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First Darboux Problem for Nonlinear Hyperbolic Equations of Second Order

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Abstract—We study the first Darboux problem for hyperbolic equations of second order with power nonlinearity. We consider the question of the existence and nonexistence of global solutions to this problem depending on the sign of the parameter before the nonlinear term and the degree of its nonlinearity. We also discuss the question of local solvability of the problem.

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1. STATEMENT OF THE PROBLEM

In the plane of independent variables x and t, consider a nonlinear hyperbolic equation of the form

$$L_\lambda u := u_{tt} - u_{xx} + a_1(x,t)u_t + a_2(x,t)u_x + a_3(x,t)u + \lambda|u|^\alpha u = f(x,t), \qquad (1)$$

where λ and α are given real constants such that $\lambda\alpha \neq 0$, $\alpha > -1$, the a_i, $i = 1,2,3$, and f are given, and u is the required real functions.

Denote by $D_T := \{(x,t) : 0 < x < t,\ 0 < t < T\}$, $T \leq \infty$, the triangular domain bounded by the characteristic segment $\gamma_{1,T}$: $x = t$, $0 \leq t \leq T$, and also by the segments $\gamma_{2,T}$: $x = 0$, $0 \leq t \leq T$ and $\gamma_{3,T}$: $t = T$, $0 \leq x \leq T$.

For Eq. (1), consider the first Darboux problem of finding its solution $u(x,t)$ in the domain D_T from the boundary conditions (see, for example, [1, p. 228]):

$$u|_{\gamma_{i,T}} = 0, \qquad i = 1,2. \qquad (2)$$

Note that, for nonlinear equations of hyperbolic type, there is a vast literature (see, for example, [2]–[11]) devoted to the existence or nonexistence of global solutions to various problems (such as initial-value, mixed, nonlocal problems of various types, including periodic ones). As is well known, in the linear case, i.e., for $\lambda\alpha = 0$, problem (1), (2), is well posed and has a global solution in the corresponding function spaces (see, for example, [1], [12]–[15]).

We shall show that, under certain conditions on the exponent of nonlinearity α and the parameter λ, problem (1), (2) in some cases is globally solvable, while, in other cases, it has no global solution, although, as will be shown, this problem is locally solvable.

Definition 1. Suppose that $a_i \in C(\overline{D}_T)$, $i = 1,2,3$, and $f \in C(\overline{D}_T)$. A function u is called a *strong generalized solution* of problem (1), (2) of class C in the domain D_T if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{D}_T)$ as $n \to \infty$, where

$$\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{u \in C^2(\overline{D}_T) : u|_{\Gamma_T} = 0\}, \qquad \Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}.$$

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Remark 1. Obviously, a classical solution of problem (1), (2) from the space $\overset{\circ}{C}^2(\overline{D}_T, \Gamma_T)$ is a strong generalized solution of this problem of class C in the domain D_T. In turn, if the strong generalized solution of problem (1), (2) of class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$,, then it is also a classical solution of this problem.

Definition 2. Suppose that $a_i \in C(\overline{D}_\infty)$, $i = 1, 2, 3$, and $f \in C(\overline{D}_\infty)$. Problem (1), (2) is said to be *globally solvable* for the class C if, for any finite $T > 0$, this problem has a strong generalized solution of class C in the domain D_T.

2. *A PRIORI* ESTIMATE OF THE SOLUTION OF PROBLEM (1), (2)

The following assertion holds.

Lemma 1. *Suppose that $-1 < \alpha < 0$ and, in the case $\alpha > 0$, it is additionally required that $\lambda > 0$. Then, for a strong generalized solution of problem (1), (2) of class C in the domain D_T, the following a priori estimate holds*:

$$\|u\|_{C(\overline{D}_T)} \leq c_1 \|f\|_{C(\overline{D}_T)} + c_2 \tag{3}$$

with positive constants $c_i(T, a_j, \alpha, \lambda)$, $i = 1, 2$, $j = 1, 2, 3$, not depending on u and f.

Proof. First, consider the case in which $\alpha > 0$ and $\lambda > 0$. Suppose that u is a strong generalized solution of problem (1), (2) of class C in the domain D_T. Then, by Definition 1, there exists a sequence of functions $u_n \in \overset{\circ}{C}^2(\overline{D}_T, \Gamma_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\overline{D}_T)} = 0, \tag{4}$$

and hence also

$$\lim_{n\to\infty} \|\lambda |u_n|^\alpha u_n - \lambda |u|^\alpha u\|_{C(\overline{D}_T)} = 0. \tag{5}$$

Consider a function $u_n \in \overset{\circ}{C}^2(\overline{D}_T, \Gamma_T)$, as a solution of the following problem:

$$L_\lambda u_n = f_n, \tag{6}$$

$$u_n|_{\Gamma_T} = 0, \qquad \Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}. \tag{7}$$

Here

$$f_n := L_\lambda u_n. \tag{8}$$

Multiplying both sides of relation (6) by $\partial u_n / \partial t$ and integrating over the domain

$$D_\tau := \{(x, t) \in D_T : 0 < t < \tau\}, \qquad 0 < \tau \leq T,$$

we see that

$$\frac{1}{2} \int_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt - \int_{D_\tau} \frac{\partial^2 u_n}{\partial x^2} \frac{\partial u_n}{\partial t}\,dx\,dt + \frac{\lambda}{\alpha + 2} \int_{D_\tau} \frac{\partial}{\partial t} |u_n|^{\alpha+2}\,dx\,dt$$
$$= \int_{D_\tau} \left(f_n - a_1 \frac{\partial u_n}{\partial t} - a_2 \frac{\partial u_n}{\partial x} - a_3 u_n\right) \frac{\partial u_n}{\partial t}\,dx\,dt.$$

Set $I_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \leq T$. Then, in view of (7), integrating by parts the left-hand side of the last equality, we obtain

$$\int_{D_\tau} \left(f_n - a_1 \frac{\partial u_n}{\partial t} - a_2 \frac{\partial u_n}{\partial x} - a_3 u_n\right) \frac{\partial u_n}{\partial t}\,dx\,dt$$

$$= \int_{\gamma_{1,\tau}} \frac{1}{2\nu_t}\left[\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2(\nu_t^2-\nu_x^2)\right] ds$$
$$+ \frac{1}{2}\int_{I_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx + \frac{\lambda}{\alpha+2}\int_{I_\tau} |u_n|^{\alpha+2}\, dx, \tag{9}$$

where $\nu := (\nu_x, \nu_t)$ is the unit vector of the outer normal to ∂D_τ and $\Gamma_\tau := \Gamma_T \cap \{t \le \tau\}$.

Taking into account the fact that the operator $\nu_t\partial/\partial x - \nu_x\partial/\partial t$ is the inner differential operator on $\gamma_{1,T}$ and using (7), we find that

$$\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)\bigg|_{\gamma_{1,\tau}} = 0. \tag{10}$$

Further, it is obvious that

$$(\nu_t^2 - \nu_x^2)|_{\gamma_{1,\tau}} = 0. \tag{11}$$

In view of (10), (11), from (9) we obtain

$$w_n(\tau) := \int_{I_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx$$
$$\le 2\int_{D_\tau}\left(f_n - a_1\frac{\partial u_n}{\partial t} - a_2\frac{\partial u_n}{\partial x} - a_3 u_n\right)\frac{\partial u_n}{\partial t}\, dx\, dt. \tag{12}$$

Taking into account the so-called ε-inequality

$$2f_n\frac{\partial u_n}{\partial t} \le \varepsilon\left(\frac{\partial u_n}{\partial t}\right)^2 + \frac{1}{\varepsilon}f_n^2$$

valid for any $\varepsilon := \text{const} > 0$ and using (12), we obtain

$$w_n(\tau) \le \varepsilon\int_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2 dx\, dt + \frac{1}{\varepsilon}\|f_n\|^2_{L_2(D_\tau)}$$
$$- 2\int_{D_\tau}\left(a_1\frac{\partial u_n}{\partial t} + a_2\frac{\partial u_n}{\partial x} + a_3 u_n\right)\frac{\partial u_n}{\partial t}\, dx\, dt. \tag{13}$$

Introducing the notation

$$A := \max_{1\le i\le 3}\ \sup_{(x,t)\in \overline{D}_T} |a_i(x,t)|,$$

and using Cauchy's inequality, we see that

$$- 2\int_{D_\tau}\left(a_1\frac{\partial u_n}{\partial t} + a_2\frac{\partial u_n}{\partial x} + a_3 u_n\right)\frac{\partial u_n}{\partial t}\, dx\, dt$$
$$\le A\left\{4\int_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2 dx\, dt + \int_{D_\tau}\left(\frac{\partial u_n}{\partial x}\right)^2 dx\, dt + \int_{D_\tau} u_n^2\, dx\, dt\right\}. \tag{14}$$

