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THE ADHESIVE CONTACT PROBLEMS IN THE PLANE THEORY OF ELASTICITY

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Abstract. The problems of constructing an exact and approximate solutions of system of singular integro-differential equations related to the problems of adhesive interaction between elastic thin finite or infinite nonhomogeneous patch and elastic plate are investigated. For the patch loaded by horizontal and vertical forces the usual model of beam bending in combination with the uniaxial stress state model is valid. Using the methods of theory of analytic functions, integral transformation or orthogonal polynomials the singular integro-differential equations reduced to the different boundary value problems (Karleman type problem with displacements, Riemann problem) of the theory of analytic functions or to the infinite system of linear algebraic equations. The asymptotic analysis of problem is carried out.

Let a finite non-homogeneous patch with modulus of elasticity $E_1(x)$, thickness $h_1(x)$ and Poisson's coefficient ν_1 be attached to the plate (E_2, v_2), which is in the condition of a plane deformation. It is assumed that the horizontal and vertical stresses with intensity $\tau_0(x)$ and $p_0(x)$ acts on the patch along the ox-axis. The patch in the vertical direction bends like a beam (has a finite bending stiffness, model A) or along the horizontal axis the vertical elastic displacements of its points are constant (model B) and besides in the horizontal direction the patch compressed or stretched like rod being in uniaxil stress state. The contact between the plate and patch is realized by a thin glue layer with width h_0 and Lame's constants λ_0, μ_0. The contact condition has the form [1]

$$u_1(x) - u_2(x,0) = k_0\tau(x), \quad v_1 s(x) - v_2(x,0) = m_0 p(x), \quad |x| < 1 \tag{1}$$

where $u_2(x,y)$, $v_2(x,y)$ are displacements of the plate points along the ox-axis, $u_1(x)$, $v_1(x)$ displacements of the patch points along the ox-axis. $k_0 := h_0/\mu_0$, $m_0 : h_0/(\lambda_0 + 2\mu_0)$.

We have to define the law of distribution of tangential and normal contact stresses $\tau(x)$ and $p(x)$ on the line of contact, the asymptotic behavior of these stresses at the end of the patches and the coefficient of stress intensity.

According to the equilibrium equation of patch elements and Hooke's law we have: in model A

$$\begin{gathered}\frac{du_1(x)}{dx} = \frac{1}{E(x)}\int\limits_{-1}^{x}[\tau(t) - \tau_0(t)]dt, \\ \frac{d^2}{dx^2}D(x)\frac{d^2v_1(x)}{dx^2} = p_0(x) - p(x), \quad |x| < 1\end{gathered} \tag{2}$$

the equilibrium equation of the patch has the form

$$\int\limits_{-1}^{1}[\tau(t) - \tau_0(t)]dt = 0, \quad \int\limits_{-1}^{1}[p(t) - p_0(t)]dt = 0, \quad \int\limits_{-1}^{1}t[p(t) - p_0(t)]dt = 0, \tag{3}$$

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and in model B

$$\frac{du_1(x)}{dx} = \frac{1}{E(x)} \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt,$$
$$\frac{dv_1(x)}{dx} = 0, \quad |x| < 1 \tag{2'}$$

where $E(x) = \dfrac{E_1(x)h_1(x)}{1-\nu_1^2}$, $D(x) = \dfrac{E_1(x)h_1^3(x)}{1-\nu_1^2}$. According to known results [2], the horizontal and vertical deformations of the points of the ox axis have the form

$$\frac{du_2(x,0)}{dx} = -ap(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{\tau(t)dt}{t-x}, \quad \frac{dv_2(x,0)}{dx} = -\frac{b}{\pi}\int_{-1}^{1} \frac{p(t)dt}{t-x} - a\tau(x) \tag{4}$$
$$a = \frac{(1+\nu_2)(1-2\nu_2)}{E_2}, \quad b = \frac{2(1-\nu_2^2)}{E_2}.$$

