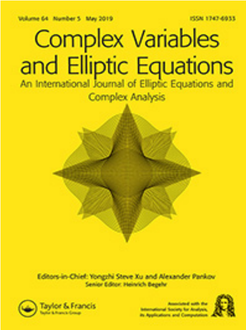

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S. Kharibegashvili & B. Midodashvili

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A boundary value problem for higher-order semilinear partial differential equations

S. Kharibegashvili[a,b] and B. Midodashvili[c,d]

[a]I. Javakhishvili Tbilisi State University A. Razmadze Mathematical Institute, Tbilisi, Georgia; [b]Georgian Technical University Department of Mathematics, Tbilisi, Georgia; [c]I. Javakhishvili Tbilisi State University Faculty of Exact and Natural Sciences, Tbilisi, Georgia; [d]Gori State Teaching University Faculty of Education, Exact and Natural Sciences, Gori, Georgia

ABSTRACT
One boundary value problem for a class of higher-order semilinear partial differential equations is considered. Theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

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1. Statetement of the problem

In the Euclidian space $\mathbb{R}^n$ of the variables $x = (x_1, \ldots, x_n)$ and t we consider the semilinear equation of the type

$$L_f := \frac{\partial^{4k} u}{\partial t^{4k}} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + f(u) = F, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $a_{ij} = a_{ji} = a_{ij}(x)$, $i, j = 1, \ldots, n$, $F = F(x, t)$ are given, and $u = u(x, t)$ is an unknown real functions, k is a natural number, $n \geq 2$.

For the equation (1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega \times (0, T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u = u(x, t)$ of that equation according to the boundary conditions

$$u|_{\Gamma} = 0, \tag{2}$$

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{\Omega_0 \cup \Omega_T} = 0, \quad i = 0, \ldots, 2k - 1, \tag{3}$$

CONTACT S. Kharibegashvili kharibegashvili@yahoo.com

where $\Gamma := \partial\Omega \times (0,T)$ is the lateral face of the cylinder D_T, $\Omega_0 : x \in \Omega,\ t = 0$ and $\Omega_T : x \in \Omega,\ t = T$ are upper and lower bases of this cylinder, respectively.

A numerous literature is dedicated to the research of initial and mixed problems for the high order semilinear hyperbolic equations having a structure different from (1) for example, see works [1–11] and works cited there). Note that some of the results in this direction have been discussed in the workshop materials [12].

Denote by $C^{2,4k}(\bar{D}_T, \partial D_T)$ the space of functions u continuous in $\bar{D}_T$, having continuous partial derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$, $\partial^l u/\partial t^l$, $i,j = 1,\ldots,n$; $l = 1,\ldots,4k$, in $\bar{D}_T$. Assume

$$C_0^{2,4k}(\bar{D}_T, \partial D_T) := \left\{ u \in C^{2,4k}(\bar{D}_T) : u|_\Gamma = 0, \quad \left.\frac{\partial^i u}{\partial t^i}\right|_{\Omega_0 \cup \Omega_T} = 0,\ i = 0,\ldots,2k-1 \right\}.$$

Let $a_{ij} = a_{ij}(x) \in C^1(\bar{\Omega})$, $i,j = 1,\ldots,n$, and $u \in C_0^{2,4k}(\bar{D}_T, \partial D_T)$ be a classical solution of the problem (1)–(3). Multiplying both parts of the equation (1) by an arbitrary function $\varphi \in C_0^{2,4k}(\bar{D}_T, \partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$\int_{D_T} \left[\frac{\partial^{2k} u}{\partial t^{2k}} \cdot \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right] \mathrm{d}x\mathrm{d}t + \int_{D_T} f(u)\varphi\, \mathrm{d}x\mathrm{d}t$$
$$= \int_{D_T} F\varphi\, \mathrm{d}x\mathrm{d}t \quad \forall\, \varphi \in C_0^{2,4k}(\bar{D}_T, \partial D_T). \tag{4}$$

Below, we assume that the operator $K := \sum_{i,j=1}^{n} \partial/\partial x_j (a_{ij}(x)(\partial u/\partial x_i))$ is strongly elliptic in $\bar{\Omega}$, i.e.

$$k_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le k_1|\xi|^2 \quad \forall x \in \bar{\Omega}, \quad \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n, \tag{5}$$

where $k_0, k_1 = \mathrm{const} > 0$, $|\xi|^2 = \sum_{i=1}^{n} \xi_i{}^2$. Note that (5) implies the hypoellipticity of the linear part of the operator from (1), i.e. L_0 is hyppoelliptic for each $x = x_0 \in \bar{\Omega}$ [13].

