ISSN 0012-2661, Differential Equations, 2007, Vol. 43, No. 3, pp. 1–15. © Pleiades Publishing, Ltd., 2007.
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PARTIAL DIFFERENTIAL EQUATIONS

On the Existence or Absence of Global Solutions for the Multidimensional Version of the Second Darboux Problem for Some Nonlinear Hyperbolic Equations

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Received August 25, 2005

DOI: 10.1134/S001226610703010X

1. STATEMENT OF THE PROBLEM

Consider the nonlinear wave equation of the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u + mu = f(u) + F, \tag{1}$$

where f and F are given real functions, f is nonlinear, and u is the unknown real function; $m = \text{const} \geq 0$, $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$, and $n \geq 2$.

Let D be a conical domain in the space R^{n+1} of the variables $x = (x_1, \ldots, x_n)$ and t; i.e., if D contains a point (x, t), then it contains the entire ray $\ell\colon (\tau x, \tau t)$, $0 < \tau < \infty$. By S we denote the cone ∂D. We assume that the domain D is homeomorphic to the conical domain ω: $t > |x|$ and $S \backslash O$ is a connected n-dimensional manifold of the class C^∞, where $O = (0, \ldots, 0, 0)$ is the vertex of the cone S. We also assume that the domain D lies in the half-space $t > 0$ and set

$$D_T = \{(x,t) \in D : \ t < T\}, \qquad S_T = \{(x,t) \in S : \ t \leq T\}, \qquad T > 0.$$

If $T = \infty$, then, obviously, $D_\infty = D$ and $S_\infty = S$.

Consider the following problem: find a solution $u(x, t)$ of Eq. (1) in the domain D_T with the boundary condition

$$u|_{S_T} = g, \tag{2}$$

where g is a given real-valued function on S_T.

If the cone $S = \partial D$ is timelike and is the graph of a function of the variables $x_1, \ldots, x_n$, i.e., if

$$\left.\left(\nu_0^2 - \sum_{i=1}^{n} \nu_i^2\right)\right|_S < 0, \qquad \nu_0|_S < 0, \tag{3}$$

where $\nu = (\nu_1, \ldots, \nu_n, \nu_0)$ is the unit outward normal to $S \backslash 0$, then problem (1), (2) is a multidimensional version of the second Darboux problem [1, pp. 228, 233] for the nonlinear equation (1).

In what follows, we assume that condition (3) is satisfied.

The existence or absence of a global solution of the Cauchy problem for semilinear equations of the form (1) with initial conditions $u|_{t=0} = u_0$ and $\partial u/\partial t|_{t=0} = u_1$ was studied in [2–7]. As to multidimensional variants of the second Darboux problem for linear equations of order ≥ 2, they are well-posed and globally solvable in appropriate function spaces [18–20].

In the present paper, we single out special cases of the nonlinear function $f = f(u)$; problem (1), (2) is globally solvable in some of these cases and is not globally solvable in the other cases.

2. GLOBAL SOLVABILITY OF THE PROBLEM

Consider the case in which $f(u) = -\lambda|u|^p u$, where $\lambda \neq 0$ and $p > 0$ are given real numbers. Then Eq. (1) acquires the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u + mu = -\lambda|u|^p u + F. \tag{4}$$

This equation arises in relativistic quantum mechanics [21–24].

Let us restrict our considerations to the case in which condition (2) is homogeneous, i.e.,

$$u|_{S_T} = 0. \tag{5}$$

We set $\overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) : \ u|_{S_T} = 0\}$, where $W_2^1(D_T)$ is the well-known Sobolev space [25, p. 56].

Remark 1. The embedding $I : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < 2(n+1)/(n-1)$ provided that $n > 1$ [25, p. 81]. At the same time, the Nemytskii operator $K : L_q(D_T) \to L_2(D_T)$ acting by the formula $Ku := -\lambda|u|^p u$ is continuous and bounded if $q \geq 2(p+1)$ [26, p. 349; 27, pp. 66–67 of the Russian translation]. Therefore, if $p < 2/(n-1)$, i.e., $2(p+1) < 2(n+1)/(n-1)$, then there exists a number q such that $1 < 2(p+1) \leq q < 2(n+1)/(n-1)$ and hence the operator

$$K_0 = KI : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T) \tag{6}$$

is continuous and compact. The inclusion $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ implies that so much the more $u \in L_{p+1}(D_T)$. As was mentioned above, we everywhere assume that $p > 0$.

Definition 1. Let $F \in L_2(D_T)$ and $0 < p < 2/(n-1)$. A function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is called a *strong generalized solution* of the nonlinear problem (4), (5) in the domain D_T if there exists a sequence $u_k \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T) := \{u \in C^2(\bar{D}_T) : \ u|_{S_T} = 0\}$ of functions such that $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ and $[Lu_k + \lambda|u_k|^p u_k] \to F$ in the space $L_2(D_T)$. The convergence of the sequence $\{\lambda|u_k|^p u_k\}$ to the function $\lambda|u|^p u$ in the space $L_2(D_T)$ under the condition that $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows from Remark 1. Note that since $|u|^{p+1} \in L_2(D_T)$ and the domain D_T is bounded, we so much the more have $u \in L_{p+1}(D_T)$.

Definition 2. Let $0 < p < 2/(n-1)$, $F \in L_{2,\mathrm{loc}}(D)$, and $F \in L_2(D_T)$ for every $T > 0$. We say that problem (4), (5) is *globally solvable* if for each $T > 0$ it has a strong generalized solution in the domain D_T in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

Lemma 1. *Let $\lambda > 0$, $0 < p < 2/(n-1)$, and $F \in L_2(D_T)$. Then each strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of problem (4), (5) in the domain D_T admits the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{e/2}\, T \|F\|_{L_2(D_T)}. \tag{7}$$

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ be a strong generalized solution of problem (4), (5). By Definition 1, there exists a sequence $u_k \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ of functions such that

$$\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \qquad \lim_{k\to\infty} \|Lu_k + \lambda|u_k|^p u_k - F\|_{L_2(D_T)} = 0. \tag{8}$$

Consider the function $u_k \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ defined as the solution of the problem

$$Lu_k + \lambda|u_k|^p u_k = F_k, \tag{9}$$

$$u_k|_{S_T} = 0;(10)$$

here

$$F_k = Lu_k + \lambda \left|u_k\right|^p u_k. \tag{11}$$

By multiplying both sides of Eq. (9) by $\partial u_k/\partial t$ and by integrating over the domain

$$D_\tau = \{(x,t) \in D:\ t < \tau\}, \qquad 0 < \tau \le T,$$

we obtain

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt - \int\limits_{D_\tau} \Delta u_k \frac{\partial u_k}{\partial t} dx\,dt + \frac{m}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t} u_k^2 dx\,dt + \frac{\lambda}{p+2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\left|u_k\right|^{p+2} dx\,dt = \int\limits_{D_\tau} F_k \frac{\partial u_k}{\partial t} dx\,dt. \tag{12}$$

We set $\Omega_\tau := D \cap \{t = \tau\}$. Obviously, $\Omega_\tau = D_\tau \cap \{t = \tau\}$ for $0 < \tau < T$. Then, by using (10) and the argument in [25, pp. 202–203] and by integrating the left-hand side of (12) by parts, we obtain

$$\int\limits_{D_\tau} F_k \frac{\partial u_k}{\partial t} dx\,dt = \int\limits_{S_\tau} \frac{1}{2\nu_0}\left[\sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\nu_0 - \frac{\partial u_k}{\partial t}\nu_i\right)^2 + \left(\frac{\partial u_k}{\partial t}\right)^2\left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\right] ds + \frac{1}{2}\int\limits_{\Omega_\tau}\left[m u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2\right] dx + \frac{\lambda}{p+2}\int\limits_{\Omega_\tau} \left|u_k\right|^{p+2} dx, \tag{13}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit outward normal on ∂D_τ.