Further, using relations (7) and the equality

$$u_n(x,t) = \int_x^t (\partial u_n(x,\tau)/\partial t)\, d\tau, \qquad (x,t) \in \overline{D}_T,$$

after standard arguments, we obtain the inequality (see, for example, [16, p. 63])

$$\int_{D_\tau} u_n^2\, dx\, dt \le \tau^2\int_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2 dx\, dt. \tag{15}$$

Hence, using (13) and (14), we see that

$$w_n(\tau) \le (\varepsilon + A(\tau^2+4))\int_0^\tau w_n(\sigma)\, d\sigma + \frac{1}{\varepsilon}\|f_n\|^2_{L_2(D_\tau)}, \qquad 0 < \tau \le T.$$

Taking into account the fact that the norm $\|f_n\|^2_{L_2(D_\tau)}$ is nondecreasing as a function of τ and using Gronwall's lemma (see, for example, [17, p. 13 (Russian transl.)]), we see that this inequality implies that

$$w_n(\tau) \le \frac{1}{\varepsilon}\|f_n\|^2_{L_2(D_\tau)} \exp(\tau(\varepsilon + A(\tau^2+4))).$$

Hence, noting the equality

$$\inf_{\varepsilon>0} \frac{\exp(\tau\varepsilon)}{\varepsilon} = e\tau,$$

which corresponds to $\varepsilon = 1/\tau$, we obtain

$$w_n(\tau) \le \tau\|f_n\|^2_{L_2(D_\tau)} \exp(A\tau(\tau^2+4)+1), \qquad 0<\tau\le T. \tag{16}$$

If $(x,t) \in \overline{D}_T$, then, in view of (7), the following relation holds:

$$u_n(x,t) = u_n(x,t) - u_n(0,t) = \int_0^x \frac{\partial u_n(\sigma,t)}{\partial x}\,d\sigma,$$

whence, by (16), we have

$$\begin{aligned}|u_n(x,t)|^2 &\le \int_0^x d\sigma \int_0^x \left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \le x\int_{I_t}\left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \le x w_n(t) \le t w_n(t)\\ &\le t^2\|f_n\|^2_{L_2(D_t)} \exp(At(t^2+4)+1) \le t^2\|f_n\|^2_{C(\overline{D}_t)} \operatorname{mes} D_t \exp(At(t^2+4)+1)\\ &\le 2^{-1}t^4\|f_n\|^2_{C(\overline{D}_T)} \exp(At(t^2+4)+1), \qquad (x,t)\in\overline{D}_T.\end{aligned} \tag{17}$$

It follows from (17) that

$$\|u_n\|_{C(\overline{D}_T)} \le \sqrt{2^{-1}}\,T^2\|f_n\|_{C(\overline{D}_T)} \exp(2^{-1}(AT(T^2+4)+1)).$$

By (4), (8), passing to the limit as $n\to\infty$ in the last inequality, we obtain

$$\|u\|_{C(\overline{D}_T)} \le \sqrt{2^{-1}}\,T^2\|f\|_{C(\overline{D}_T)} \exp(2^{-1}(AT(T^2+4)+1)). \tag{18}$$

Estimate (18) implies (3) in the case $\alpha>0$ and $\lambda>0$.

Now consider the case $-1<\alpha<0$ for an arbitrary λ. In this case, $1<\alpha+2<2$, and applying the well-known inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad a = |u_n|^{\alpha+2}, \quad b=1, \quad p=\frac{2}{\alpha+2}>1, \quad q=-\frac{2}{\alpha}>1, \quad \frac{1}{p}+\frac{1}{q}=1,$$

we obtain

$$\int_{I_\tau} |u_n|^{\alpha+2}\,dx \le \int_{I_\tau}\left[\frac{\alpha+2}{2}|u_n|^2 - \frac{\alpha}{2}\right]dx = \frac{\alpha+2}{2}\int_{I_\tau}|u_n|^2\,dx + \frac{|\alpha|\tau}{2}.$$

Hence, taking into account the form of the function $w_n(\tau)$ and (9)–(11), we see that (12) implies that

$$w_n(\tau) \le |\lambda|\int_{I_\tau}|u_n|^2\,dx + \frac{|\lambda\alpha|\tau}{\alpha+2} + 2\int_{D_\tau}\left(f_n - a_1\frac{\partial u_n}{\partial t} - a_2\frac{\partial u_n}{\partial x} - a_3u_n\right)\frac{\partial u_n}{\partial t}\,dx\,dt. \tag{19}$$

According to the theory of the trace, the following estimate holds (see, for example, [16, pp. 77, 86]):

$$\|u_n\|_{L_2(I_\tau)} \le \sqrt{\tau}\,\|u_n\|_{\overset{\circ}{W}{}^1_2(D_\tau,\Gamma_\tau)}, \qquad 0<\tau\le T, \tag{20}$$

where $\overset{\circ}{W}{}^1_2(D_\tau,\Gamma_\tau) := \{u\in W^1_2(D_\tau) : u|_{\Gamma_\tau}=0\}$, $W^1_2(D_\tau)$ is the well-known Sobolev space, and

$$\|u_n\|^2_{\overset{\circ}{W}{}^1_2(D_\tau,\Gamma_\tau)} := \int_{D_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right]dx\,dt.$$

Since

$$2f_n\frac{\partial u_n}{\partial t}\le f_n^2+\left(\frac{\partial u_n}{\partial t}\right)^2,$$

in view of (14), (15), and (20), it follows from (19) that

$$w_n(\tau)\le (A(\tau^2+4)+|\lambda|\tau+1)\int_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2dx\,dt$$
$$+(A+|\lambda|\tau)\int_{D_\tau}\left(\frac{\partial u_n}{\partial x}\right)^2dx\,dt+\int_{D_\tau}f_n^2\,dx\,dt+\frac{|\lambda\alpha|\tau}{\alpha+2}.$$

This yields

$$w_n(\tau)\le (A(\tau^2+4)+|\lambda|\tau+1)\int_0^\tau w_n(\sigma)\,d\sigma+\|f_n\|^2_{L_2(D_\tau)}+\frac{|\lambda\alpha|\tau}{\alpha+2}.$$

Applying Gronwall's lemma (see, for example, [17, p. 13 (Russian transl.]), from the last inequality we obtain

$$w_n(\tau)\le\left[\|f_n\|^2_{L_2(D_T)}+\frac{|\lambda\alpha|T}{\alpha+2}\right]\exp\{(A(T^2+4)+|\lambda|T+1)T\}. \tag{21}$$

Just as (16) yields (17), inequality (21) implies

$$|u_n(x,t)|^2\le tw_n(t)\le T\left[\|f_n\|^2_{C(\overline{D}_T)}\operatorname{mes}D_T+\frac{|\lambda\alpha|T}{\alpha+2}\right]\exp\{(A(T^2+4)+|\lambda|T+1)T\}$$
$$=T^2\left[\frac{T}{2}\|f_n\|^2_{C(\overline{D}_T)}+\frac{|\lambda\alpha|}{\alpha+2}\right]\exp\{(A(T^2+4)+|\lambda|T+1)T\}.$$

It follows from this inequality that

$$\|u_n\|_{C(\overline{D}_T)}\le T\left[\sqrt{\frac{T}{2}}\|f_n\|_{C(\overline{D}_T)}+\sqrt{\frac{|\lambda\alpha|}{\alpha+2}}\right]\exp\left\{\frac{T}{2}(A(T^2+4)+|\lambda|T+1)\right\},$$

whence, by (4), (8), passing to the limit as $n\to\infty$, we obtain the estimate

$$\|u\|_{C(\overline{D}_T)}\le T\sqrt{\frac{T}{2}}\exp\left\{\frac{T}{2}(A(T^2+4)+|\lambda|T+1)\right\}\|f\|_{C(\overline{D}_T)}$$
$$+T\sqrt{\frac{|\lambda\alpha|}{\alpha+2}}\exp\left\{\frac{T}{2}(A(T^2+4)+|\lambda|T+1)\right\}. \tag{22}$$

The proof of estimate (3) is now complete. □

Remark 2. It follows from (18) and (22) that the constants c_1 and c_2 in estimate (3) are as follows:

$$c_1=\sqrt{2^{-1}}T^2\exp(2^{-1}(AT(T^2+4)+1)),\quad c_2=0,\qquad\text{for}\quad \alpha>0,\ \lambda>0; \tag{23}$$

$$c_1=T\sqrt{\frac{T}{2}}\exp\left\{\frac{T}{2}(A(T^2+4)+|\lambda|T+1)\right\},\quad c_2=T\sqrt{\frac{|\lambda\alpha|}{\alpha+2}}\exp\left\{\frac{T}{2}(A(T^2+4)+|\lambda|T+1)\right\},$$
$$\text{for}\quad -1<\alpha<0,\ -\infty<\lambda<+\infty. \tag{24}$$

3. EQUIVALENT REDUCTION OF PROBLEM (1), (2) TO A NONLINEAR INTEGRAL EQUATION OF VOLTERRA TYPE

Suppose that $P:=P(x,t)$ is an arbitrary point of the domain D_T. Denote by $D_{x,t}$ the quadrangle with vertices at the points $O:=O(0,0)$, P as well as at the points P_1 and P_3 lying, respectively, on the supports of $\gamma_{2,T}$ and $\gamma_{1,T}$ i.e.,

$$P_1:=P_1(0,t-x),\qquad P_3:=P_3\left(\frac{x+t}{2},\frac{x+t}{2}\right).$$

Obviously, the domain $D_{x,t}$ consists of the characteristic rectangle $D_{1;x,t} := PP_1P_2P_3$ and the triangle $D_{2;x,t} := OP_1P_2$, where $P_2 := P_2((t-x)/2, (t-x)/2)$.