Introduce the Hilbert space $W_0^{1,2k}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int_{D_T} \left[u^2 + \sum_{i=1}^{2k} \left(\frac{\partial^i u}{\partial t^i} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right] \mathrm{d}x\mathrm{d}t \tag{6}$$

of the classical space $C_0^{2,4k}(\bar{D}_T, \partial D_T)$.

Remark 1.1: It follows from (6) that if $u \in W_0^{1,2k}(D_T)$, then $u \in \overset{o}{W}{}_2^1(D_T)$ and $\partial^i u/\partial t^i \in L_2(D_T)$, $i = 2,\ldots,2k$. Here $W_2^1(D_T)$ is the well-known Sobolev space [14] consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$, and $\overset{o}{W}{}_2^1(D_T) = \{u \in W_2^1(D_T) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory [14].

We take the equality (4) as a basis for our definition of the weak generalized solution u of the problem (1), (2), (3).

Below, on the function $f = f(u)$ we impose the following requirements

$$f \in C(\mathbb{R}), \quad |f(u)| \leq M_1 + M_2|u|^{\alpha}, \quad u \in \mathbb{R}, \tag{7}$$

where $M_i =$ const ≥ 0, $i = 1, 2$, and

$$0 \leq \alpha = \text{const} < \frac{n+1}{n-1}. \tag{8}$$

Remark 1.2: The embedding operator $I : W_2^1(\bar{D}_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < 2(n+1)/(n-1)$, when $n > 1$ [14]. At the same time the Nemytsky operator $N : L_q(D_T) \to L_2(D_T)$, acting by the formula $Nu = -f(u)$, due to (7) is continuous and bounded if $q \geq 2\alpha$ [15]. Thus, since due to (8) we have $2\alpha < 2(n+1)/(n-1)$, then there exists a number q such that $1 < q < 2(n+1)/(n-1)$ and $q \geq 2\alpha$. Therefore, in this case the operator

$$N_0 = NI : \overset{0}{W_2^1}(D_T) \to L_2(D_T) \tag{9}$$

will be continuous and compact. Besides, from $u \in W_0^{1,2k}(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u_m \to u$ in the space $W_0^{1,2k}(D_T)$, then $f(u_m) \to f(u)$ in the space $L_2(D_T)$.

Definition 1.1: Let the function f satisfy the conditions (7) and (8), $F \in L_2(D_T)$. The function $u \in W_0^{1,2k}(D_T)$ is said to be a weak generalized solution of the problem (1)–(3), if for any $\varphi \in W_0^{1,2k}(D_T)$ the integral equality (4) is valid.

It is not difficult to verify that if the solution of the problem (1)–(3) in the sense of Definition 1.1 belongs to the class $C_0^{2,4k}(D_T, \partial D_T)$, then it will also be a classical solution of this problem.

2. The solvability of problem (1)–(3)

In the space $C_0^{2,4k}(\bar{D}_T, \partial D_T)$, together with the scalar product

$$(u, v)_o = \int_{D_T} \left[u \cdot v + \sum_{i=1}^{2k} \frac{\partial^i u}{\partial t^i} \frac{\partial^i v}{\partial t^i} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right] \mathrm{d}x\mathrm{d}t \tag{10}$$

with norm $||\cdot||_0 = ||u||_{W_0^{1,2k}(D_T)}$ defined by the right-hand side part of equality (6), let us introduce the following scalar product

$$(u, v)_1 = \int_{D_T} \left[\frac{\partial^{2k} u}{\partial t^{2k}} \frac{\partial^{2k} v}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right] \mathrm{d}x\mathrm{d}t \tag{11}$$

with norm

$$||u||_1^2 = \int_{D_T} \left[\left(\frac{\partial^{2k} u}{\partial t^{2k}} \right)^2 + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right] \mathrm{d}x\mathrm{d}t, \tag{12}$$

where $u, v \in C_0^{2,4k}(\bar{D}_T, \partial D_T)$.