Since $\left(\nu_0 \dfrac{\partial}{\partial x_i} - \nu_i \dfrac{\partial}{\partial t}\right)$, $i = 1, \dots, n$, is an intrinsic differential operator on S_T, it follows from (10) that

$$\left.\left(\frac{\partial u_k}{\partial x_i}\nu_0 - \frac{\partial u_k}{\partial t}\nu_i\right)\right|_{S_\tau} = 0, \qquad i = 1, \dots, n. \tag{14}$$

By using (3) and (14), from (13), we obtain the inequality

$$\int\limits_{\Omega_\tau}\left[m u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2\right] dx \le 2\int\limits_{D_\tau} F_k \frac{\partial u_k}{\partial t} dx\,dt. \tag{15}$$

By using the notation

$$w(\delta) = \int\limits_{\Omega_\delta}\left[m u_k^2 + (\partial u_k/\partial t)^2 + \sum_{i=1}^n (\partial u_k/\partial x_i)^2\right] dx$$

and by taking into account the inequality

$$2F_k \frac{\partial u_k}{\partial t} \le \varepsilon \left(\frac{\partial u_k}{\partial t}\right)^2 + \frac{1}{\varepsilon} F_k^2,$$

which is valid for every $\varepsilon = \text{const} > 0$, from (15), we obtain the inequality

$$w(\delta) \le \varepsilon \int\limits_0^\delta w(\sigma) d\sigma + \frac{1}{\varepsilon}\left\|F_k\right\|^2_{L_2(D_\delta)}, \qquad 0 < \delta \le T. \tag{16}$$

Since the quantity $\|F_k\|^2_{L_2(D_\delta)}$ is a nondecreasing function of δ, it follows from (16) and the Gronwall lemma [28, p. 13 of the Russian translation] that

$$w(\delta) \le \frac{1}{\varepsilon}\|F_k\|^2_{L_2(D_\delta)} \exp \delta\varepsilon.$$

This, together with the relation $\inf_{\varepsilon>0} \dfrac{\exp \delta\varepsilon}{\varepsilon} = e\delta$ attained for $\varepsilon = 1/\delta$, implies the inequality

$$w(\delta) \le e\delta \|F_k\|^2_{L_2(D_\delta)}, \qquad 0 < \delta \le T. \tag{17}$$

In turn, it follows from (17) that

$$\|u_k\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = \int\limits_{D_T}\left[mu_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx\,dt = \int\limits_0^T w(\delta)d\delta \le \frac{e}{2}T^2\|F_k\|^2_{L_2(D_T)}. \tag{18}$$

Here we have used the fact that one of the equivalent norms in the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$ is given by the expression

$$\left\{\int\limits_{D_T}\left[mu^2 + (\partial u/\partial t)^2 + \sum_{i=1}^n (\partial u/\partial x_i)^2\right]dx\,dt\right\}^{1/2}$$

regardless of whether $m = 0$ or $m > 0$. Indeed, it follows by a standard argument from the relations $u|_{S_T} = 0$ and

$$u(x,t) = \int\limits_{\varphi(x)}^t \frac{\partial u(x,\tau)}{\partial t}d\tau, \qquad (x,t) \in \bar{D}_T,$$

where $t - \varphi(x) = 0$ is the equation of the cone S, that the following inequality holds [25, p. 63]:

$$\int\limits_{D_\tau} u^2(x,t)dx\,dt \le T^2 \int\limits_{D_\tau}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt.$$

By using (8) and (11) and by passing to the limit as $k \to \infty$ in (8), we obtain the estimate (7), which completes the proof of the lemma.

Theorem 1. *Let* $\lambda > 0$, $0 < p < 2/(n-1)$, $F \in L_{2,\mathrm{loc}}(D)$, *and* $F \in L_2(D_T)$ *for each* $T > 0$. *Then problem* (4), (5) *is globally solvable*; *i.e., for each* $T > 0$, *this problem has a strong generalized solution* $u \in \overset{\circ}{W}{}^1_2(D_T,S_T)$ *in the domain* D_T.

Proof. First, in the form needed by us, we study the solvability of the linear problem corresponding to (4), (5) for the case in which $\lambda = 0$ in (4), i.e., for the problem

$$Lu(x,t) = F(x,t), \qquad (x,t) \in D_T, \qquad u(x,t) = 0, \qquad (x,t) \in S_T. \tag{19}$$

In this case, if $F \in L_2(D_T)$, then, in a similar way, one can introduce the notion of a strong generalized solution $u \in \overset{\circ}{W}{}^1_2(D_T,S_T)$ of problem (19) for which there exists a sequence $u_k \in \overset{\circ}{C}{}^2(\bar{D}_T,S_T)$ of functions such that

$$\lim_{k\to\infty}\|u_k - u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0, \qquad \lim_{k\to\infty}\|Lu_k - F\|_{L_2(D_T)} = 0.$$

It follows from the proof of Lemma 1 that the a priori estimate (7) is also valid for a strong generalized solution of problem (19).

We introduce the weighted Sobolev space $W^k_{2,\alpha}(D)$, $0<\alpha<\infty$, $k=1,2,\dots$, of functions of the class $W^k_{2,\mathrm{loc}}(D)$ with finite norm

$$\|u\|^2_{W^k_{2,\alpha}(D)} = \sum_{i=0}^{k} \int_D r^{-2\alpha-2(k-i)} \left|\frac{\partial^i u}{\partial x^{i'}\partial t^{i_0}}\right|^2 dx\,dt,$$

where

$$r=\left(\sum_{j=1}^{n} x_j^2+t^2\right)^{1/2}, \qquad \frac{\partial^i u}{\partial x^{i'}\partial t^{i_0}}=\frac{\partial^i u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}\partial t^{i_0}}, \qquad i=i_1+\cdots+i_n+i_0.$$

We set $\mathring{W}^k_{2,\alpha}(D,S)=\left\{u\in W^k_{2,\alpha}(D):\ u|_S=0\right\}$. Along with problem (19) in the domain D_T, we consider a similar problem in the infinite cone D in the following setting:

$$\tilde{L}u(x,t)=F(x,t), \qquad (x,t)\in D, \qquad u(x,t)=0, \qquad (x,t)\in S. \tag{20}$$

Here $\tilde{L}u:=\partial^2u/\partial t^2-\Delta u+\tilde{m}u$, and the coefficient $\tilde{m}=\tilde{m}(x,t)$ has the properties

$$\tilde{m}\in C^\infty\left(\bar{D}\right), \qquad \tilde{m}|_{D_T}=m, \qquad \operatorname{diam}\operatorname{supp}\tilde{m}<+\infty. \tag{21}$$

The existence of a function $\tilde{m}$ with the above-mentioned properties is obvious. If $m=0$, then, obviously, we set $\tilde{m}\equiv 0$.