Consider the smoothness conditions imposed on the coefficients of Eq. (1):

$$a_i \in C^{k+1}(\overline{D}_\infty), \quad i = 1, 2, \quad a_3 \in C^k(\overline{D}_\infty), \qquad k \geq 1. \tag{25}$$

Remark 3. It is well known that, under conditions (25), the Green–Hadamard function $G(x, t; x', t')$ for problem (1), (2) for $\lambda = 0$ is well defined and, together with its partial derivatives up to $(k+1)$th order inclusive, is bounded and piecewise continuous, with discontinuities of the first kind only in passing through the singular manifold $t' + x' - t + x = 0$ (see, for example, [18], [19, p. 230], [20, p. 38]).

Below, unless otherwise stated, we assume that, in condition (25), the smoothness exponent $k = 1$.

Moreover, if $u \in C^2(\overline{D}_T)$ is a classical solution of problem (1), (2), then it satisfies the following integral equality:

$$u(x, t) + \lambda \int_{D_{x,t}} G(x', t'; x, t)|u|^\alpha u \, dx' \, dt' = \int_{D_{x,t}} G(x', t'; x, t) f(x', t') \, dx' \, dt', \qquad (x, t) \in \overline{D}_T. \tag{26}$$

Remark 4. Relation (26) can be regarded as a nonlinear integral equation of Volterra type; it can be rewritten in the form

$$u(x, t) + \lambda(L_0^{-1}|u|^\alpha u)(x, t) = F(x, t), \qquad (x, t) \in \overline{D}_T. \tag{27}$$

Here L_0^{-1} is the linear operator acting by the formula

$$(L_0^{-1}v)(x, t) := \int_{D_{x,t}} G(x', t'; x, t) v(x', t') \, dx' \, dt', \qquad (x, t) \in \overline{D}_T, \tag{28}$$

$$F(x, t) := (L_0^{-1} f)(x, t), \qquad (x, t) \in \overline{D}_T. \tag{29}$$

Lemma 2. *The function $u \in C(\overline{D}_T)$ is a strong generalized solution of problem* (1), (2) *of class C in the domain D_T if and only if it is a continuous solution of the nonlinear integral equation* (27).

Proof. Indeed, suppose that $u \in C(\overline{D}_T)$ is a solution of Eq. (27). Since $f \in C(\overline{D}_T)$, and the space $C^2(\overline{D}_T)$ is dense in $C(\overline{D}_T)$ (see, for example, [21, p. 37]), there exists a sequence of functions $f_n \in C^2(\overline{D}_T)$ such that $f_n \to f$ in the space $C(\overline{D}_T)$ as $n \to \infty$. Similarly, since $u \in C(\overline{D}_T)$, there exists a sequence of functions $w_n \in C^2(\overline{D}_T)$ such that $w_n \to u$ in the space $C(\overline{D}_T)$ as $n \to \infty$. Set

$$u_n := -\lambda(L_0^{-1}|w_n|^\alpha w_n) + L_0^{-1} f_n, \qquad n = 1, 2, \dots .$$

It is easily verified that $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, and since L_0^{-1} is a linear continuous operator in the space $C(\overline{D}_T)$ and

$$\lim_{n\to\infty} \|w_n - u\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|f_n - f\|_{C(\overline{D}_T)} = 0,$$

we have

$$u_n \to -\lambda(L_0^{-1}|u|^\alpha u) + L_0^{-1} f$$

in the space $C(\overline{D}_T)$ as $n \to \infty$. But it follows from relation (26) that

$$-\lambda(L_0^{-1}|u|^\alpha u) + L_0^{-1} f = u.$$

Thus, we have

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0.$$

On the other hand, $L_0u_n=-\lambda|w_n|^\alpha w_n+f_n$; hence, noting the equalities

$$\lim_{n\to\infty}\|u_n-u\|_{C(\overline{D}_T)}=0,\qquad \lim_{n\to\infty}\|w_n-u\|_{C(\overline{D}_T)}=0,\qquad \lim_{n\to\infty}\|f_n-f\|_{C(\overline{D}_T)}=0,$$

we obtain

$$\begin{aligned}L_\lambda u_n&=L_0u_n+\lambda|u_n|^\alpha u_n=-\lambda|w_n|^\alpha w_n+f_n+\lambda|u_n|^\alpha u_n\\&=-\lambda[|w_n|^\alpha w_n-|u|^\alpha u]+\lambda[|u_n|^\alpha u_n-|u|^\alpha u]+f_n\to f\end{aligned}$$

in the space $C(\overline{D}_T)$ as $n\to\infty$. The converse statement is obvious. □

4. THE CASE OF GLOBAL SOLVABILITY OF PROBLEM (1), (2) FOR THE CLASS OF CONTINUOUS FUNCTIONS

As noted above, the operator L_0^{-1} from (28) is a linear continuous operator acting in the space $C(\overline{D}_T)$. Let us now show that, in fact, this operator is a linear and continuous operator from the space $C(\overline{D}_T)$ to the space $C^1(\overline{D}_T)$ of continuously differentiable functions. To do this, let us pass to the plane of the variables ξ, τ by using the linear nonsingular transformation of the independent variables $t=\xi+\tau$ and $x=\xi-\tau$. As a result of this transformation:

1) the triangular domain D_T becomes the triangle Ω_T with vertices at the points with coordinates $(0,0)$, $(T,0)$ and $(T/2,T/2)$;

2) the quadrangle $D_{x,t}$ becomes the quadrangle $\Omega_{\xi,\tau}$ with vertices at the points $Q(\xi,\tau)$, $Q_1(\tau,\tau)$, $Q_2(\tau,0)$, $Q_3(\xi,0)$;

3) the characteristic rectangle $D_{1;x,t}$ becomes the rectangle $\Omega_{1;\xi,\tau}$ with vertices at the points Q, Q_1, Q_2, and Q_3;

4) the triangular domain $D_{2;x,t}$ becomes the triangle $\Omega_{2;\xi,\tau}:=OQ_1Q_2$.

We preserve the old notation $G(\xi,\tau;\xi',\tau')$ for the Green–Hadamard function $G(x,t;x',t')$ in the new variables $\xi,\tau;\xi',\tau'$ ($t'=\xi'+\tau'$, $x'=\xi'-\tau'$).

Moreover, the operator L_0^{-1} from (28) becomes the operator K acting in the space $C(\overline{\Omega}_T)$ by the formula

$$\begin{aligned}(Kw)(\xi,\tau)&=2\int_{\Omega_{\xi,\tau}}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\xi'\,d\tau'\\&=2\int_{\Omega_{1;\xi,\tau}}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\xi'\,d\tau'+2\int_{\Omega_{2;\xi,\tau}}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\xi'\,d\tau'\\&=2\int_\tau^\xi d\xi'\int_0^\tau G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\tau'\\&\qquad+2\int_0^\tau d\xi'\int_0^{\xi'}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\tau',\qquad(\xi,\tau)\in\overline{\Omega}_T.\end{aligned}\tag{30}$$

If $w\in C(\overline{\Omega}_T)$, then, in view of Remark 3, it immediately follows from (30) that

$$\begin{aligned}\frac{\partial}{\partial\xi}(Kw)(\xi,\tau)&=2\int_0^\tau G(\xi,\tau';\xi,\tau)w(\xi,\tau')\,d\tau'+2\int_\tau^\xi d\xi'\int_0^\tau\frac{\partial}{\partial\xi'}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\tau'\\&\qquad+2\int_0^\tau d\xi'\int_0^{\xi'}\frac{\partial}{\partial\xi'}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\tau',\qquad(\xi,\tau)\in\overline{\Omega}_T,\end{aligned}\tag{31}$$

$$\frac{\partial}{\partial\tau}(Kw)(\xi,\tau)=-2\int_0^\tau G(\tau,\tau';\xi,\tau)w(\tau,\tau')\,d\tau'+2\int_\tau^\xi G(\xi',\tau;\xi,\tau)w(\xi',\tau)\,d\xi'$$

$$+\,2\int_\tau^\xi d\xi'\int_0^\tau \frac{\partial}{\partial\tau'}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\tau' + 2\int_0^\tau G(\tau,\tau';\xi,\tau)w(\tau,\tau')\,d\tau'$$
$$+\,2\int_0^\tau d\xi'\int_0^{\xi'} \frac{\partial}{\partial\tau'}G(\xi',\tau';\xi,\tau)w(\xi',\tau')\,d\tau', \qquad (\xi,\tau)\in\overline{\Omega}_T. \tag{32}$$

