi'(l) = 0. \tag{6.2}$$

Similar arguments show that if $u = u_\tau$ is a classical solution of the problem (6.1) for any $\tau \in [0, 1]$, then according to Remark 3.7, it is natural to require that the smoothness condition (3.28) and the following equalities (instead of (1.4)) be fulfilled:

$$\varphi'(0) = F[\tau\varphi(0)], \quad \psi'(0) = \tau F'[\tau\varphi(0)]\psi(0), \quad \varphi'(l) = \tau\beta(0)\varphi(l), \quad \psi'(l) = \tau\beta'(0)\varphi(l) + \tau\beta(0)\psi(l).$$

It is easy to see that these conditions will be satisfied for any $\tau \in [0, 1]$, if, for instance, along with (6.2) will be satisfied the following conditions:

$$\psi(0) = 0,\ \psi'(0) = 0,\ \psi(l) = 0,\ \psi'(l) = 0. \tag{6.3}$$

**Remark 6.1.** Note that for $\tau = 1$, the problems (6.1) and (1.1)-(1.3) coincide, and similar to Definition 1.1, it can be defined the notion of strong generalized solution of the problem (6.1) of the class $C$ in the domain $D_T$, provided that the consistency condition (6.2) is satisfied.

**Remark 6.2.** In view of Remark 3.8, the problem (6.1) in the class of continuous functions can be reduced the following equivalent nonlinear operator equation:

$$(u, \mu_1, \mu_2) = \tau A_0(u, \mu_1, \mu_2), \tag{6.4}$$

where the operator $A_0$ is as in (3.27) and, by Remark 3.5, is compact.

As a consequence of Remarks 6.1, 6.2 and Leray-Schauder theorem (see [12], p. 375), we can state the following result.

**Lemma 6.1.** *Let the conditions (1.5) and (6.2) be fulfilled. If for any strong generalized solution $u = u_\tau$ of problem (6.1) of the class $C$ in the domain $D_T$ for any $\tau \in [0,1]$ the following a priori estimate holds:*

$$\|u\|_{C(\overline{D}_T)} \le M_*, \tag{6.5}$$

*where $M_* = M_*(g, f, \varphi, \psi, F, \alpha, \beta, \gamma)$ is a nonnegative constant independent of $\tau$, then the problem (1.1)-(1.3) has at least one strong generalized solution of the class $C$ in the domain $D_T$.*

**Proof.** Observe first that in view of Remarks 6.1 and 6.2, a function $u \in C(\overline{D}_T)$ is a strong generalized solution of problem (1.1)-(1.3) of the class $C$ in the domain $D_T$ if and only if it is a continuous solution of the nonlinear operator equation (6.4) for $\tau = 1$. On the other hand, according to conditions of the lemma, for any solution $u \in C(\overline{D}_T)$ of the equation (6.4) with compact operator $A_0$, for any $\tau \in [0,1]$ the a priori estimate (6.5) holds, and hence, according to Leray-Schauder theorem, the equation (6.4) for $\tau = 1$ has at least one solution $u \in C(\overline{D}_T)$, which is also a strong generalized solution of problem (1.1)-(1.3) of the class $C$ in the domain $D_T$. Lemma 6.1 is proved.

As a consequence of Lemmas 2.1 and 6.1 and Theorem 5.1, we have the following result.

**Theorem 6.1.** *Let $T \le l$, and let (1.5), (6.2) and the conditions of Lemma 2.1 be fulfilled. Then the problem (1.1)-(1.3) has at least one strong generalized solution of the class $C$ in the domain $D_T$, which in the case $g, F \in C^1(\mathbb{R})$ is unique. Moreover, if the smoothness condition (3.28) and the equalities (6.2), (6.3) are also satisfied, then this solution will also be classical.*

**Proof.** Observe first that if the given functions $g, f, \varphi, \psi, F$ of problem (1.1)-(1.3) we replace by the functions $\tau g, \tau f, \tau\varphi, \tau\psi, \tau F,\ \tau \in [0,1]$, then by (2.3) and (2.18), for any strong generalized solution $u = u_\tau$ of the class $C$ in the domain $D_T$ of the obtained problem the following a priori estimate holds:

$$\|u\|_{C(\overline{D}_T)} \le c_1\tau\|f\|_{C(\overline{D}_T)} + c_2\tau\|\varphi\|_{C^1(\omega_0)} + c_3\tau\|\psi\|_{C(\omega_0)} + c_4\|G(|g|;|\tau\varphi|)\|^{1/2}_{C(\omega_0)}$$

$$+c_5\tau\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + c_6$$

$$\le c_1\|f\|_{C(\overline{D}_T)} + c_2\|\varphi\|_{C(\omega_0)} + c_3\|\psi\|_{C(\omega_0)} + c_4\|G(|g|;|\varphi|)\|^{1/2}_{C(\omega_0)} + c_5\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + c_6.$$

Hence, the first assertion of the theorem follows from Lemma 6.1 and Theorem 5.1. The assertion that under conditions (3.28) and (6.3) the solution is classical, follows from Remark 3.7. Theorem 6.1 is proved.