Lemma 2.1: *The inequalities*

$$c_1||u||_0 \leq ||u||_1 \leq c_2||u||_0 \quad \forall\, u \in C_0^{2,4k}(\bar{D}_T, \partial D_T) \tag{13}$$

hold, where the positive constants c_1 and c_2 do not depend on u.

Proof: If $u \in C_0^{2,4k}(\bar{D}_T, \partial D_T)$ then for fixed $t \in [0,T]$ the function $u(\cdot,t) \in \overset{o}{W}{}_2^1(\Omega)$ and due to a known inequality [14]

$$||u(\cdot,t)||^2_{L_2(\Omega)} \leq c_0 \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 (x,t)\,\mathrm{d}x, \tag{14}$$

whence, in view of (5), we have

$$||u(\cdot,t)||^2_{L_2(\Omega)} \leq \frac{c_0}{k_0} \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}(x,t)\,\mathrm{d}x, \tag{15}$$

where the positive constants k_0 and $c_0 = c_0(\Omega)$ do not depend on $t \in [0,T]$ and u. Integrating inequalities (14) and (15) on $t \in [0,T]$ we obtain

$$||u||^2_{L_2(D_T)} \leq c_0 \int_{D_T} \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 (x,t)\,\mathrm{d}x\mathrm{d}t, \tag{16}$$

$$||u||^2_{L_2(D_T)} \leq \frac{c_0}{k_0} \int_{D_T} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}(x,t)\,\mathrm{d}x\mathrm{d}t. \tag{17}$$

Let us evaluate the norms $\|\partial^i u/\partial t^i\|_{L_2(D_T)}$ for $i = 1,\ldots,2k-1$ through $\|\partial^{2k}u/\partial t^{2k}\|_{L_2(D_T)}$. Since $u \in C_0^{2,4k}(\bar{D}_T, \partial D_T)$ satisfies equalities (3), then it is easy to see that

$$\frac{\partial^i u(\cdot,t)}{\partial t^i} = \frac{1}{(2k-i-1)!}\int_0^t (t-\tau)^{2k-i-1}\frac{\partial^{2k}u(\cdot,\tau)}{\partial t^{2k}}\,\mathrm{d}\tau, \quad i = 1,\ldots,2k-1. \tag{18}$$

From (18), using Cauchy inequality, we obtain

$$\begin{aligned}
\left(\frac{\partial^i u(\cdot,t)}{\partial t^i}\right)^2 &\leq \frac{1}{((2k-i-1)!)^2}\int_0^t (t-\tau)^{2(2k-i-1)}\,\mathrm{d}\tau \int_0^t \left(\frac{\partial^{2k}u(\cdot,t)}{\partial t^{2k}}\right)^2 \mathrm{d}\tau \\
&= \frac{t^{4k-2i-1}}{((2k-i-1)!)^2\,(4k-2i-1)}\int_0^t \left(\frac{\partial^{2k}u(\cdot,t)}{\partial t^{2k}}\right)^2 \mathrm{d}\tau \\
&\leq T^{4k-2i-1}\int_0^T \left(\frac{\partial^{2k}u(\cdot,\tau)}{\partial t^{2k}}\right)^2 \mathrm{d}\tau,
\end{aligned}$$

whence

$$\int_0^T \left(\frac{\partial^i u(\cdot,t)}{\partial t^i}\right)^2 \mathrm{d}t \leq T^{4k-2i}\int_0^T \left(\frac{\partial^{2k}u(\cdot,\tau)}{\partial t^{2k}}\right)^2 \mathrm{d}\tau, \quad i = 1,\ldots,2k-1. \tag{19}$$

Integrating both parts of inequality (19) over the domain Ω we obtain

$$\left\|\frac{\partial^i u}{\partial t^i}\right\|^2_{L_2(D_T)} \leq T^{4k-2i}\left\|\frac{\partial^{2k} u}{\partial t^{2k}}\right\|^2_{L_2(D_T)}, \quad i=1,\ldots,2k-1. \tag{20}$$

Due to (5) we have

$$k_0\int_{D_T}\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x\mathrm{d}t \leq \int_{D_T}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x\mathrm{d}t \leq k_1\int_{D_T}\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x\mathrm{d}t. \tag{21}$$

Finally, from (6), (12), (16), (17), (20) and (21) we easily obtain (13). Lemma 2.1 is proved. ■

Remark 2.1: If we complete the space $C_0^{2,4k}(\bar{D}_T, \partial D_T)$ under the norm [12] due to Lemma 2.1, then in view of (10) we obtain the same Hilbert space $W_0^{1,2k}(D_T)$ with equivalent scalar products (10) and (11).