Now, taking into account the fact that, for $(\xi,\tau)\in\overline{\Omega}_T$, we have $0\le\xi\le T$ and $0\le\tau\le T/2$ and using (30)–(32), we obtain

$$\begin{aligned}
&|(Kw)(\xi,\tau)| + \left|\frac{\partial}{\partial\xi}(Kw)(\xi,\tau)\right| + \left|\frac{\partial}{\partial\tau}(Kw)(\xi,\tau)\right| \\
&\le 2\tau(\xi-\tau)G_0\|w\|_{C(\overline{\Omega}_T)} + \tau^2G_0\|w\|_{C(\overline{\Omega}_T)} + 2\tau G_0\|w\|_{C(\overline{\Omega}_T)} \\
&\qquad + 2\tau(\xi-\tau)G_1\|w\|_{C(\overline{\Omega}_T)} + \tau^2G_1\|w\|_{C(\overline{\Omega}_T)} + 2\tau G_0\|w\|_{C(\overline{\Omega}_T)} \\
&\qquad + 2(\xi-\tau)G_0\|w\|_{C(\overline{\Omega}_T)} + 2\tau(\xi-\tau)G_2\|w\|_{C(\overline{\Omega}_T)} \\
&\qquad + 2\tau G_0\|w\|_{C(\overline{\Omega}_T)} + 2\tau^2G_2\|w\|_{C(\overline{\Omega}_T)} \\
&\le 2(3\tau\xi-\tau^2+2\tau+\xi)G_3\|w\|_{C(\overline{\Omega}_T)} \le (3T^2+4T)G_3\|w\|_{C(\overline{\Omega}_T)},
\end{aligned}$$

where

$$G_0 := \sup_{t'+x'-t+x\ne0}|G|, \qquad G_1 := \sup_{t'+x'-t+x\ne0}|\partial G/\partial x'|,$$
$$G_2 := \sup_{t'+x'-t+x\ne0}|\partial G/\partial t'|$$

and $G_3 := G_0+G_1+G_2 < +\infty$ by Remark 3. Thus,

$$\|K\|_{C(\overline{\Omega}_T)\to C^1(\overline{\Omega}_T)} \le (3T^2+4T)G_3, \tag{33}$$

which proves the assertion.

Further, since the space $C^1(\overline{\Omega}_T)$ is compactly embedded in the space $C(\overline{\Omega}_T)$ (see, for example, [22, p. 135 (Russian transl.], in view of (33), the operator $K\colon C(\overline{\Omega}_T)\to C(\overline{\Omega}_T)$ is a linear and compact operator. Thus, returning now from the variables ξ and τ to the variables x and t, we obtain the following statement for the operator K from (28).

Lemma 3. *The operator $K\colon C(\overline{D}_T)\to C(\overline{D}_T)$ acting by formula* (32)*, is a linear compact operator. Moreover, by* (33)*, the same operator takes the space $C(\overline{D}_T)$ to the space $C^1(\overline{D}_T)$ and is bounded.*

In view of (29), Eq. (27) can be rewritten as

$$u = Au := K(-\lambda|u|^\alpha u + f), \tag{34}$$

where the operator $A\colon C(\overline{D}_T)\to C(\overline{D}_T)$ is continuous and compact, because the nonlinear operator $K\colon C(\overline{D}_T)\to C(\overline{D}_T)$, acting by the formula

$$Ku := -\lambda|u|^\alpha u + f, \qquad \alpha > -1$$

is bounded and continuous, while the linear operator $K\colon C(\overline{D}_T)\to C(\overline{D}_T)$ is compact by Lemma 3. At the same time, by Lemmas 1 and 2 and relations (23) and (24), the *a priori* estimate

$$\|u\|_{C(\overline{D}_T)} \le \widetilde{c}_1\|f\|_{C(\overline{D}_T)} + \widetilde{c}_2$$

with positive constants $\widetilde{c}_1$ and $\widetilde{c}_2$ not depending on u, τ, and f holds for any parameter $\tau\in[0,1]$ and for any solution $u\in C(\overline{D}_T)$ of the equation $u=\tau Au$. Therefore, by the Leray–Schauder theorem (see, for example, [23, p. 375]) under the conditions of Lemma 1, Eq. (34) has at least one solution $u\in C(\overline{D}_T)$. Thus, by Lemma 2, we have proved the following statement.

Theorem 1. *Suppose that $-1<\alpha<0$, while, in the case $\alpha>0$, the parameter λ is positive. Then problem* (1), (2) *is globally solvable for the class C in the sense of Definition* 2*, i.e., the inclusion $f\in C(\overline{D}_\infty)$ implies that, for any $T>0$, problem* (1), (2) *has a strong generalized solution of class C in the domain D_T.*

5. SMOOTHNESS AND UNIQUENESS OF THE SOLUTION OF PROBLEM (1), (2). EXISTENCE OF A GLOBAL SOLUTION IN D_∞

By Lemmas 2 and 3 and Remark 3, relations (27)–(29) imply the following statement.

Lemma 4. *Suppose that u is a strong generalized solution of problem* (1), (2) *of class C in the domain D_T in the sense of Definition* 1. *In that case, if $a_1, a_2 \in C^{k+1}(\overline{D}_T)$, $a_3, f \in C^k(\overline{D}_T)$, then $u \in C^{k+1}(\overline{D}_T)$, $k \geq 1$.*

In particular, it follows from this lemma that, for $k \geq 1$, the strong generalized solution of problem (1), (2) of class C in the domain D_T is a classical solution of this problem.

Lemma 5. *For $\alpha > 0$, problem* (1), (2) *cannot have more than one strong generalized solution of class C in the domain D_T.*

Proof. Indeed, suppose that problem (1), (2) has two strong generalized solutions u_1 and u_2 of class C in the domain D_T. By Definition 1, there exists a sequence of functions

$$u_{in} \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T), \qquad i = 1, 2$$

such that

$$\lim_{n\to\infty} \|u_{in} - u_i\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_{in} - f\|_{C(\overline{D}_T)} = 0, \qquad i = 1, 2. \tag{35}$$

Let $\omega_{nm} := u_{2n} - u_{1m}$. We can easily see that the function $\omega_{nm} \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ satisfies the following identities:

$$L_0\omega_{nm} + g_{nm}\omega_{nm} = f_{nm}, \tag{36}$$

$$\omega_{nm}|_{\Gamma_T} = 0. \tag{37}$$

Here

$$g_{nm} := \lambda(1+\alpha)\int_0^1 |u_{1m} + t(u_{2n} - u_{1m})|^\alpha \, dt, \tag{38}$$

$$f_{nm} := L_\lambda u_{2n} - L_\lambda u_{1m}, \tag{39}$$

where we have used the obvious equality

$$\varphi(x_2) - \varphi(x_1) = (x_2 - x_1)\int_0^1 \varphi'(x_1 + t(x_2 - x_1))\, dt$$

for the function $\varphi(x) := |x|^\alpha x$ for $x_2 = u_{2n}$, $x_1 = u_{1m}$, $\alpha > 0$. By the first equality in (35), there exists a number $M := \mathrm{const} > 0$ not depending on the indices i and n such that $\|u_{in}\|_{C(\overline{D}_T)} \leq M$; hence, in turn, by (38) we have

$$\|g_{n,m}\|_{C(\overline{D}_T)} \leq |\lambda|(1+\alpha)M^\alpha \qquad \forall\, n, m. \tag{40}$$

In view of (39) and the second equality, it follows from (35) that

$$\lim_{n,m\to\infty} \|f_{nm}\|_{C(\overline{D}_T)} = 0. \tag{41}$$

Multiplying both sides of relation (36) by $\partial\omega_{nm}/\partial t$, integrating over the domain

$$D_\tau := \{(x,t) \in D_T : 0 < t < \tau\}, \qquad 0 < \tau \leq T,$$

using the boundary conditions (37), just as in the derivation of inequality (12), from (6), (7) we obtain

$$\begin{aligned} w_{nm}(\tau) &:= \int_{I_\tau} \left[\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2 + \left(\frac{\partial\omega_{nm}}{\partial x}\right)^2\right] dx \\ &\leq 2\int_{D_\tau} \left(f_{nm} - a_1\frac{\partial\omega_{nm}}{\partial t} - a_2\frac{\partial\omega_{nm}}{\partial x} - a_3\omega_{nm} - g_{nm}\omega_{nm}\right)\frac{\partial\omega_{nm}}{\partial t}\, dx\, dt, \end{aligned} \tag{42}$$

where $I_\tau := \overline{D}_\infty \cap \{t=\tau\}$, $0<\tau\le T$.