Introducing the notations

$$\varphi(x) = \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \quad \psi(x) = \int_{-1}^{x} dt \int_{-1}^{t} [p_0(\tau) - p(\tau)]d\tau$$

from (1), (2) and (4) we obtain the following system of integro-differential equations

$$\frac{\varphi(x)}{E(x)} - \frac{b}{\pi}\int_{-1}^{1} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) - a\psi''(x) = f_1(x), \tag{5}$$

$$\frac{\psi(x)}{D(x)} - \frac{b}{\pi}\frac{d}{dx}\int_{-1}^{1} \frac{\psi''(t)dt}{t-x} + a\varphi''(x) + m_0\psi^{(\mathrm{IV})}(x) = f_2(x), \quad |x| < 1 \tag{6}$$

where

$$f_1(x) = -ap_0(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{\tau_0(t)dt}{t-x} + k_0\tau_0'(x),$$

$$f_2(x) = -a\tau_0'(x) - \frac{b}{\pi}\frac{d}{dx}\int_{-1}^{1} \frac{p_0(t)dt}{t-x} + m_0p_0''(x),$$

or from (1), (2′) and (4) we have

$$\frac{\varphi(x)}{E(x)} - \frac{b}{\pi}\int_{-1}^{1} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) - a\psi''(x) = f_1(x), \tag{5'}$$

$$m_0\psi^{m}(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{\psi''(t)dt}{t-x} + a\varphi'(x) = f_3(x), \quad |x| < 1 \tag{6'}$$

where

$$f_3(x) = -a\tau_0(x) + m_0p_0'(x)$$

and from (3) we have the conditions

$$\varphi(1) = 0, \quad \psi(1) = 0, \quad \psi'(1) = 0. \tag{7}$$

Thus, the above posed boundary contact problem reduced to the system of singular integro-differential equation (5), (6) or ((5′), (6′)) with the condition (7). From the symmetry of the problem, we assume, that $E(x)$, $D(x)$ and $p_0(x)$ are even functions and $\tau_0(x)$ is uneven function, the solutions of equation (5), (6), ((5′), (6′)) under the condition (7) can be sought in the class of even functions, besides

$$\varphi, \varphi' \in H[-1,1], \quad \varphi'' \in H^*(-1,1), \quad \psi, \psi', \psi'' \in H[-1,1], \quad \psi''', \psi^{(\mathrm{IV})} \in H^*(-1,1).$$

We assume that the function $\tau_0(x)$ and $p_0(x)$ is continuous in the Holder's sense, $\tau_0(x)$ has a continuous first order derivative, $p_0(x)$ has a continuous first and second order derivatives on the interval [-1,1]. Under the assumption that

$$E(x) = (1-x^2)^\gamma b_0(x), \quad \gamma \geq 0, \quad b_0(x) = b_0(-x), \quad b_0 \in C([-1,1]) \tag{8}$$
$$b_0(x) \geq c_0 = \text{const} > 0$$
$$D(x) = (1-x^2)^\delta b_1(x), \quad \delta > 0, \quad b_1(x) = b_1(-x), \quad b_1 \in C([-1,1]), \tag{9}$$
$$b_1(x) \geq c_1 = \text{const} > 0$$

a solutions of problem (5)–(7) will be sought in the class of even function whose derivatives are representable in the form

$$\begin{aligned} \varphi'(x) &= (1-x^2)^a g_1(x), \quad \alpha > -1 \\ \psi''(x) &= (1-x^2)^\beta g_2(x), \quad \beta > -1 \end{aligned} \tag{10}$$

where

$$g_1(x) = -g_1(-x), \quad g_1 \in C'([-1,1]), \quad g_1(x) \neq 0, \quad x \in [-1,1]$$
$$g_2(x) = g_2(-x), \quad g_2 \in C'([-1,1]), \quad g_2(x) \neq 0, \quad x \in [-1,1].$$

There is valid the following

Theorem. *In condition* (8)*,* (9)*, if the problems* (5)–(7) ((5′)–(7)) *has the solutions in the form* (10)*, then:*
If $\gamma > 2$ *and* $\delta > 4$*, then* $\alpha = \gamma - 1$ *and* $\beta = \delta - 2$*,*
If $\gamma \leq 2$ *and* $\delta \leq 4$*, then* $0 \leq \alpha \leq 1$ *and* $0 \leq \beta \leq 2$*.*

Remark 1. Numbers and cannot be negative, which corresponds to the physical meaning of the condition (1).