By virtue of inequality (3), which, by [20, p. 114], is valid for the equation $\tilde{L}u=F$, there exists a constant $\alpha_0=\alpha_0(k)>1$ such that if $\alpha\ge\alpha_0$, then problem (20) has a unique solution $u\in\mathring{W}^k_{2,\alpha}(D,S)$ for each function $F\in W^{k-1}_{\alpha-1}(D)$.

Since the space $C_0^\infty(D_T)$ of compactly supported and infinitely differentiable functions in D_T is dense in $L_2(D_T)$, it follows that for a given function $F\in L_2(D_T)$ there exists a sequence of functions $F_\ell\in C_0^\infty(D_T)$ such that $\lim_{\ell\to\infty}\|F_\ell-F\|_{L_2(D_T)}=0$. We fix ℓ, continue the function F_ℓ by zero outside D_T, and keep the same notation for the resulting function; then $F_\ell\in C_0^\infty(D)$. Obviously, $F_\ell\in W^{k-1}_{\alpha-1}(D)$ for any $k\ge 1$ and $\alpha>1$ and hence for $\alpha\ge\alpha_0=\alpha_0(k)$. By virtue of preceding considerations, there exists a solution $\tilde{u}_\ell\in\mathring{W}^k_{2,\alpha}(D,S)$ of problem (20) for $F=F_\ell$. By virtue of (21), the function $u_\ell=\tilde{u}_\ell|_{D_T}$ is a solution of problem (19) for $F=F_\ell$; i.e., $Lu_\ell=F_\ell$ and $u_\ell|_{S_T}=0$. Since $u_\ell\in\mathring{W}^k_2(D_T,S_T)=\left\{u\in W^k_2(D_T):\ u|_{S_T}=0\right\}$, it follows from the embedding theorem [25, p. 84] that $u_\ell\in\mathring{C}^2\left(\bar{D}_T,S_T\right)$ for sufficiently large k, namely, for $k>(n+1)/2+2$. Since the a priori estimate (7) is also valid for a strong generalized solution of problem (19), we have

$$\|u_\ell-u_{\ell'}\|_{\mathring{W}^1_2(D_T,S_T)}\le\sqrt{e/2}T\,\|F_\ell-F_{\ell'}\|_{L_2(D_T)}. \tag{22}$$

Since $\{F_\ell\}$ is a Cauchy sequence in $L_2(D_T)$, it follows from (22) that $\{u_\ell\}$ is a Cauchy sequence in $\mathring{W}^1_2\left(\bar{D}_T,S_T\right)$. Since the space $\mathring{W}^1_2\left(\bar{D}_T,S_T\right)$ is complete, it follows that there exists a function $u\in\mathring{W}^1_2(D_T,S_T)$ such that

$$\lim_{\ell\to\infty}\|u_\ell-u\|_{\mathring{W}^1_2(D_T,S_T)}=0,$$

and since $Lu_\ell=F_\ell\to F$ in the space $L_2(D_T)$, we find that this function, by definition, is a strong generalized solution of problem (19). The uniqueness of this solution in the space $\mathring{W}^1_2(D_T,S_T)$ follows from the a priori estimate (7). Consequently, for the solution u of problem (19), we can write out $u=L^{-1}F$, where $L^{-1}:L_2(D_T)\to\mathring{W}^1_2(D_T,S_T)$ is a linear continuous operator whose norm, by (7), can be estimated as

$$\left\|L^{-1}\right\|_{L_2(D_T)\to\mathring{W}^1_2(D_T,S_T)}\le\sqrt{e/2}\,T. \tag{23}$$

Note that if $F \in L_2(D_T)$, $0 < p < 2/(n-1)$, then, by virtue of (23) and Remark 1, for the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ to be a strong generalized solution of problem (4), (5) it is necessary and sufficient that u is a solution of the functional equation

$$u = L^{-1}\left(-\lambda|u|^p u + F\right) \tag{24}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

We rewrite Eq. (24) in the form

$$u = Au := L^{-1}\left(K_0 u + F\right), \tag{25}$$

where, by Remark 1, the operator K_0 in (6) is a continuous compact operator. Consequently, by virtue of the estimate (23), $A : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is also a continuous compact operator. At the same time, by Lemma 1, the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c\|F\|_{L_2(D_T)}$ with a positive constant c independent of u, τ, and F is valid for any parameter value $\tau \in [0,1]$ and for any solution of the parametric equation $u = \tau A u$. Therefore, by the Leray–Schauder theorem [29, p. 375], Eq. (25) and hence problem (4), (5) have at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$. The proof of the theorem is complete.

3. ABSENCE OF THE GLOBAL SOLVABILITY OF THE PROBLEM

Below we restrict our consideration of Eq. (4) to the case in which $\lambda < 0$ and the spatial dimension is $n = 2$. To simplify the argument, we assume that $m = 0$ and

$$S : \ t = k_0|x|, \qquad k_0 = \text{const} > 1. \tag{26}$$

Obviously, condition (3) is valid for the cone S given by (26). In this case, we have

$$D_T = \{(x,t) \in R^3 : \ k|x| < t < T\}.$$

For $(x^0, t^0) \in D_T$, we introduce the domain $D_{x^0,t^0} = \{(x,t) \in R^3 : \ k|x| < t < t^0 - |x - x^0|\}$, which is bounded below by the cone S and above by the past light cone $S^-_{x^0,t^0} : \ t = t^0 - |x - x^0|$ with vertex (x^0, t^0).

The following assertion is valid for any $n \ge 2$.

Lemma 2. *Let $F \in C\left(\bar{D}_T\right)$, and let $u \in C^2\left(\bar{D}_T\right)$ be a classical solution of problem* (4), (5)*. If $F|_{D_{x^0,t^0}} = 0$ for some point $(x^0, t^0) \in D_T$, then $u|_{D_{x^0,t^0}} = 0$.*

Proof. Since the proof of this lemma reproduces, in a sense, the proof of Lemma 1, we only outline key points of the proof.

We set

$$D_{x^0,t^0,\tau} := D_{x^0,t^0} \cap \{t < \tau\}, \qquad \Omega_{x^0,t^0,\tau} := D_{x^0,t^0} \cap \{t = \tau\}, \qquad 0 < t < \tau.$$

Then $\partial D_{x^0,t^0,\tau} = S_{1,\tau} \cup S_{2,\tau} \cup S_{3,\tau}$, where $S_{1,\tau} = \partial D_{x^0,t^0,\tau} \cap S$, $S_{2,\tau} = \partial D_{x^0,t^0,\tau} \cap S^-_{x^0,t^0}$, and $S_{3,\tau} = \partial D_{x^0,t^0,\tau} \cap \bar{\Omega}_{x^0,t^0,\tau}$. Just as in the derivation of (13), by multiplying both sides of Eq. (4) by $\partial u/\partial t$ and by integrating the resulting relation over the domain $D_{x^0,t^0,\tau}$, $0 < \tau < t^0$, in view of (4) and the relation $F|_{D_{x^0,t^0}} = 0$, we obtain

$$0 = \int\limits_{S_{1,\tau}\cup S_{2,\tau}} \frac{1}{2\nu_0}\left[\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\nu_0 - \frac{\partial u}{\partial t}\nu_i\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\right] ds$$
$$+ \int\limits_{S_{2,\tau}\cup S_{3,\tau}} \frac{\lambda}{p+2}|u|^{p+2}\nu_0 ds + \int\limits_{S_{3,\tau}} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2\right] dx. \tag{27}$$