By estimate (40) and Cauchy's inequality, we have

$$\begin{aligned}
2\int_{D_\tau}\Big(f_{nm}-a_1\frac{\partial\omega_{nm}}{\partial t}-a_2\frac{\partial\omega_{nm}}{\partial x}-a_3\omega_{nm}-g_{nm}\omega_{nm}\Big)\frac{\partial\omega_{nm}}{\partial t}\,dx\,dt&\\
=-2\int_{D_\tau}\Big(a_1\frac{\partial\omega_{nm}}{\partial t}+a_2\frac{\partial\omega_{nm}}{\partial x}+a_3\omega_{nm}\Big)\frac{\partial\omega_{nm}}{\partial t}\,dx\,dt&\\
+2\int_{D_\tau}\Big(f_{nm}-g_{nm}\omega_{nm}\Big)\frac{\partial\omega_{nm}}{\partial t}\,dx\,dt&\\
\le 2A\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial t}\Big)^2dx\,dt+A\Big\{\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial t}\Big)^2dx\,dt+\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial x}\Big)^2dx\,dt\Big\}&\\
+A\Big\{\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial t}\Big)^2dx\,dt+\int_{D_\tau}\omega_{nm}^2\,dx\,dt\Big\}&\\
+2\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial t}\Big)^2dx\,dt+\int_{D_\tau}f_{nm}^2\,dx\,dt+\int_{D_\tau}g_{nm}^2\omega_{nm}^2\,dx\,dt&\\
=2(2A+1)\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial t}\Big)^2dx\,dt+A\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial x}\Big)^2dx\,dt&\\
+\int_{D_\tau}f_{nm}^2\,dx\,dt+(\lambda^2(1+\alpha)^2M^{2\alpha}+A)\int_{D_\tau}\omega_{nm}^2\,dx\,dt.&\qquad(43)
\end{aligned}$$

Since the function ω_{nm} satisfies the same homogeneous boundary conditions as u_n, by the same arguments, we obtain estimate (15) for it. Taking into account this estimate as well as inequalities (42), (43), we see that

$$\begin{aligned}
w_{nm}(\tau)&\le(A(\tau^2+4)+2+\lambda^2(1+\alpha)^2M^{2\alpha}\tau^2)\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial t}\Big)^2dx\,dt\\
&\quad+A\int_{D_\tau}\Big(\frac{\partial\omega_{nm}}{\partial x}\Big)^2dx\,dt+\int_{D_\tau}f_{nm}^2\,dx\,dt\\
&\le(A(\tau^2+4)+2+\lambda^2(1+\alpha)^2M^{2\alpha}\tau^2)\int_0^\tau w_{nm}(\sigma)\,d\sigma+\int_{D_T}f_{nm}^2\,dx\,dt.\qquad(44)
\end{aligned}$$

Hence by Gronwall's lemma (see, for example, [17, p. 13 (Russian transl.]), we find that

$$w_{nm}(\tau)\le c\|f_{nm}\|^2_{L_2(D_T)},\qquad 0<\tau\le T,\qquad(45)$$

where

$$c:=\exp(A(\tau^2+4)+2+\lambda^2(1+\alpha)^2M^{2\alpha}T^2)T.$$

Using the same arguments as those leading to inequality (17) and taking into account the obvious inequality

$$\|f_{nm}\|^2_{L_2(D_T)}\le\|f_{nm}\|^2_{C(\overline{D}_T)}\operatorname{mes}D_T,$$

from (45) we obtain

$$|\omega_{nm}(x,t)|^2\le tw_{nm}(t)\le cT\operatorname{mes}D_T\|f_{nm}\|^2_{C(\overline{D}_T)}=2^{-1}cT^3\|f_{nm}\|^2_{C(\overline{D}_T)},\qquad(x,t)\in\overline{D}_T.$$

This implies that

$$\|\omega_{nm}\|_{C(\overline{D}_T)}\le T\sqrt{2^{-1}cT}\,\|f_{nm}\|_{C(\overline{D}_T)}.\qquad(46)$$

Since $\omega_{nm}:=u_{2n}-u_{1m}$, by the first equality from (35), we have

$$\lim_{n,m\to\infty}\|\omega_{nm}\|_{C(\overline{D}_T)}=\|u_2-u_1\|_{C(\overline{D}_T)}.$$

Hence by (41), passing to the limit as $n, m \to \infty$ in inequality (46), we obtain $\|u_2 - u_1\|_{C(\overline{D}_T)} = 0$, i.e., $u_1 = u_2$, which proves Lemma 5. □

Theorem 2. *Suppose that $\alpha > 0$ and $\lambda > 0$. Then, under condition (25), in the case $k = 1$ and for any $f \in C^1(\overline{D}_\infty)$, problem (1), (2) has a unique global classical solution $u \in \overset{\circ}{C}{}^2(\overline{D}_\infty, \Gamma_\infty)$ in the domain D_∞.*

Proof. If $\alpha > 0$, $\lambda > 0$ and $a_1, a_2 \in C^2(\overline{D}_\infty)$, $a_3, f \in C^1(\overline{D}_\infty)$, then, by Theorem 1 and Lemmas 4 and 5, there exists a unique classical solution $u_n \in \overset{\circ}{C}{}^2(\overline{D}_n, \Gamma_n)$ of problem (1), (2) for $T = n$ in the domain D_T. Since u_{n+1} is also a classical solution of problem (1), (2) in the domain D_n, by Lemma 5 we have $u_{n+1}|_{D_n} = u_n$. Therefore, the function u constructed in the domain D_∞ by means of the rule $u(x,t) = u_n(x,t)$ for $n = [t] + 1$, where $[t]$ is the integer part of the number t, and the point (x,t) belongs to D_∞, is the unique classical solution of problem (1), (2) in the domain D_∞ of class $\overset{\circ}{C}{}^2(\overline{D}_\infty, \Gamma_\infty)$. The proof of Theorem 2 is complete. □

6. SOME PROPERTIES OF THE GREEN–HADAMARD FUNCTION OF PROBLEM (1), (2) FOR $\lambda = 0$

Let us present a sufficient condition imposed on the coefficients a_1, a_2, and a_3 of Eq. (1) guaranteeing that the Green–Hadamard function of problem (1) (2) is nonnegative for $\lambda = 0$, i.e.,

$$G(x,t;x',t') \ge 0, \qquad (x,t) \in \overline{D}_T, \quad (x',t') \in \overline{D}_{x,t}. \tag{47}$$

To do this, let us rewrite problem (1), (2) for $\lambda = 0$ in the characteristic variables ξ and τ given in Sec. 4:

$$LU := U_{\xi\tau} + A(\xi,\tau)U_\xi + B(\xi,\tau)U_\tau + C(\xi,\tau)U = F(\xi,\tau), \qquad (\xi,\tau) \in \overline{\Omega}_T, \tag{48}$$

$$U(\xi,0) = 0, \quad 0 \le \xi \le T, \qquad U(\tau,\tau) = 0, \quad 0 \le \tau \le \frac{T}{2}. \tag{49}$$

Here

$$U(\xi,\tau) := u(\xi - \tau, \xi + \tau), \qquad A := \frac{a_1 + a_2}{2}, \quad B := \frac{a_1 - a_2}{2}, \quad C := a_3, \qquad F := f. \tag{50}$$

In addition, note that, by relation (30), the solution $U(\xi,\tau)$ of problem (48), (49) can be expressed as

$$U(\xi,\tau) = 2\int_{\Omega_{\xi,\tau}} G(\xi',\tau';\xi,\tau)F(\xi',\tau')\,d\xi'\,d\tau', \qquad (\xi,\tau) \in \overline{\Omega}_T. \tag{51}$$

Lemma 6. *Suppose that*

$$k \ge 0, \tag{52}$$

where $k := B_\tau + AB - C$ is the Laplace invariant of Eq. (48). Then condition (47) holds.

Proof. The operator L from (48) can be expressed as

$$LU = lU - kU, \tag{53}$$

where

$$lU := \left(\frac{\partial}{\partial\tau} + A\right)\left(\frac{\partial}{\partial\xi} + B\right)U. \tag{54}$$

By direct integration, we can easily verify that the solution of the problem

$$lV = F_1, \qquad V(\xi,0) = 0, \quad 0 \le \xi \le T, \qquad V(\tau,\tau) = 0, \quad 0 \le \tau \le \frac{T}{2} \tag{55}$$

can be expressed as

$$V(\xi,\tau)=\int_{\Omega_{1;\xi,\tau}} R(\xi',\tau';\xi,\tau)F_1(\xi',\tau')\,d\xi'\,d\tau'. \tag{56}$$

Here

$$R(\xi,\tau;\xi',\tau') := \exp\biggl\{\int_{\tau'}^{\tau} A(\xi,\tau_1)\,d\tau_1+\int_{\xi'}^{\xi} B(\xi_1,\tau')\,d\xi_1\biggr\}\ge 0 \tag{57}$$

is the Riemann function of the operator l acting by formula (54) (see, for example, [24, p. 16]).