Consider the following condition

$$\lim_{|u|\to\infty}\inf\frac{f(u)}{u} \geq 0. \tag{22}$$

Lemma 2.2: *Let $F \in L_2(D_T)$ and the conditions* (7), (8) *and* (22) *be fulfilled. Then for a weak generalized solution $u \in W_0^{1,2k}(D_T)$ of the problem* (1)–(3) *the a priori estimate*

$$||u||_0 = ||u||_{W_0^{1,2k}(D_T)} \leq c_3||F||_{L_2(D_T)} + c_4 \tag{23}$$

is valid with constants $c_3 > 0$ and $c_4 \geq 0$, independent of u and F.

Proof: Since $f \in C(\mathbb{R})$, then from (22) it follows that for each $\varepsilon > 0$ there exists a number $M_\varepsilon \geq 0$ such that

$$uf(u) \geq -M_\varepsilon - \varepsilon u^2 \quad \forall u \in \mathbb{R}. \tag{24}$$

Assuming that $\varphi = u \in W_0^{1,2k}(D_T)$ in equality (4) and taking into account (24) and (12), for each $\varepsilon > 0$ we have

$$\begin{aligned}||u||_1^2 &= -\int_{D_T} uf(u)\,\mathrm{d}x\mathrm{d}t + \int_{D_T} Fu\,\mathrm{d}x\mathrm{d}t \leq M_\varepsilon\, mesD_T + \varepsilon\int_{D_T} u^2\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{D_T}\left(\frac{1}{4\varepsilon}F^2 + \varepsilon u^2\right)\mathrm{d}x\mathrm{d}t = \frac{1}{4\varepsilon}||F||^2_{L_2(D_T)} + M_\varepsilon\, mesD_T + 2\varepsilon||u||^2_{L_2(D_T)} \\ &\leq \frac{1}{4\varepsilon}||F||^2_{L_2(D_T)} + M_\varepsilon\, mesD_T + 2\varepsilon||u||_0^2.\end{aligned} \tag{25}$$

Due to (13) from (25) we have

$$c_1^2||u||_0^2 \leq ||u||_1^2 \leq \frac{1}{4\varepsilon}||F||^2_{L_2(D_T)} + M_\varepsilon\, mesD_T + 2\varepsilon||u||_0^2,$$

whence, for $\varepsilon = \frac{1}{4}c_1^2$ we obtain

$$||u||_0^2 \leq 2c_1^{-4}||F||^2_{L_2(D_T)} + 2c_1^{-2}M_\varepsilon \, mesD_T.$$

From the last inequality follows (23) for $c_3 = 2c_1^{-4}$ and $c_4 = 2c_1^{-2}M_\varepsilon \, mesD_T$, where $\varepsilon = \frac{1}{4}c_1^2$. Lemma 2.2 is proved. ■

Remark 2.2: First we consider a linear problem correspondent to (1)–(3), i.e. when $f = 0$. In this case for $F \in L_2(D_T)$ we analogously introduce a notion of a weak generalized solution $u \in W_0^{1,2k}(D_T)$ of this problem for which it is valid the integral equality

$$\begin{aligned}(u,\varphi)_1 &= \int_{D_T}\left[\frac{\partial^{2k}u}{\partial t^{2k}}\frac{\partial^{2k}\varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\right]\mathrm{d}x\mathrm{d}t \\ &= \int_{D_T} F\varphi \,\mathrm{d}x\mathrm{d}t \quad \forall \varphi \in W_0^{1,2k}(D_T). \end{aligned} \tag{26}$$