By virtue of (3), (5), and the relation

$$\left(\nu_0^2-\sum_{i=1}^n \nu_i^2\right)\Bigg|_{S_{1,\tau}}<0, \qquad \nu_0|_{S_{1,\tau}}<0, \qquad \left(\nu_0^2-\sum_{i=1}^n \nu_i^2\right)\Bigg|_{S_{2,\tau}}=0, \qquad \nu_0|_{S_{2,\tau}}=\frac{1}{\sqrt{2}}>0,$$

$$\left(\frac{\partial u}{\partial x_i}\nu_0-\frac{\partial u}{\partial t}\nu_i\right)\Bigg|_{S_{1,\tau}}=0, \qquad \left(\frac{\partial u}{\partial x_i}\nu_0-\frac{\partial u}{\partial t}\nu_i\right)^2\Bigg|_{S_{2,\tau}}\geq 0, \qquad i=1,\ldots,n,$$

we have the inequality

$$\int\limits_{S_{1,\tau}\cup S_{2,\tau}} \frac{1}{2\nu_0}\left[\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\nu_0-\frac{\partial u}{\partial t}\nu_i\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\left(\nu_0^2-\sum_{j=1}^n \nu_j^2\right)\right]ds\geq 0, \tag{28}$$

which, together with (27), implies that

$$\int\limits_{S_{3,\tau}}\left[\left(\frac{\partial u}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2\right]dx\leq M\int\limits_{S_{2,\tau}\cup S_{3,\tau}} u^2ds, \qquad 0<\tau<t^0. \tag{29}$$

Here, by virtue of the inclusion $u\in C^2\left(\bar D_T\right)$, $|\nu_0|\leq 1$, there exists a nonnegative constant M independent of the parameter τ, which can be taken in the form

$$M=\frac{|\lambda|}{p+2}\|u\|^p_{C(\bar D_T)}<+\infty. \tag{30}$$

Since $u|_{S_T}=0$, it follows from (26) that

$$u(x,t)=\int\limits_{k_0|x|}^{t}\frac{\partial u(x,\sigma)}{\partial t}d\sigma, \qquad (x,t)\in S_{2,\tau}\cup S_{3,\tau}, \tag{31}$$

which, after standard considerations, implies the inequality [25, p. 63]

$$\int\limits_{S_{2,\tau}\cup S_{3,\tau}} u^2ds\leq 2t^0\int\limits_{D_{x^0,t^0,\tau}}\left(\frac{\partial u}{\partial t}\right)^2dx, \qquad 0<\tau<t^0. \tag{32}$$

By setting $w(\tau)=\int_{S_{3,\tau}}\left[(\partial u/\partial t)^2+\sum_{i=1}^n\left(\partial u/\partial x_i\right)^2\right]dx$, from (29) and (32), one can readily obtain

$$w(\tau)\leq 2t^0M\int\limits_0^\tau w(\delta)d\delta, \qquad 0<\tau<t^0.$$

This, together with (30) and the Gronwall lemma, readily implies that $w(\tau)=0$, $0<\tau<t^0$, and hence $\partial u/\partial t=\partial u/\partial x_1=\cdots=\partial u/\partial x_n=0$ in the domain D_{x^0,t^0}. Therefore, $u|_{D_{x^0,t^0}}=\text{const}$, and, by using the homogeneous boundary condition (5), we finally obtain $u|_{D_{x^0,t^0}}=0$. The proof of the lemma is complete.

Let G_a: $t>|x|+a$ be the future light cone with vertex $(0,0,a)$, where $a=\text{const}>0$. Then, by (26), obviously, $D\backslash G_a=\{(x,t)\in R^3:\ k_0|x|<t<|x|+a,\ |x|<a/\left(k_0-1\right)\}$; moreover,

$$D\backslash\bar G_a\subset\left\{(x,t)\in R^3:\ 0<t<b\right\}, \qquad b=\frac{ak_0}{k_0-1}. \tag{33}$$

One can readily see that $D_T\backslash\bar G_a=D\backslash\bar G_a$ for $T>b=ak_0/\left(k_0-1\right)$.

Lemma 3. *Let* $n = 2$, $\lambda < 0$, $F \in C\left(\bar{D}_T\right)$, $T \geq b = ak_0/(k_0 - 1)$, $\operatorname{supp} F \subset \bar{G}_a$, *and* $F \geq 0$. *If* $u \in C^2\left(\bar{D}_T\right)$ *is a classical solution of problem* (4), (5), *then* $u|_{D_b} \geq 0$.

Proof. First, let us show that $u|_{D_T \setminus \bar{G}_a} = 0$. Indeed, let $(x^0, t^0) \in D_T \setminus \bar{G}_a$. Since $\operatorname{supp} F \subset \bar{G}_a$, we have $F|_{D_{x^0,t^0}} = 0$, and, by Lemma 2, $u|_{D_{x^0,t^0}} = 0$. Therefore, by using (33), by continuing the functions u and F by zero outside D_b in the strip $\Sigma_b = \{(x,t) \in R^3 : \ 0 < t < b\}$, and by using the same notation for the resulting functions, we find that $u \in C^2\left(\bar{\Sigma}_b\right)$ is a classical solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = -\lambda |u|^p u + F, \qquad u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \tag{34}$$

in the strip Σ_b. It is known that a solution $u \in C^2\left(\bar{\Sigma}_b\right)$ of problem (34) admits the integral representation [30, pp. 213–216]

$$u(x,t) = -\frac{\lambda}{2\pi} \int\limits_{\Omega_{x,t}} \frac{|u|^p u}{\sqrt{(t-\tau)^2 + |x-\xi|^2}} d\xi \, d\tau + F_0(x,t), \qquad (x,t) \in \Sigma_b. \tag{35}$$

Here

$$F_0(x,t) = \frac{1}{2\pi} \int\limits_{\Omega_{x,t}} \frac{F(\xi,\tau)}{\sqrt{(t-\tau)^2 + |x-\xi|^2}} d\xi \, d\tau, \tag{36}$$

where $\Omega_{x,t} = \{(\xi,\tau) \in R^3 : \ |\xi - x| < t, \ 0 < \tau < t - |\xi - x|\}$ is a circular cone with vertex (x,t) and with base in the form of the disk d: $|\xi - x| < t$, $\tau = 0$ in the plane $\tau = 0$ of the variables ξ_1 and ξ_2, $\xi = (\xi_1, \xi_2)$.