Remark 2. If the inclusion rigidity varies by the law

$$E(x) = (1-x^2)^{n+\frac{1}{2}} b_0(x), \tag{11}$$

where $n \geq 0$ is integer, then from the above asymptotic analysis we obtain:

$$\alpha = n - \frac{1}{2}, \quad \text{for} \quad n = 2, 3, \ldots$$

and $0 < \alpha < 1$, for $n = 0$, $n = 1$ (the same result is obtained for $E(x) = b_0(x) > 0$).

Remark 3. If the inclusion bending rigidity varies by the law

$$D(x) = (1-x^2)^{m+\frac{1}{2}} b_1(x), \tag{12}$$

where $m \geq 0$ is integer, then from the above asymptotic analysis we obtain:

$$\beta = m - \frac{3}{2}, \quad \text{for} \quad m = 4, 5, \ldots$$

and $0 < \beta < 2$, for $m = 0, 1, 2, 3$ (the same result is obtained for $D(x = b_1(x) > 0)$).

Based on the above asymptotic analysis (see. (11), (12)) in cases

$$n = 0; 1, \quad E(x) = b_0(x) > 0,$$
$$m = 0; 1; 2; 3, \quad D(x) = b_1(x) > 0, \quad |x| \leq 1$$

the solution of system of equations (5), (6) we will be sought in the form

$$\varphi'(x) = \sqrt{1-x^2}\sum_{k=1}^{\infty} X_k P_k^{(1/2,1/2\ )}(x),$$

$$\psi''(x) = (1-x^2)^{3/2}\sum_{k=1}^{\infty} \frac{Y_k}{k+1} P_k^{(3/2,3/2)}(x)$$

and for the case $n=2$, $m=4$, the solution of this system will be present in follows

$$\varphi'(x) = (1-x^2)^{3/2}\sum_{k=1}^{\infty} X_k P_k^{(3/2,3/2)}(x), \quad \psi'(x) = (1-x^2)^{5/2}\sum_{k=1}^{\infty} \frac{Y_k}{k+1} P_k^{(3/2,3/2)}(x),$$

where the numbers X_k, Y_k are subject to determination. $P_k^{(\alpha,\beta)}(z)$ are Jacob's orthogonal polynomials, $(k=1,2,\dots)$ [3].

For determination of the numbers X_k, Y_k the pair of infinite systems of linear algebraic equations is obtained and quasi-completely regularity of this system is proved in the class of bounded sequences for any positive values of the parameters k_0, m_0, a, b [4].

In some specific cases the system of equations (5), (6) ((5′), (6′)) is splitting and the integro-differential equations of the following types are obtained:

$$\frac{\varphi(x)}{E_0} + b\int_0^{\infty} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) = 0, \quad x>0 \tag{13}$$

$$\varphi(0) = T, \quad \varphi(\infty) = 0$$

or

$$b\int_0^{\infty} \frac{p(t)dt}{t-x} = m_0 p'(x), \quad x>0, \tag{14}$$

$$\int_0^{\infty} p(t)dt = P_0.$$

The solution of equation (13) or (14) is sought in the class of functions φ, $\varphi' \in H([0,\infty))$, $\varphi'' \in H((0,\infty))$, $p \in H([0,\infty)))$, $p' \in H((0,\infty))$, $p(x) = O(x^{-(2+\omega)})$, $x\to\infty$, $\omega>0$.

$E(x) = E_0 = \text{const}$, T and P_0 are known constants.

Based to the different boundary value problems of the theory of analytic functions the equations (13), (14) are solved effectively and asymptotic estimates are obtained [5].

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