In view of (13) we have

$$\begin{aligned}\left|\int_{D_T} F\varphi \,\mathrm{d}x\mathrm{d}t\right| &\leq ||F||_{L_2(D_T)}||\varphi||_{L_2(D_T)} \\ ||F||_{L_2(D_T)}||\varphi||_0 &\leq c_1^{-1}||F||_{L_2(D_T)}||\varphi||_1. \end{aligned} \tag{27}$$

Due to Remark 2.1, (26) and (27) from the Riess theorem it follows the existence of a unique function $u \in W_0^{1,2k}(D_T)$ which satisfies equality (26) for any $\varphi \in W_0^{1,2k}(D_T)$ and for its norm is valid the estimate

$$||u||_1 \leq c_1^{-1}||F||_{L_2(D_T)}. \tag{28}$$

Due to (13) from (28) we obtain

$$||u||_0 = ||u||_{W_0^{1,2k}(D_T)} \leq c_1^{-2}||F||_{L_2(D_T)}. \tag{29}$$

Thus, introducing the notation $u = L_0^{-1}F$, we find that to the linear problem corresponding to (1)–(3), i.e. when $f = 0$, there corresponds the linear bounded operator

$$L_0^{-1} : L_2(D_T) \to W_0^{1,2k}(D_T)$$

and for its norm the estimate

$$||L_0^{-1}||_{L_2(D_T)\to W_0^{1,2k}(D_T)} \leq c_1^{-2} \tag{30}$$

holds by virtue of (29).

Taking into account Definition 1.1 and Remark 2.2, we can rewrite the equality (4), equivalent to the problem (1)–(3) in the form

$$u = L_0^{-1}\left[-f(u) + F\right] \tag{31}$$

in the Hilbert space $W_0^{1,2k}(D_T)$.

Remark 2.3: Since due to (6) and Remark 1.1 the space $\overset{\circ}{W}{}^{1,2k}_0(D_T)$ is continuously embedded into the space $W_2^{1,0}(D_T)$, taking into account (9) from Remark 1.2, when the conditions (7) and (8) are fulfilled, we see that the operator

$$N_1 = NII_1 : W_0^{1,2k}(D_T) \to L_2(D_T),$$

where $I_1 : W_0^{1,2k}(D_T) \to W_2^{1,0}(D_T)$ is the embedding operator, is likewise continuous and compact.

We rewrite the equation (31) as

$$u = Au := L_0^{-1}(N_1 u + F). \tag{32}$$

Then, taking into account (30) and Remark 2.3, we conclude that the operator $A : W_0^{1,2k}(D_T) \to W_0^{1,2k}(D_T)$ from (32) is continuous and compact. At the same time according to the a priori estimate (23) of Lemma 2.2 in which the constants $c_3 = 2c_1^{-4}$ and $c_4 = 2c_1^{-2} M_\varepsilon \, mesD_T$, $\varepsilon = \frac{1}{4}c_1^2$ for any parameter $\tau \in [0,1]$ and for every solution $u \in W_0^{1,2k}(D_T)$ of equation $u = \tau Au$ with the above-mentioned parameter the a priori estimate (23) is valid with the same constants $c_3 > 0$ and $c_4 \geq 0$, independent of u, F and τ. Therefore, by the Schaefer's fixed point theorem [16] equation (32) and hence the problem (1)–(3) has at least one weak generalized solution u from the space $W_0^{1,2k}(D_T)$. Thus, the following theorem is valid.

Theorem 2.1: *Let the conditions* (7), (8) *and* (22) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *has at least one weak generalized solution* $u \in W_0^{1,2k}(D_T)$.

3. Uniqueness of the solution of problem (1)–(3)

Theorem 3.1: *Let f be a monotone function and satisfy the conditions* (7), (8). *Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *cannot have more than one weak generalized solution in the space* $W_0^{1,2k}(D_T)$.

Proof: Let $F \in L_2(D_T)$, and moreover, let u_1 and u_2 be two weak generalized solutions of the problem (1)–(3) from the space $W_0^{1,2k}(D_T)$, i.e. according to (4) the equalities

$$\int_{D_T} \left[\frac{\partial^{2k} u_m}{\partial t^{2k}} \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u_m}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right] \mathrm{d}x\mathrm{d}t$$
$$= -\int_{D_T} f(u_m)\varphi \, \mathrm{d}x\mathrm{d}t + \int_{D_T} F\varphi \, \mathrm{d}x\mathrm{d}t \quad \forall \, \varphi \in W_0^{1,2k}(D_T), \tag{33}$$

are valid, $m = 1,2$.