Let $(x^0, t^0) \in D_b$ and $\tilde{\psi}_0 = \tilde{\psi}_0(x,t) \in C\left(\bar{\Omega}_{x^0,t^0}\right)$. Then the linear operator $\Psi : C\left(\bar{\Omega}_{x^0,t^0}\right) \to C\left(\bar{\Omega}_{x^0,t^0}\right)$ acting by the formula

$$\Psi v(x,t) = \frac{1}{2\pi} \int\limits_{\Omega_{x,t}} \frac{\tilde{\psi}_0(\xi,\tau) v(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi \, d\tau, \qquad (x,t) \in \bar{\Omega}_{x^0,t^0},$$

is continuous, and its norm can be estimated as [30, p. 215]

$$\|\Psi\|_{C(\bar{\Omega}_{x^0,t^0}) \to C(\bar{\Omega}_{x^0,t^0})} \leq \frac{\left(t^0\right)^2}{2} \left\|\tilde{\psi}_0\right\|_{C(\bar{\Omega}_{x^0,t^0})} \leq \frac{T^2}{2} \left\|\tilde{\psi}_0\right\|_{C(\bar{\Omega}_{x^0,t^0})}.$$

Consider the integral equation

$$v(x,t) = \int\limits_{\Omega_{x,t}} \frac{\psi_0(\xi,\tau) v(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi \, d\tau + F_0(x,t), \qquad (x,t) \in \bar{\Omega}_{x^0,t^0}, \tag{37}$$

for the unknown function v. Here

$$\psi_0(\xi,\tau) = -\frac{\lambda}{2\pi} |u(\xi,\tau)|^p \in C\left(\bar{\Omega}_{x^0,t^0}\right), \tag{38}$$

where u is the classical solution of problem (4), (5) occurring in Lemma 3. Since ψ_0, $F_0 \in C\left(\bar{\Omega}_{x^0,t^0}\right)$; and the operator occurring on the right-hand side in (37) is a Volterra type integral equation (with respect to the variable t) with a weak singularity, it follows that Eq. (37) is uniquely solvable in the space $C\left(\bar{\Omega}_{x^0,t^0}\right)$. In this case, a solution v of Eq. (37) can be obtained by the method of Picard sequential approximations:

$$v_0 = 0, \qquad v_{k+1}(x,t) = \int\limits_{\Omega_{x,t}} \frac{\psi_0(\xi,\tau)v_k(\xi,\tau)}{\sqrt{(t-\tau)^2-|x-\xi|^2}} d\xi\, d\tau + F_0(x,t), \qquad k = 1,2,\dots \tag{39}$$

Indeed, let $\omega_\tau = \Omega_{x^0,t^0} \cap \{t=\tau\}$, $w_m|_{\bar{\Omega}_{x^0,t^0}} = v_{m+1} - v_m$ $(w_0|_{\bar{\Omega}_{x^0,t^0}} = F_0)$, $\lambda_m(t) = \max_{x\in\bar{\omega}_t} |w_m(x,t)|$, $m = 0,1,\dots$;

$$\delta = \int\limits_{|\eta|<1} \left(1-|\eta|^2\right)^{-1/2} d\eta_1 d\eta_2 \,\|\psi_0\|_{C(\bar{\Omega}_{x^0,t^0})} = 2\pi\, \|\psi_0\|_{C(\bar{\Omega}_{x^0,t^0})}.$$

If $B_\beta\varphi(t) = \delta\int_0^t (t-\tau)^{\beta-1}\varphi(\tau)d\tau$, $\beta > 0$, then, by taking into account (39) and the relation [28, p. 206 of the Russian translation]

$$B_\beta^m\varphi(t) = \frac{1}{\Gamma(m\beta)}\int\limits_0^t (\delta\Gamma(\beta))^m (t-\tau)^{m\beta-1}\varphi(\tau)d\tau,$$

we obtain

$$|w_m(x,t)| = \left|\int\limits_{\Omega_{x,t}} \frac{\psi_0 w_{m-1}}{\sqrt{(t-\tau)^2-|x-\xi|^2}} d\xi\, d\tau\right| \le \int\limits_0^t d\tau \int\limits_{|x-\xi|<t-\tau} \frac{|\psi_0|\;|w_{m-1}|}{\sqrt{(t-\tau)^2-|x-\xi|^2}} d\xi$$

$$\le \|\psi_0\|_{C(\bar{\Omega}_{x^0,t^0})} \int\limits_0^t d\tau \int\limits_{|x-\xi|<t-\tau} \frac{\lambda_{m-1}(\tau)}{\sqrt{(t-\tau)^2-|x-\xi|^2}} d\xi$$

$$= \|\psi_0\|_{C(\bar{\Omega}_{x^0,t^0})} \int\limits_0^t (t-\tau)\lambda_{m-1}(\tau)d\tau \int\limits_{|\eta|<1} \frac{d\eta_1 d\eta_2}{\sqrt{1-|\eta|^2}} = B_2\lambda_{m-1}(t), \qquad (x,t)\in\Omega_{x^0,t^0}.$$

It follows that

$$\lambda_m(t) \le B_2\lambda_{m-1}(t) \le \dots \le B_2^m\lambda_0(t) = \frac{1}{\Gamma(2m)}\int\limits_0^t (\delta\Gamma(2))^m (t-\tau)^{2m-1}\lambda_0(\tau)d\tau$$

$$\le \frac{\delta^m}{\Gamma(2m)}\int\limits_0^t (t-\tau)^{2m-1}\,\|w_0\|_{C(\bar{\Omega}_{x^0,t^0})}\, d\tau = \frac{(\delta T^2)^m}{\Gamma(2m)\times 2m}\|F\|_{C(\bar{\Omega}_{x^0,t^0})}$$

$$= \frac{(\delta T^2)^m}{(2m)!}\,\|F_0\|_{C(\bar{\Omega}_{x^0,t^0})}$$

and hence

$$\|w_m\|_{C(\bar{\Omega}_{x^0,t^0})} = \|\lambda_m\|_{C([0,t^0])} \le \frac{(\delta T^2)^m}{(2m)!}\,\|F_0\|_{C(\bar{\Omega}_{x^0,t^0})}.$$

Therefore, the series $v = \lim_{m\to\infty} v_m = v_0 + \sum_{m=0}^{\infty} w_m$ is convergent in the class $C\left(\bar{\Omega}_{x^0,t^0}\right)$, and its sum is a solution of Eq. (37). In a similar way, one can show that the solution of Eq. (37) is unique in the space $C\left(\bar{\Omega}_{x^0,t^0}\right)$.

Since $\lambda < 0$, it follows from (38) that

$$\psi_0(\xi,\tau) = -(2\pi)^{-1}\lambda|u(\xi,\tau)|^p \ge 0,$$

and, by (36) $F_0(x,t) \geq 0$, since, by assumption, $F(x,t) \geq 0$. Therefore, the successive approximations v_k given by (39) are nonnegative; and since

$$\lim_{k\to\infty} \|v_k - v\|_{C(\bar{\Omega}_{x^0,t^0})} = 0,$$

we have $v \geq 0$ in the closed domain $\bar{\Omega}_{x^0,t^0}$. Now it remains to note that, by (35), (37), and (38), the function u is a solution of Eq. (37); and, by virtue of the unique solvability of this equation, $u = v \geq 0$ in $\bar{\Omega}_{x^0,t^0}$. Therefore, $u\left(x^0,t^0\right) \geq 0$ for any point $(x^0,t^0) \in D_b$, which completes the proof.