In view of (53) and (56), the solution of problem (48), (49) satisfies the following integral equation:

$$\begin{aligned} U(\xi,\tau)=\int_{\Omega_{1;\xi,\tau}} & R(\xi',\tau';\xi,\tau)k(\xi',\tau')U(\xi',\tau')\,d\xi'\,d\tau' \\ &+\int_{\Omega_{1;\xi,\tau}} R(\xi',\tau';\xi,\tau)F(\xi',\tau')\,d\xi'\,d\tau', \qquad (\xi,\tau)\in\overline{\Omega}_T. \end{aligned} \tag{58}$$

As is well known, the integral Volterra equation (58) can be solved by the method of successive approximations:

$$\begin{aligned} U_0=0,\qquad U_n(\xi,\tau)=\int_{\Omega_{1;\xi,\tau}} & R(\xi',\tau';\xi,\tau)k(\xi',\tau')U_{n-1}(\xi',\tau')\,d\xi'\,d\tau' \\ &+\int_{\Omega_{1;\xi,\tau}} R(\xi',\tau';\xi,\tau)F(\xi',\tau')\,d\xi'\,d\tau', \qquad n\ge 1,\quad (\xi,\tau)\in\overline{\Omega}_T. \end{aligned} \tag{59}$$

In view of (52) and (57), it follows from the recurrence relations (59) for $F\ge 0$ that

$$U_n(\xi,\tau)\ge 0,\quad n=0,1,\dots,\qquad (\xi,\tau)\in\overline{\Omega}_T. \tag{60}$$

Since

$$\lim_{n\to\infty}\|U_n-U\|_{C(\overline{\Omega}_T)}=0,$$

by (60), we have

$$U(\xi,\tau)\ge 0 \qquad \text{for}\quad F(\xi,\tau)\ge 0,\ \ (\xi,\tau)\in\overline{\Omega}_T. \tag{61}$$

Thus, by (61), for any nonnegative function $F\in C(\overline{\Omega}_T)$, the right-hand sides of relation (51) is also nonnegative. This implies the validity of condition (47). The proof of lemma 7 is complete. □

Remark 5. As is well known (see, for example, [19, p. 230]), the Green–Hadamard function in the characteristic rectangle $\Omega_{1;\xi,\tau}$ is identical with the Riemann function of the operator L from (48). At the same time, it follows from expression (56) that the Green–Hadamard function of problem (55) in the triangular part $\Omega_{2;\xi,\tau}$ of the domain $\Omega_{\xi,\tau}$ is zero.

Remark 6. Note that, for the case in which the coefficients of the operator L_0 are constant,, the sufficient condition (52) for the for the nonnegativity of the Green–Hadamard function of problem (1), (2) for $\lambda=0$ is also a necessary one.

Indeed, suppose that the coefficients A, B, and C of Eq. (48) are constant and the Laplace invariant is

$$k:=AB-C<0. \tag{62}$$

Problem (48), (49) with respect to the new unknown function $V=U\exp(A\tau+B\xi)$ can be rewritten as

$$V_{\xi\tau}+(C-AB)V=F\exp(A\tau+B\xi),\qquad (\xi,\tau)\in\overline{\Omega}_T, \tag{63}$$

$$V(\xi,0)=0,\quad 0\le\xi\le T,\qquad V(\tau,\tau)=0,\quad 0\le\tau\le\frac{T}{2}. \tag{64}$$

The solution of this problem can be expressed as follows:

$$V(\xi,\tau) = 2\int_{\Omega_{\xi,\tau}} \widetilde{G}(\xi',\tau';\xi,\tau)F(\xi',\tau')\exp(A\tau'+B\xi')\,d\xi'\,d\tau', \qquad (\xi,\tau)\in\overline{\Omega}_T. \tag{65}$$

where $\widetilde{G}(\xi,\tau;\xi',\tau')$ is the Green–Hadamard function of problem (63), (64).

Comparing the representations (51) and (65), we can easily establish that

$$G(\xi,\tau;\xi',\tau') = \widetilde{G}(\xi,\tau;\xi',\tau')\exp\{A(\tau-\tau')+B(\xi-\xi')\}.$$

By Remark 5, the Green–Hadamard function $\widetilde{G}(\xi,\tau;\xi',\tau')$ of problem (63), (64) in the characteristic rectangle $\Omega_{1;\xi,\tau}$ is identical with the Riemann function of Eq. (63), which can be expressed by using formula [25, p. 455 (Russian transl.] involving the Bessel function J_0:

$$\widetilde{G}(\xi,\tau;\xi',\tau') = J_0\big(2\sqrt{k(\xi-\xi')(\tau-\tau')}\big), \qquad (\xi',\tau')\in\Omega_{1;\xi,\tau}.$$

It only remains to note that the Bessel function J_0 is one with alternating signs, having infinitely many zeros.

Similar results concerning the nonnegativity of the Riemann function were obtained in [18].

7. THE CASE OF THE NONEXISTENCE OF GLOBAL SOLUTIONS OF PROBLEM (1), (2)

Consider the case for which, in Eq. (1), the parameter $\lambda<0$, the exponent of nonlinearity $\alpha>0$, and conditions (25) hold for $k=1$.

Lemma 7. *Suppose that u is a strong generalized solution of problem* (1), (2) *of class C in the domain D_T in the sense of Definition* 1. *Then the following integral equality holds*:

$$\int_{D_T} uL_0^*\varphi\,dx\,dt = -\lambda\int_{D_T}|u|^\alpha u\varphi\,dx\,dt + \int_{D_T} f\varphi\,dx\,dt \tag{66}$$

for any function φ such that

$$\varphi\in C^2(\overline{D}_T), \qquad \varphi|_{\gamma_{3,T}}=0, \quad \varphi_t|_{\gamma_{3,T}}=0, \quad \varphi|_{\gamma_{2,T}}=0, \tag{67}$$

where L_0^* is the operator conjugate in the sense of Lagrange and acting by the formula

$$L_0^*\varphi := \varphi_{tt}-\varphi_{xx}-(a_1\varphi)_t-(a_2\varphi)_x+a_3\varphi.$$

Proof. By the definition of a strong generalized solution u of problem (1), (2) of class C in the domain D_T, the function u belongs to $C(\overline{D}_T)$ and there exists a sequence of functions $u_n\in \overset{\circ}{C}{}^2(\overline{D}_T,\Gamma_T)$ such that relation (4) hold.

Set $f_n := L_\lambda u_n$. Let us multiply both sides of the equality $L_\lambda u_n = f_n$ by the function φ and integrate the resulting equality over the domain D_T. Integrating by parts the left-hand side of this equality, noting (67), and using the boundary conditions $u_n|_{\gamma_i,T}=0$, $i=1,2$, we find that

$$\int_{D_T} u_nL_0^*\varphi\,dx\,dt = -\lambda\int_{D_T}|u_n|^\alpha u_n\varphi\,dx\,dt + \int_{D_T} f_n\varphi\,dx\,dt.$$

Passing to the limit as $n\to\infty$ in this equality and taking (4) into account, we obtain (66). Lemma 7 is proved. □

In what follows, condition (52) will be considered in the original variables x, t according to formulas (50).

Lemma 8. *Suppose that $\lambda<0$ and $\alpha>0$, while the function $u\in C(\overline{D}_T)$ is a strong generalized solution of problem* (1), (2) *of class C in the domain D_T. In that case, if condition* (52) *holds and $f\geq 0$, then $u\geq 0$ in the domain D_T.*

Proof. By Lemma 2 and relations (27)–(29), the function u is a solution of the following integral equation of Volterra type:

$$u(x,t)=\int_{D_{x,t}} K(x,t;x',t')u(x',t')\,dx'\,dt'+F(x,t), \qquad (x,t)\in \overline{D}_T. \tag{68}$$

Here

$$K(x,t;x',t'):=-\frac{\lambda}{2}G(x',t';x,t)|u(x',t')|^{\alpha},$$
$$F(x,t):=\frac{1}{2}\int_{D_{x,t}} G(x',t';x,t)F(x',t')\,dx'\,dt'.$$

Taking into account the assumptions of Lemma 8 and Lemma 6, we obtain inequality (47) and, therefore,

$$K(x,t;x',t')\ge 0, \quad (x,t)\in\overline{D}_T,\ \ (x',t')\in\overline{D}_{x,t}, \qquad F(x,t)\ge 0, \quad (x,t)\in\overline{D}_T. \tag{69}$$

Given the function $K(x,t;x',t')$, consider the following linear integral equation of Volterra type:

$$v(x,t)=\int_{D_{x,t}} K(x,t;x',t')v(x',t')\,dx'\,dt'+F(x,t), \qquad (x,t)\in \overline{D}_T, \tag{70}$$

for the class $C(\overline{D}_T)$ with respect to the unknown function $v(x,t)$. As is well known (see, for example, [15]), Eq. (70) has a unique continuous solution $v(x,t)$ for the class $C(\overline{D}_T)$, so that, for $(x,t)\in\overline{D}_T$, we can use the method of successive approximations, obtaining

$$v_0(x,t)=0, \qquad v_{n+1}(x,t)=\int_{D_{x,t}} K(x,t;x',t')v_n(x',t')\,dx'\,dt'+F(x,t), \quad n=0,1,\dots. \tag{71}$$