From (33), for the difference $v = u_2 - u_1$ we have

$$\int_{D_T} \left[\frac{\partial^{2k} v}{\partial t^{2k}} \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right] \mathrm{d}x\mathrm{d}t$$
$$= - \int_{D_T} \big(f(u_2) - f(u_1)\big)\, \varphi \, \mathrm{d}x\mathrm{d}t \quad \forall \varphi \in \overset{\circ}{W}{}_0^{1,2k}(D_T). \tag{34}$$

Putting $\varphi = v \in W_0^{1,2k}(D_T)$ in the equality (34), in view of (12) we obtain

$$||v||_1 = - \int_{D_T} \big(f(u_2) - f(u_1)\big)\,(u_2 - u_1)\,\mathrm{d}x\mathrm{d}t. \tag{35}$$

Since f is a monotone function, we have

$$\big(f(s_2) - f(s_1)\big)\,(s_2 - s_1) \geq 0 \quad \forall\, s_1, s_2 \in \mathbb{R}^n. \tag{36}$$

From (13), (35) and (36) it follows that

$$c_1 ||v||_0 \leq ||v||_1 \leq 0,$$

whence we find that $v = 0$, i.e. $u_2 = u_1$, and hence the proof of the Theorem 3.1 is complete. ■

From Theorem 2.1 and 3.1 in its turn it follows

Theorem 3.2: *Let f be a monotone function and satisfy the conditions* (7), (8) *and* (22). *Then for any $F \in L_2(D_T)$ the problem* (1)–(3) *has a unique weak generalized solution in the space $W_0^{1,2k}(D_T)$.*

4. Nonexistence of a solution of problem (1)–(3)

Let for simplicity $\Omega : |x| < 1$.

Theorem 4.1: *Let $F^0 \in L_2(D_T)$, $||F^0||_{L_2(D_T)} \neq 0$, $F^0 \geq 0$ and $F = \mu F^0$, $\mu = \mathrm{const} > 0$. Then, if conditions* (7), (8) *are fulfilled and $f(u) \leq -|u|^\alpha \; \forall u \in \mathbb{R}^n$, $\alpha > 1$, there exist a number $\mu_0 = \mu_0(F^0, \alpha) > 0$ such that for $\mu > \mu_0$ the problem* (1)–(3) *cannot have a weak generalized solution in the space $W_0^{1,2k}(D_T)$.*

Proof: Assume that the conditions of the theorem are fulfilled and the solution $u \in W_0^{1,2k}(D_T)$ of the problem (1)–(3) exists for any fixed $\mu > 0$. Then the equality (4) takes the form

$$\int_{D_T}\left[\frac{\partial^{2k}u}{\partial t^{2k}}\cdot\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\right]\mathrm{d}x\mathrm{d}t=-\int_{D_T}f(u)\varphi\,\mathrm{d}x\mathrm{d}t$$
$$+\mu\int_{D_T}F^0\varphi\,\mathrm{d}x\mathrm{d}t\quad\forall\,\varphi\in W_0^{1,2k}(D_T).\tag{37}$$

By integration by parts it can be easily verified that

$$\int_{D_T}\left[\frac{\partial^{2k}u}{\partial t^{2k}}\cdot\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\right]\mathrm{d}x\mathrm{d}t$$
$$=\int_{D_T}u\left[\frac{\partial^{4k}\varphi}{\partial t^{4k}}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial\varphi}{\partial x_i}\right)\right]\varphi\,\mathrm{d}x\mathrm{d}t$$
$$=\int_{D_T}uL_0\,\varphi\,\mathrm{d}x\mathrm{d}t\quad\forall\varphi\in C_0^{2,4k}(\bar{D}_T,\partial D_T),\tag{38}$$

where the space $C_0^{2,4k}(\bar{D}_T,\partial D_T)$ was introduced in the first section, besides

$$C_0^{2,4k}(\bar{D}_T,\partial D_T)\subset W_0^{1,2k}(D_T).$$

In view of (38) and conditions of the theorem from (37) we obtain

$$\int_{D_T}|u|^{\alpha}\varphi\,\mathrm{d}x\mathrm{d}t\le\int_{D_T}uL_0\,\varphi\,\mathrm{d}x\mathrm{d}t-\mu\int_{D_T}F^0\varphi\,\mathrm{d}x\mathrm{d}t\quad\forall\varphi\in C_0^{2,4k}(\bar{D}_T,\partial D_T).\tag{39}$$