Let c_R and $\varphi_R(x)$ be the first eigenvalue and eigenfunction, respectively, of the Dirichlet problem in the disk $\omega_R:\ x_1^2 + x_2^2 < R^2$. Consequently,

$$(\Delta\varphi_R + c_R\varphi_R)|_{\omega_R} = 0, \qquad \varphi_R|_{\partial\omega_R} = 0. \tag{40}$$

It is known that $c_R > 0$, and, by changing the sign and by performing related normalization, one can possibly assume that [31, p. 25]

$$\varphi_R|_{\omega_R} > 0, \qquad \int\limits_{\omega_R} \varphi_R dx = 1. \tag{41}$$

Below we suppose that the assumptions of Lemma 3 are valid. As was shown in the proof of that lemma, by continuing the functions u and F by zero outside D_b in the strip $\Sigma_b = \{(x,t) \in R^3 :\ 0 < t < b\}$ and by using the same notation for the resulting function, we have found that $u \in C^2\left(\bar{\Sigma}_b\right)$ is a classical solution of the Cauchy problem (34) in the strip Σ_b.

Remark 2. Without loss of generality, in (4), one can assume that $\lambda = -1$, since, by virtue of the condition $p > 0$, the case in which $\lambda < 0$ and $\lambda \neq -1$ can reduced to the case in which $\lambda = -1$ by the reduction of the new unknown function $v = |\lambda|^{1/p}u$. Therefore, the function v satisfies the equation

$$v_{tt} - \Delta v = v^{p+1} + |\lambda|^{1/p}F(x,t), \qquad (x,t) \in \Sigma_b.$$

In accordance with this remark, instead of (34), we consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = u^{p+1} + F(x,t), \quad (x,t) \in \Sigma_b, \qquad u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{42}$$

where $u|_{\Sigma_b} \geq 0$ and $u \in C^2\left(\overline{\Sigma_b}\right)$. In this case, as was shown in the proof of Lemma 3,

$$u|_{\Sigma_b \setminus \bar{G}_a} = 0. \tag{43}$$

We choose $R \geq b > a/(k_0 - 1)$, where the number $a/(k_0 - 1)$ is the radius of the disk obtained as the intersection of the domain $D\colon\ t > k_0|x|$ with the plane $t = b$. We introduce the functions

$$E(t) = \int\limits_{\omega_R} u(x,t)\varphi_R(x)dx, \qquad f_R(t) = \int\limits_{\omega_R} F(x,t)\varphi_R(x)dx, \qquad 0 \leq t \leq b. \tag{44}$$

Since $u|_{\Sigma_b} \geq 0$, $u \in C^2\left(\overline{\Sigma_b}\right)$, and $F \in C\left(\overline{\Sigma_b}\right)$, we have $E \geq 0$, $E \in C^2([0,b])$, and $f_R \in C([0,b])$. By using (40), (43), and (44) and by integrating by parts, we obtain

$$\int\limits_{\omega_R} \Delta u\varphi_R dx = \int\limits_{\omega_R} u\Delta\varphi_R dx = -c_R \int\limits_{\omega_R} u\varphi_R dx = -c_R E. \tag{45}$$

Now, by using (41), the inequalities $p > 0$ and $u|_{\Sigma_b} \geq 0$, and the Jensen inequality [31, p. 26], we obtain

$$\int\limits_{\omega_R} u^{p+1}\varphi_R dx \geq \left(\int\limits_{\omega_R} u\varphi_R dx\right)^{p+1} = E^{p+1}. \tag{46}$$

It readily follows from (42)–(46) that

$$E'' + c_R E \geq E^{p+1} + f_R, \qquad 0 \leq t \leq b, \tag{47}$$
$$E(0) = 0, \qquad E'(0) = 0. \tag{48}$$

To study problem (47), (48), we use the method of test functions [14, pp. 10–12]. To this end, we choose b_1, $0 < b_1 < b$, and consider a nonnegative function $\psi \in C^2([0,b])$ such that

$$0 \leq \psi \leq 1, \qquad \psi(t) = 1, \qquad 0 \leq t \leq b, \qquad \psi^{(i)}(b) = 0, \qquad i = 0,1,2. \tag{49}$$

It follows from (47)–(49) that

$$\int\limits_0^b E^{p+1}(t)\psi(t)dt \leq \int\limits_0^b E(t)\left[\psi''(t) + c_R\psi(t)\right]dt - \int\limits_0^b f_R(t)\psi(t)dt. \tag{50}$$

If in the Young inequality

$$yz \leq \frac{\varepsilon}{\alpha} y^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} z^{\alpha'}, \qquad y, z \geq 0, \qquad \alpha' = \frac{\alpha}{\alpha - 1}$$

with parameter $\varepsilon > 0$ we take $\alpha = p+1$, $\alpha' = (p+1)/p$, $y = E\psi^{1/(p+1)}$, and $z = |\psi'' + c_R\psi| / \psi^{1/(p+1)}$ and use the relation $\alpha'/\alpha = 1/(\alpha - 1) = \alpha' - 1$, then we obtain

$$E\left|\psi'' + c_R\psi\right| = E\psi^{1/\alpha}\frac{|\psi'' + c_R\psi|}{\psi^{1/\alpha}} \leq \frac{\varepsilon}{\alpha}E^\alpha\psi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|\psi'' + c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}. \tag{51}$$

By virtue of (51), from (50), we have

$$\left(1 - \frac{\varepsilon}{\alpha}\right)\int\limits_0^b E^\alpha\psi\, dt \leq \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int\limits_0^b \frac{|\psi'' + c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}dt - \int\limits_0^b f_R(t)\psi(t)dt. \tag{52}$$

By using the relation $\inf_{0<\varepsilon<\alpha}\left[\frac{\alpha-1}{\alpha-\varepsilon}\frac{1}{\varepsilon^{\alpha'-1}}\right] = 1$, which is attained for $\varepsilon = 1$, and relation (52), from (49), we obtain

$$\int\limits_0^{b_1} E^\alpha dt \leq \int\limits_0^b \frac{|\psi'' + c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}dt - \alpha'\int\limits_0^b f_R(t)\psi(t)dt. \tag{53}$$

Now for the test function ψ, we take the function

$$\psi(t) = \psi_0(\tau), \qquad \tau = \frac{t}{b_1}, \qquad 0 \leq \tau \leq \tau_1 = \frac{b}{b_1}. \tag{54}$$

Here

$$\begin{gathered} \psi_0 \in C^2\left([0,\tau_1]\right), \qquad 0 \leq \psi_0 \leq 1, \\ \psi_0(\tau) = 1, \qquad 0 \leq \tau \leq 1, \qquad \psi_0^{(i)}\left(\tau_1\right) = 0, \qquad i = 0,1,2. \end{gathered} \tag{55}$$

One can readily see that

$$c_R = \frac{c_1}{R^2} \leq \frac{c_1}{b^2} \leq \frac{c_1}{b_1^2}, \qquad \varphi_R(x) = \frac{1}{R^2}\varphi_1\left(\frac{x}{R}\right). \tag{56}$$