In view of (69), from (71) we find $v_n(x,t)\ge 0$ in $\overline{D}_T$ for all $n=0,1,\dots$. But, for the class $C(\overline{D}_T)$, $v_n\to v$ as $n\to\infty$. Therefore, the limit function v is nonnegative in the domain D_T. It only remains to note that, by relation (68), the function u is also a solution of Eq. (70) and, therefore, by the uniqueness of the solution of this equation, we finally obtain $u=v\ge 0$ in the domain D_T. Lemma 8 is proved. □

Under the assumptions of Lemma 8, relation (66) can be rewritten as

$$\int_{D_T}|u|L_0^*\varphi\,dx\,dt=|\lambda|\int_{D_T}|u|^p\varphi\,dx\,dt+\int_{D_T} f\varphi\,dx\,dt, \qquad p:=\alpha+1. \tag{72}$$

Let us use the method of trial functions [10, pp. 10–12]. Consider a function $\varphi^0:=\varphi^0(x,t)$ such that

$$\begin{gathered}\varphi^0\in C^2(\overline{D}_\infty),\\ \varphi^0|_{D_{T=1}}>0, \quad \varphi^0_x|_{D_{T=1}}\ge 0, \quad \varphi^0_t|_{D_{T=1}}\le 0, \quad \varphi^0|_{\gamma_{2,\infty}}=0, \quad \varphi^0|_{t\ge 1}=0\end{gathered} \tag{73}$$

and

$$\kappa_0:=\int_{D_{T=1}}\frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx\,dt<+\infty, \qquad p'=1+\frac{1}{\alpha}, \tag{74}$$

where $\Box:=\partial^2/\partial t^2-\partial^2/\partial x^2$.

It is easily verified that the function φ^0 satisfying conditions (73) and (74) can be taken to be

$$\varphi^0(x,t)=\begin{cases}x^n(1-t)^m, & (x,t)\in D_{T=1},\\ 0, & t\ge 1,\end{cases}$$

for sufficiently large positive constants n and m.

Setting $\varphi_T(x,t):=\varphi^0(x/T,t/T)$, $T>0$, and taking (73) into account, we can easily see that

$$\varphi_T\in C^2(\overline{D}_T), \qquad \varphi_T|_{D_T}>0, \quad \left.\frac{\partial\varphi_T}{\partial x}\right|_{D_T}\ge 0, \quad \left.\frac{\partial\varphi_T}{\partial t}\right|_{D_T}\le 0, \tag{75}$$

$$\varphi_T|_{\gamma_{2,T}} = 0, \quad \varphi_T|_{t=T} = 0, \quad \frac{\partial \varphi_T}{\partial t}\bigg|_{t=T} = 0. \tag{76}$$

Assuming the function f to be fixed, consider the function of one variable T,

$$\zeta(T) := \int_{D_T} f\varphi_T \, dx \, dt, \qquad T > 0. \tag{77}$$

The following theorem on the nonexistence of global solvability of problem (1), (2) is valid.

Theorem 3. *Suppose that condition* (52) *is satisfied, $\lambda < 0$, $\alpha > 0$, the function $f \in C(\overline{D}_\infty)$ is nonnegative, and $a_1 \le 0$, $a_2 \ge 0$, $a_3 - \partial a_1/\partial t - \partial a_2/\partial x \le 0$ in the domain D_∞. In that case, if*

$$\liminf_{T \to +\infty} \zeta(T) > 0, \tag{78}$$

then there exists a positive number $T_0 := T_0(f)$ such that, for $T > T_0$, problem (1), (2) *cannot have a strong generalized solution of class C in the domain D_T.*

Proof. Suppose that, under the assumptions of this theorem, there exists a strong generalized solution u of problem (1), (2) of class C in the domain D_T. Then, by Lemma 7 and 8, relation (72) holds in which, by (75), (76), the function φ can be taken as the function $\varphi = \varphi_T$ i.e.,

$$\int_{D_T} |u| L_0^* \varphi_T \, dx \, dt = |\lambda| \int_{D_T} |u|^p \varphi_T \, dx \, dt + \int_{D_T} f\varphi_T \, dx \, dt.$$

In view of the notation (77), the definitions of the operators L_0^* and $\Box$, we can rewrite the last equality in the form

$$|\lambda| \int_{D_T} |u|^p \varphi_T \, dx \, dt = \int_{D_T} |u| \Box \varphi_T \, dx \, dt - \int_{D_T} |u| \left(a_1 \frac{\partial \varphi_T}{\partial t} + a_2 \frac{\partial \varphi_T}{\partial x} \right) dx \, dt$$
$$+ \int_{D_T} |u| \left(a_3 - \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial x} \right) \varphi_T \, dx \, dt - \zeta(T);$$

hence, by the assumptions of Theorem 3 and (75), we obtain the inequality

$$|\lambda| \int_{D_T} |u|^p \varphi_T \, dx \, dt \le \int_{D_T} |u| \Box \varphi_T \, dx \, dt - \zeta(T). \tag{79}$$

If, in Young's inequality with the parameter $\varepsilon > 0$,

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{p' \varepsilon^{p'-1}} b^{p'}, \qquad a, b \ge 0, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p := \alpha + 1 > 1,$$

we take $a = |u| \varphi_T^{1/p}$, $b = |\Box \varphi_T| / \varphi_T^{1/p}$, then, in view of the fact that $p'/p = p' - 1$, we obtain

$$|u \Box \varphi_T| = |u| \varphi_T^{1/p} \frac{|\Box \varphi_T|}{\varphi_T^{1/p}} \le \frac{\varepsilon}{p} |u|^p \varphi_T + \frac{1}{p' \varepsilon^{p'-1}} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}}.$$

By (79) and the last inequality, we have

$$\left(|\lambda| - \frac{\varepsilon}{p} \right) \int_{D_T} |u|^p \varphi_T \, dx \, dt \le \frac{1}{p' \varepsilon^{p'-1}} \int_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} \, dx \, dt - \zeta(T),$$

whence, for $\varepsilon < |\lambda| p$, we obtain

$$\int_{D_T} |u|^p \varphi_T \, dx \, dt \le \frac{p}{(|\lambda| p - \varepsilon) p' \varepsilon^{p'-1}} \int_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} \, dx \, dt - \frac{p}{|\lambda| p - \varepsilon} \zeta(T). \tag{80}$$

In view of the equalities $p'=p/(p-1)$, $p=p'/(p'-1)$, and the relation

$$\min_{0<\varepsilon<|\lambda|p}\frac{p}{(|\lambda|p-\varepsilon)p'\varepsilon^{p'-1}}=\frac{1}{|\lambda|^{p'}},$$

which is attained at $\varepsilon=|\lambda|$, inequality (80) implies that

$$\int_{D_T}|u|^p\varphi_T\,dx\,dt\le\frac{1}{|\lambda|^{p'}}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt-\frac{p'}{|\lambda|}\zeta(T). \tag{81}$$

Since $\varphi_T(x,t):=\varphi^0(x/T,t/T)$, using (73), (74), and making the change of variables $x=Tx'$, $t=Tt'$, we can easily verify that

$$\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt=T^{-2(p'-1)}\int_{D_{T=1}}\frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx'\,dt'=T^{-2(p'-1)}\kappa_0<+\infty.$$

Hence, using (75) and inequality (81), we obtain

$$0\le\int_{D_T}|u|^p\varphi_T\,dx\,dt\le\frac{1}{|\lambda|^{p'}}T^{-2(p'-1)}\kappa_0-\frac{p'}{|\lambda|}\zeta(T). \tag{82}$$

Since $p'=p/(p-1)>1$, it follows that $-2(p'-1)<0$, and, by (74), we have

$$\lim_{T\to\infty}\frac{1}{|\lambda|^{p'}}T^{-2(p'-1)}\kappa_0=0.$$

Therefore, in view of (78), there exists a positive number $T_0:=T_0(f)$ such that, for $T>T_0$, the right-hand side of inequality (82) is negative, while the left-hand side of this inequality is nonnegative. This implies that if there exists a strong generalized solution u of problem (1), (2) of class C in the domain D_T, then necessarily $T\le T_0$, which proves Theorem 3. □

Remark 7. Note that the conditions imposed on the coefficients a_1, a_2, and a_3 in Theorem 3, hold if, for example, $a_1=a_2=0$ and $a_3\le 0$.