Below we use the method of test functions [17]. As a test function we take $\varphi\in C_0^{2,4k}(\bar{D}_T,\partial D_T)$ such that $\varphi|_{D_T}>0$. If in Young's inequality with parameter $\varepsilon>0$

$$ab\le\frac{\varepsilon}{\alpha}a^{\alpha}+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}b^{\alpha'};a,b\ge 0,\quad \alpha'=\frac{\alpha}{\alpha-1}$$

we take $a=|u|\varphi^{1/\alpha}$, $b=|L_0\varphi|/\varphi^{1/\alpha}$, then taking into account that $\alpha'/\alpha=\alpha'-1$ we have

$$|uL_0\varphi|=|u|\varphi^{1/\alpha}\frac{|L_0\varphi|}{\varphi^{1/\alpha}}\le\frac{\varepsilon}{\alpha}|u|^{\alpha}\varphi+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}.\tag{40}$$

From (39), (40) we have the inequality

$$\left(1-\frac{\varepsilon}{\alpha}\right)\int_{D_T}|u|^{\alpha}\varphi\,\mathrm{d}x\mathrm{d}t=\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int_{D_T}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\mathrm{d}x\mathrm{d}t-\mu\int_{D_T}F^0\varphi\,\mathrm{d}x\mathrm{d}t,$$

whence for $\varepsilon<\alpha$ we get

$$\int_{D_T}|u|^{\alpha}\varphi\,\mathrm{d}x\mathrm{d}t\le\frac{\alpha}{(\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}}\int_{D_T}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\mathrm{d}x\mathrm{d}t-\frac{\alpha\mu}{\alpha-\varepsilon}\int_{D_T}F^0\varphi\,\mathrm{d}x\mathrm{d}t.\tag{41}$$

Taking into account the equalities $\alpha' = \alpha/(\alpha-1), \alpha = \alpha'/(\alpha'-1)$ and $\min_{0<\varepsilon<\alpha} \alpha/((\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}) = 1$ which is achieved at $\varepsilon = 1$, from (41) we find that

$$\int_{D_T} |u|^\alpha \varphi \, dxdt \le \int_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dxdt - \alpha'\mu \int_{D_T} F^0 \varphi \, dxdt. \tag{42}$$

Note that it is not difficult to show the existence of a test function φ such that

$$\varphi \in C_0^{2,4k}(\bar{D}_T, \partial D_T), \ \varphi|_{D_T} > 0, \kappa_0 = \int_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dxdt < +\infty. \tag{43}$$

Indeed, as it can be easily verified, the function

$$\varphi(x,t) = [(1-|x|^2)t(T-t)]^m$$

for a sufficiently large positive m satisfies conditions (43).

Since by the condition of the theorem $F^0 \in L_2(D_T)$, $||F^0||_{L_2(D_T)} \neq 0$, $F^0 \ge 0$, and $mes\, D_T < +\infty$, due to the fact that $\varphi|_{D_T} > 0$ we have

$$0 < \kappa_1 = \int_{D_T} F^0 \varphi \, dxdt < +\infty. \tag{44}$$

Denote by $g(\mu)$ the right-hand side of the inequality (42) which is a linear function with respect to μ. From (43) and (44) we have

$$g(\mu) < 0 \quad \text{for } \mu > \mu_0 \quad \text{and} \quad g(\mu) > 0 \quad \text{for } \mu < \mu_0, \tag{45}$$

where

$$g(\mu) = \kappa_0 - \alpha'\mu\kappa_1, \quad \mu_0 = \frac{\kappa_0}{\alpha'\kappa_1} > 0.$$

Owing to (45) for $\mu > \mu_0$, the right-hand side of the inequality (42) is negative, whereas the left-hand side of that inequality is nonnegative. The obtained contradiction proves the theorem. ■

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No potential conflict of interest was reported by the authors.

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