Since $\psi''(t)=0$ for $0\le t\le b_1$ and $f_R\ge 0$ (because $F\ge 0$), it follows from (54)–(56), the well-known inequality $|y+z|^{\alpha'}\le 2^{\alpha'-1}\left(|y|^{\alpha'}+|z|^{\alpha'}\right)$, and in (53) that

$$\begin{aligned}\int\limits_0^{b_1} E^\alpha dt &\le \int\limits_0^{b_1}\frac{c_R^{\alpha'}\psi^{\alpha'}}{\psi^{\alpha'-1}}dt+\int\limits_{b_1}^{b}\frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}dt-\alpha'\int\limits_0^b f_R(t)\psi(t)dt\\ &\le c_R^{\alpha'}\int\limits_0^{b_1}\psi\,dt+b_1\int\limits_1^{\tau_1}\frac{\left|b_1^{-2}\psi_0''(\tau)+c_R\psi_0(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau-\alpha'\int\limits_0^{b_1}f_R(t)dt\\ &\le c_R^{\alpha'}b_1+\frac{2^{\alpha'-1}}{b_1^{2\alpha'-1}}\int\limits_1^{\tau_1}\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau+b_1\times 2^{\alpha'-1}c_R^{\alpha'}\int\limits_1^{\tau_1}\psi_0(\tau)d\tau-\alpha'\int\limits_0^{b_1}f_R(t)dt\\ &\le \frac{c_1^{\alpha'}}{b_1^{2\alpha'-1}}+\frac{2^{\alpha'-1}}{b_1^{2\alpha'-1}}\int\limits_1^{\tau_1}\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau+\frac{2^{\alpha'-1}c_1^{\alpha'}}{b_1^{2\alpha'-1}}(\tau_1-1)-\alpha'\int\limits_0^{b_1}f_R(t)dt.\end{aligned}\tag{57}$$

Now, by setting $R=b=ak_0/(k_0-1)$ and by choosing a number $\tau_1>1$ such that

$$b_1=\frac{b}{\tau_1}=a+2\frac{b-a}{3}=\frac{a+2b}{3}=\frac{a}{3}\left(\frac{3k_0-1}{k_0-1}\right),\tag{58}$$

from (57), we obtain

$$\int\limits_0^{b_1}E^\alpha dt\le b_1^{1-2\alpha'}\left[c_1^{\alpha'}\left(1+2^{\alpha'-1}(\tau_1-1)\right)+2^{\alpha'-1}\int\limits_1^{\tau_1}\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau-\alpha' b_1^{2\alpha'-1}\int\limits_0^{b_1}f_b(t)dt\right],\tag{59}$$

$2\alpha'-1=(p+2)/p.$

By [14, p. 11], the function ψ_0 with properties (55) such that the integral

$$d(\psi_0)=\int\limits_1^{\tau_1}\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau<+\infty\tag{60}$$

is finite exists.

By (44) and (56), we have

$$\begin{aligned}J(b)=\int\limits_0^{b_1}f_b(t)dt&=\int\limits_0^{b_1}dt\int\limits_{\omega_b}F(x,t)\varphi_b(x)dx=\int\limits_0^{b_1}dt\int\limits_{\omega_b}F(x,t)\frac{1}{b^2}\varphi_1\left(\frac{x}{b}\right)dx\\ &=\int\limits_0^{b_1}dt\int\limits_{\omega_1}F(b\xi,t)\varphi_1(\xi)d\xi.\end{aligned}\tag{61}$$

By virtue of (60), the quantity

$$\varkappa_0=\varkappa_0(c_1,\alpha',\psi_0)=\frac{\tau_1^{2\alpha'-1}}{\alpha'}\left[c_1^{\alpha'}\left(1+2^{\alpha'-1}(\tau_1-1)\right)+2^{\alpha'-1}d(\psi_0)\right]\tag{62}$$

is also finite.

The above-represented considerations imply the following assertion.

Theorem 2. *Let* $n = 2$, $m = 0$, $\lambda = -1$, $F \in C\left(\bar{D}\right)$, $F \geq 0$, *and* $\operatorname{supp} F \subset \bar{G}_a : t \geq |x| + a$, $a = \text{const} > 0$. *If*

$$b^{(p+2)/p} \int\limits_0^{b/\tau_1} dt \int\limits_{\omega_1} F(b\xi, t)\varphi_1(\xi)d\xi > \varkappa_0, \qquad b = \frac{ak_0}{k_0 - 1}, \qquad \tau_1 = \frac{3k_0}{3k_0 - 1}, \tag{63}$$

then for $T \geq b$ *problem* (4), (5) *cannot have a classical solution* $u \in C^2\left(\bar{D}_T\right)$ *in the domain* D_T.

Proof. Indeed, by virtue of (58) and (16)–(63), the right-hand side of inequality (59) is negative, which is impossible, since the left-hand side of this inequality is nonnegative. Therefore, if $T \geq b$, then problem (4), (5) cannot have a classical solution $u \in C^2\left(\bar{D}_T\right)$ in the domain D_T. The proof of the theorem is complete.

Remark 3. It follows from the proof of Theorem 2 that if its assumptions are valid; and problem (4), (5) has a solution $u \in C^2\left(\bar{D}_T\right)$ in the domain D_T, then the quantity T lies in the interval $(0, b)$, i.e., $0 < T < b = ak_0/(k_0 - 1)$.

If $\varepsilon = (b - a)/3 > 0$, then by

$$G_{a,\varepsilon} = \left\{(x,t) \in R^3 : \ |x| < \varepsilon/2, \ a + \varepsilon < t < b_1\right\}$$

we denote the cylinder lying in the domain $D_b \cap G_a$ together with its closure, where

$$G_a = \left\{(x,t) \in R^3 : \ t > |x| + a\right\}.$$

For fixed positive constants a and δ for a real number k, we introduce the function space

$$C_a^{\delta,k}\left(\bar{D}\right) = \left\{F \in C\left(\bar{D}\right) : \ F \geq 0, \ \operatorname{supp} F \subset \bar{G}_a, \ F|_{G_{a,\varepsilon}} \geq \delta b^{-k}\right\}, \tag{64}$$

where $b = ak_0/(k_0 - 1)$ and $\varepsilon = (b - a)/3$.

Corollary 1. *Let* $n = 2$, $m = 0$, $\lambda = -1$, *and* $F \in C_a^{\delta,k}\left(\bar{D}\right)$. *Then for* $k > (p-2)/2$, *there exists a positive number* $a_0 = a_0\left(\varkappa_0, p, k, \delta\right)$ *such that if* $a < a_0$, *then problem* (4), (5) *cannot have a classical solution* $u \in C^2\left(\bar{D}_T\right)$ *for* $T \geq b = ak_0/(k_0 - 1)$.