Remark 8. It is easily verified that if

$$f\in C(\overline{D}_\infty),\quad f\ge 0,\qquad\text{and}\qquad f(x,t)\ge ct^{-m},\quad t\ge 1,$$

where $c:=\text{const}>0$, $0\le m:=\text{const}\le 2$, then condition (77) holds, and thus, for

$$\lambda<0,\quad\alpha>0,\quad k\ge 0,\qquad a_1\le 0,\quad a_2\ge 0,\quad a_3-\frac{\partial a_1}{\partial t}-\frac{\partial a_2}{\partial x}\le 0,$$

for sufficiently large T, problem (1), (2) does not have a strong generalized solution u of class C in the domain D_T.

Indeed, in (77), introducing the transformation of the independent variables x and t by the formula $x=Tx'$, $t=Tt'$, after a few manipulations, we obtain

$$\begin{aligned}\zeta(T)&=T^2\int_{D_{T=1}}f(Tx',Tt')\varphi^0(x',t')\,dx'\,dt'\\&\ge cT^{2-m}\int_{D_{T=1}\cap\{t'\ge T^{-1}\}}t'^{-m}\varphi^0(x',t')\,dx'\,dt'+T^2\int_{D_{T=1}\cap\{t'<T^{-1}\}}f(Tx',Tt')\varphi^0(x',t')\,dx'\,dt'\end{aligned}$$

under the assumption that $T>1$. Further, suppose that $T_1>1$ is an arbitrary fixed number. Then the last inequality for the function ζ implies

$$\zeta(T)\ge cT^{2-m}\int_{D_{T=1}\cap\{t'\ge T^{-1}\}}t'^{-m}\varphi^0(x',t')\,dx'\,dt'\ge c\int_{D_{T=1}\cap\{t'\ge T_1^{-1}\}}t'^{-m}\varphi^0(x',t')\,dx'\,dt' \tag{83}$$

if $T\ge T_1>1$ and $m\le 2$. In view of (75), inequality (78) immediately follows from (83).

8. LOCAL SOLVABILITY OF PROBLEM (1), (2) IN THE CASE $\lambda < 0$ AND $\alpha > 0$

Theorem 4. *Suppose that condition* (25) *holds for* $k = 1$, $\lambda < 0$, $\alpha > 0$, *and* $f \in C(\overline{D}_\infty)$, $f \not\equiv 0$. *Then there exists a positive number* $T_* := T_*(f)$ *such that, for* $T \leq T_*$, *problem* (1), (2) *has at least one strong generalized solution* u *of class* C *in the domain* D_T.

Proof. In Sec. 4, problem (1), (2) considered in the space $C(\overline{D}_T)$ was reduced, in an equivalent way, to the functional equation (34), where the operator $A\colon C(\overline{D}_T) \to C(\overline{D}_T)$ is continuous and compact. Therefore, by Schauder's theorem, in order to prove the solvability of Eq. (34), it suffices to show that the operator A takes some ball

$$B_R := \{v \in C(\overline{D}_T) : \|v\|_{C(\overline{D}_T)} \leq R\}$$

of radius $R > 0$ which is a closed and convex set in the Banach space $C(\overline{D}_T)$ into itself. Let us show that this holds for sufficiently small T.

Indeed,, by (28) and (34), for $\|u\|_{C(\overline{D}_T)} \leq R$ we have

$$\begin{aligned}
\|Au\|_{C(\overline{D}_T)} &\leq \|L_0^{-1}\|_{C(\overline{D}_T)\to C(\overline{D}_T)}[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}] \\
&\leq G_T \sup_{(x,t)\in\overline{D}_T} \operatorname{mes} D_{x,t}[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}] \\
&\leq G_T \operatorname{mes} D_T[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}] \\
&= \frac{1}{2}G_T T^2[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}] \leq \frac{1}{2}G_T T^2[|\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_T)}], \qquad (84)
\end{aligned}$$

where

$$G_T := \sup_{(x,t)\in\overline{D}_T,\,(x',t')\in\overline{D}_{x,t}} |G(x,t;x',t')| < +\infty.$$

Choose arbitrarily a positive number T_0. Then, by estimate (84), for $0 < T \leq T_0$, we have

$$\|Au\|_{C(\overline{D}_T)} \leq \frac{1}{2}G_{T_0}T^2[|\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_{T_0})}].$$

Hence, in turn, it follows that if

$$T_*^2 := \min\biggl\{T_0^2, \frac{2RG_{T_0}}{|\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_{T_0})}}\biggr\},$$

then

$$\|Au\|_{C(\overline{D}_T)} \leq R \qquad \text{for} \quad \|u\|_{C(\overline{D}_T)} \leq R, \quad 0 < T \leq T_*.$$

The proof of Theorem 4 is complete. □

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REFERENCES

1. A. V. Bitsadze, *Some Classes of Partial Differential Equations* (Nauka, Moscow, 1981) [in Russian].
2. F. John, "Blow-up of solutions of nonlinear wave equation in three space dimensions," Manuscripta Math. **28** (1–3), 235–268 (1979).
3. F. John, "Blow-up for quasi-linear wave equations in three space dimensions," Comm. Pure Appl. Math. **34** (1), 29–51 (1981).
4. F. John and S. Klainerman, "Almost global existence to nonlinear wave equations in three space dimensions," Comm. Pure Appl. Math. **37** (4), 443–455 (1984).

5. T. Kato, “Blow-up of solutions of some nonlinear hyperbolic equations,” Comm. Pure Appl. Math. **33** (4), 501–505 (1980).
6. V. Georgiev, H. Lindblad, and C. Sogge, “Weighted Strichartz estimates and global existence for semilinear wave equations,” Amer. J. Math. **119** (6), 1291–1319 (1977).
7. T. C. Sideris, “Nonexistence of global solutions to semilinear wave equations in high dimensions,” J. Differential Equations **52** (3), 378–406 (1984).
8. L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, in *Math. Appl. (Berlin)* (Springer-Verlag, Berlin, 1997), Vol. 26.
9. S. I. Pohozaev (S. I. Pokhozhaev) and L. Véron, “Blow-up results for nonlinear hyperbolic inequaties,” Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2), 393–420 (2000).
10. É. Mitidieri and S. I. Pokhozhaev, *A Priori Estimates and Blow-Up of Solutions to Nonlinear Partial Differential Equations and Inequalities*, in *Proc. Steklov Inst. Math.* (Nauka, Moscow, 2001), Vol. 234.
11. G. Todorova and E. Vitillaro, “Blow-up for nonlinear dissipative wave equations in $\mathbb{R}^n$,” J. Math. Anal. Appl. **303** (1), 242–257 (2005).
12. E. Goursat, *Cours d’analyse mathématique*, Tome III: *Integrales infiniment voisines. Equations aux derivees du second Course of mathematical analysis*, Repr. of the 3rd ed. 1923, (Gostekhizdat, Moscow–Leningrad, 1933; Gauthier-Villars, 1992).
13. E. I. Moiseev, “Approximation of the classical solution of a Darboux problem by smooth solutions,” Differentsial’nye Uravneniya **20** (1), 73–87 (1984) [Differential Equations **20** (1), 59–70 (1984)].
14. E. I. Moiseev, *Equations of Mixed Type with a Spectral Parameter* (Izd. Moskov. Univ., Moscow, 1988) [in Russian].
15. S. Kharibegashvili, *Goursat and Darboux Type Problems for Linear Hyperbolic Partial Differential Equations and Systems*, in *Mem. Differential Equations Math. Phys.* (Georgian Acad. Sci., Tbilisi, 1995), Vol. 4.
16. O. A. Ladyzhenskaya, *Boundary-Value Problems of Mathematical Physics* (Nauka, Moscow, 1973) [in Russian].
17. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, in *Lecture Notes in Mathematics* (Springer-Verlag, Berlin–Heidelberg–New York, 1981; Mir, Moscow, 1985), Vol. 840.
18. M. E. Lerner, “Qualitative properties of the Riemann function,” Differentsial’nye Uravneniya the equations **27** (12), 2106–2120 (1991) [Differential Equations **27** (12), 1495–1508 (1991)].
19. A. M. Nakhushev, *Equations of Mathematical Biology* (Vysshaya Shkola, Moscow, 1995) [in Russian].
20. M. M. Smirnov *Equations of Mixed Type* (Nauka, Moscow, 1970) [in Russian].
21. R. Narasimhan, *Analysis on Real and Complex Manifolds*, in *Advanced Studies in Pure Mathematics* (North-Holland, Amsterdam, 1968; Mir, Moscow, 1971), Vol. 1.
22. D. Gil’barg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Nauka, Moscow, 1989) [in Russian].
23. V. A. Trenogin, *Functional Analysis* (Nauka, Moscow, 1993) [in Russian].
24. V. I. Zhegalov and A. N. Mironov, *Differential Equations with Higher Partial Derivatives*, in *Monographs in Mathematics* (Kazan Mat. Obshch., Kazan, 2001), Vol. 5 [in Russian].
25. R. Courant and D. Hilbert *Methods of Mathematical Physics*, Vol. II: *Partial Differential Equations* (Interscience Publishers, New York–London, 1962; Mir, Moscow, 1964).