**Remark 6.3.** Notice that the existence of the unique classical solution in the domain $D_{l,k} := \{(x,t) \in \mathbb{R}^2 : 0 < x < l,\ (k-1)l < t < kl\},\ k \in \mathbb{N},\ k \ge 2$, of the mixed problem

$$Lu = f(x,t), \quad (x,t) \in D_{l,k}, \quad u|_{t=(k-1)l} = \varphi, \quad u_t|_{t=(k-1)l} = \psi,$$

$$u_x(0,t) = F[u(0,t)] + \alpha(t), \quad u_x(l,t) = \beta(t)u(l,t) + \gamma(t), \quad (k-1)l \le t \le kl,$$

can be proved exactly in the same way as in the case $k = 1$, that is, in the domain $D_{l,1} = D_l$. Therefore, all the constructions of structural nature, given in the previous sections in the domain $D_T$ with $T \le l$ (such us the representations (3.7), (3.10), (3.16), (3.17) of a solution of the linear problem (3.1)-(3.3) and the nonlinear operator equations of type (3.25) as a system of Volterra type nonlinear integral equations with respect to variable $t$) analogously can be transferred to the case of domain $D_T$ for any $T \ge l$. Hence, if the conditions of Lemma 2.1, the smoothness condition (3.28) for $T = \infty$, and the consistency conditions (6.2), (6.3) are satisfied, then for any $T > 0$ (in particular, for $T = \infty$) in the domain $D_T$ there exists a unique classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1)-(1.3). Thus, we have the following result.

**Theorem 6.2.** *Let the conditions of Lemma 2.1, the smoothness condition (3.28) for $T = \infty$, and the consistency conditions (6.2), (6.3) be satisfied. Then for $T = \infty$ the problem (1.1)-(1.3) has a unique global classical solution $u \in C^2(\overline{D}_\infty)$.*

## 7. THE EXISTENCE OF BLOW-UP SOLUTION OF PROBLEM (1.1)-(1.3)

In this section, in a special case, we show that if the conditions in (2.1), imposed on the nonlinear functions $g$ and $F$ are violated, then the solution $u$ of the problem (1.1)-(1.3) can turn out to be blow-up. That is, a number $T^* \in (0, l]$ can be found such that for $T < T^*$ the problem (1.1)-(1.3) has a unique classical solution $u$, and

$$\lim_{T \to T^* - 0} \|u\|_{C(\overline{D}_T)} = \infty. \tag{7.1}$$

This, in particular, implies that the considered problem has no a classical solution in the domain $D_T$ for $T \geq T^*$. Indeed, consider the following special case of problem (1.1)-(1.3)

$$\begin{gathered} u_{tt} - u_{xx} = 0, \quad (x,t) \in D_T; \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \leq x \leq l, \\ u_x(0,t) = F[u(0,t)], \quad u_x(l,t) = 0, \quad 0 \leq t \leq T, \end{gathered} \tag{7.2}$$

where $\varphi \in C^2([0,l]),\ \varphi(0) > 0,\ \psi \in C^1([0,l])$ and $F(s) = -\delta |s|^\lambda s,\ \delta := const > 0,\ \lambda := const > 0,\ s \in \mathbb{R}$, and the corresponding consistency conditions, similar to (1.4), are satisfied. It is easy to check that in the case $\psi = -\varphi'$, the classical solution $u$ of this problem in the domain $D_T$ for $T = T^*$ is given by formula:

$$u(x,t) = \begin{cases} \varphi(x-t), \quad (x,t) \in \Delta_1 \cap \{t < T^*\}, \\ \mu_1(t-x), \qquad (x,t) \in \Delta_2 \cap \{t < T^*\}, \\ \varphi(2l - x - t) - \varphi(l) + \varphi(x-t), \quad (x,t) \in \Delta_3 \cap \{t < T^*\}, \\ \mu_1(t-x) + \varphi(2l - x - t) - \varphi(x + t - l), \\ (x,t) \in \Delta_4 \cap \{t < T^*\}, \end{cases} \tag{7.3}$$

where

$$\mu_1(t) = \frac{\varphi(0)}{[1 - \delta\lambda\varphi^\lambda(0)t]^{\frac{1}{\lambda}}}, \quad 0 \leq t < T^* := \frac{1}{\delta\lambda\varphi^\lambda(0)} < l. \tag{7.4}$$

It follows from (7.3) and (7.4) that the solution of problem (7.2) is blow-up, that is, (7.1) is satisfied. Therefore, in the considered case, in the statement of this problem it should be required that $T < T^*$.

**Remark 7.1.** In fact, formula (7.3) allows to continue the solution of the considered problem from the domain $D_{T^*}$ to domain $D_l \cap \{t < x + T^*\}$, and this solution $u(x,t)$ will unboundedly increase when the point $(x,t)$ from the domain $D_l \cap \{t < x + T^*\}$ approaches to the characteristic $t - x = T^*$, to which border on this domain by the part of its boundary.

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