Indeed, if $(x,t) \in G_{a,\varepsilon}$ for $\varepsilon = (b - a)/3$, then, by (26), we have

$$\left|\frac{x}{b}\right| < \frac{\varepsilon}{2b} = \frac{b-a}{6b} = \frac{1}{6k_0} < 1. \tag{65}$$

Further, if we introduce the number

$$m_0 = \inf_{|\eta| < 1/(6k_0)} \varphi_1(\eta),$$

then, by using the fact that, by (41), $\varphi_1(x) > 0$ in the unit disk $\omega_1 : \ |x| < 1$, we obtain $m_0 > 0$. Therefore, by taking into account relations (64) and (65) and the inclusion $F \in C_a^{\delta,k}\left(\bar{D}\right)$, from (61) with $\varepsilon = (b - a)/3$, we obtain

$$\begin{aligned} J(b) &= \int\limits_0^{b_1} dt \int\limits_{\omega_b} F(x,t)\frac{1}{b^2}\varphi_1\left(\frac{x}{b}\right)dx \geq \frac{1}{b^2}\int\limits_{a+\varepsilon}^{b_1} dt \int\limits_{|x|<\varepsilon/2} F(x,t)\varphi_1\left(\frac{x}{b}\right)dx \\ &\geq \frac{m_0}{b^2}\int\limits_{G_{a,\varepsilon}} F(x,t)dx\,dt \geq \frac{m_0\delta}{b^2}b^{-k} = m_0\delta b^{-(k+2)}. \end{aligned} \tag{66}$$

By virtue of (61), (66), and the relation $b_1 = b/\tau_1$, we obtain

$$b^{(p+2)/2} \int\limits_0^{b/\tau_1} dt \int\limits_{\omega_1} F(b\xi, t)\varphi_1(\xi) d\xi = b^{(p+2)/2} J(b) \geq m_0 \delta b^{(p+2)/2-(k+2)}. \qquad (67)$$

Since, by assumption, $k > (p-2)/2$ and hence $(p+2)/2 - (k+2) < 0$ and the number $\varkappa_0$ occurring in (62) is independent of the quantity a and $b = ak_0/(k_0-1)$, it follows from (67) that there exists a positive number $a_0 = a_0(\varkappa_0, p, k, \delta)$ such that if $a < a_0$, then inequality (63) is valid. Therefore, by Theorem 2, problem (4), (5) cannot have a classical solution $u \in C^2(\bar{D}_T)$ for $T \geq b$.

Remark 4. It was assumed in Theorem 2 that $\lambda = -1$. By using Remark 2, we find that Theorem 2 with the quantity $\varkappa_0$ on the right-hand side of (63) replaced by $|\lambda|^{-1/p}\varkappa_0$ remains valid in the case in which $\lambda < 0$. Similarly, in Corollary 1 one can consider $\lambda < 0$ instead of $\lambda = -1$.

The following assertion can be proved in an even simpler way.

Corollary 2. *Let* $n = 2$, $m = 0$, $\lambda < 0$, $F = \mu F_0$, *where* $\mu = \text{const} > 0$, $F_0 \in C(\bar{D})$, $F_0 \geq 0$, $\operatorname{supp} F_0 \subset \bar{G}_a$, *and* $F_0|_{D_b} \not\equiv 0$. *There exists a positive number* μ_0 *such that if* $\mu > \mu_0$, *then problem* (4), (5) *cannot have a classical solution* $u \in C^2(\bar{D}_T)$ *for all* $T \geq b$.

ACKNOWLEDGMENTS

The work was financially supported by the INTAS (project no. 03-51-5007).

REFERENCES

1. Bitsadze, A.V., *Nekotorye klassy uravnenii v chastnykh proizvodnykh* (Some Classes of Partial Differential Equations), Moscow: Nauka, 1981.
2. Jörgens, K., *Math. Z.*, 1961, vol. 77, pp. 295–308.
3. Levis, H.A., *Trans. Amer. Math. Soc.*, 1974, vol. 192, pp. 1–21.
4. John, F., *Manuscripta Math.*, 1979, vol. 28, pp. 235–268.
5. John, F., *Comm. Pure Appl. Math.*, 1981, vol. 34, pp. 29–51.
6. John, F. and Klainerman, S., *Comm. Pure Appl. Math.*, 1984, vol. 37, pp. 443–455.
7. Kato, T., *Comm. Pure Appl. Math.*, 1980, vol. 33, pp. 501–505.
8. Ginibre, J., Soffer, A., and Velo, G., *J. Funct. Anal.*, 1982, vol. 110, pp. 96–130.
9. Strauss, W.A., *J. Funct. Anal.*, 1981, vol. 41, pp. 110–133.
10. Georgiev, V., Lindblad, H., and Sogge, C., *Amer. J. Math.*, 1977, vol. 119, pp. 1291–1319.
11. Sideris, T.G., *J. Differential Equations*, 1984, vol. 52, pp. 378–406.
12. Hörmander, L., *Lectures on Nonlinear Hyperbolic Differential Equations*, Berlin: Springer, 1997 (Math. and Appl.; V. 26).
13. Aassila, M., *Differential Integral Equations*, 2001, vol. 14, pp. 1301–1314.
14. Mitidieri, E. and Pokhozhaev, S.I., *Tr. Mat. Inst. Steklova, 2001, vol. 234, pp. 1–384.*
15. Belchev, E., Kepka, M., and Zhou, Z., *J. Funct. Anal.*, 2002, vol. 190, pp. 233–254.
16. Guedda, M., *Electron. J. Differ. Equ.*, 2002, vol. 2002, no. 26, pp. 1–13.
17. Keel, M., Smith, H.F., and Sogge, C.D., *J. Amer. Math. Soc.*, 2004, vol. 17, pp. 109–153.
18. Sobolev, S.L., *Mat. Sb.*, 1942, vol. 11(53), no. 3, pp. 155–200.
19. Kharibegashvili, S., *Georgian Math. J.*, 1993, vol. 1, no. 2, pp. 159–169.
20. Kharibegashvili, S., *Mem. Differential Equations Math. Phys.*, 1995, vol. 4, pp. 1–127.
21. Schiff, L.I., *Phys. Rev.*, 1951, vol. 84, pp. 1–9.
22. Segal, I.E., *Bull. Soc. Math. France*, 1963, vol. 91, pp. 129–135.
23. Lions, J.-L., *Quelques méthodes de résolution des problèmes aux limites nonlinéaires*, Paris: Dunod, 1969. Translated under the title *Nekotorye metody resheniya nelineinykh kraevykh zadach*, Moscow: Mir, 1972.

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24. Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. Vol. 2: Fourier Analysis, Self-Adjointness*, New York: Academic, 1975. Translated under the title *Metody sovremennoi matematicheskoi fiziki. T.2. Garmonicheskii analiz. Samosopryazhennost'*, Moscow: Mir, 1978.
25. Ladyzhenskaya, O.A., *Kraevye zadachi matematicheskoi fiziki* (Boundary Value Problems of Mathematical Physics), Moscow: Nauka, 1973.
26. Krasnosel'skii, M.A., Zabreiko, P.P., Pustyl'nik, E.I., and Sobolevskii, P.E., *Integral'nye operatory v prostranstvakh summiruemykh funktsii* (Integral Operators in Spaces of Integrable Functions), Moscow: Nauka, 1966.
27. Fučik, S. and Kufner, A., *Nonlinear Differential Equations*, Amsterdam: Elsevier Scientific Publishing Co., 1980. Translated under the title *Nelineinye differentsial'nye uravneniya*, Moscow: Nauka, 1988.
28. Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Heidelberg: Springer-Verlag, 1981. Translated under the title *Geometricheskaya teoriya polulineinykh parabolicheskikh uravnenii*, Moscow: Mir, 1985.
29. Trenogin, V.A., *Funktsional'nyi analiz* (Functional Analysis), Moscow: Nauka, 1993.
30. Vladimirov, V.S., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow: Nauka, 1971.
31. Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., and Mikhailov, A.P., *Rezhimy s obostreniem v zadachakh dlya kvazilineinykh parabolicheskikh uravnenii* (Peaking Modes in Problems for Quasilinear Parabolic Equations), Moscow: Nauka, 1987.

 (Reg. No. 310, 21.3.2007)