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ON SOME THREE-DIMENSIONAL VARIANTS OF GOURSAT AND DARBOUX PROBLEMS FOR HIGHER-ORDER HYPERBOLIC EQUATIONS WITH DOMINATING PRINCIPAL PARTS

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ABSTRACT. Some three-dimensional variants of Goursat and Darboux problems for higher-order hyperbolic equations with dominating principal part are set and investigated. Conditions to the problems' data which in some cases ensure the well-posedness of the problem in question and in other cases, despite the solvability of the problem, imply the presence of an infinite set of linearly independent solutions of corresponding homogeneous problem, are found. We consider both generalized and classical solutions.

CONTENTS

1. Formulation of the Problem

In the Euclidean space $\mathbb{R}^3$ of independent variables x_1, x_2, and x_3, let us consider a higher-order hyperbolic equation with dominating principal part of the form (see [6, pp. 103, 183])

$$\frac{\partial^m u}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}} + \sum_{\substack{|\alpha|\le m-1\\ \alpha_i\le m_i\\ i=1,2,3}} A^\alpha \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} = F, \tag{1.1}$$

where $m \ge 3$, $1 \le m_i \le m-1$, $m = \sum\limits_{i=1}^{3} m_i$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \sum\limits_{i=1}^{3} \alpha_i$, $\alpha_i \ge 0$, $m_i, \alpha_i \in Z$, F is given, and u is an unknown function. Equations of the form (1.1) are encountered during the study of mathematical models for some natural and physical processes (see [1, 3, 4, 13, 15–18]).

Let us note that in the domain $D := \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 < x_i < 1,\ i = 1,2,3\big\}$ the three-dimensional variant of the Goursat problem for Eq. (1.1) with boundary conditions

$$\frac{\partial^i u}{\partial x_j^i}\bigg|_{x_j=0} = f_{ji}, \quad j = 1,2,3, \quad i = 0,\dots,m_j-1, \tag{1.2}$$

was considered by many authors (see [5, 7, 10, 18]).

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For Eq. (1.1), a characteristic problem in the domain D with general boundary conditions was considered and studied in [8, 11].

When $m = 4$, $m_1 = 2$, and $m_2 = m_3 = 1$, for Eq. (1.1) with multiple characteristics, the well-posedness of Darboux-type spatial problems in the domain D with the conditions

$$\left.\frac{\partial^i u}{\partial x_1^i}\right|_{x_1=0} = f_{1i}, \quad i = 0, 1,$$

and also

$$u|_{x_3=k_1x_2} = f_2, \quad u|_{x_2=k_2x_3} = f_3, \quad 0 < k_i < 1, \quad i = 1, 2,$$

or

$$u|_{x_3=k_1x_2} = f_2, \quad u|_{x_1=k_2x_2} = f_3, \quad 0 < k_i < 1, \quad i = 1, 2,$$

was established in [12].

In the present paper, for Eq. (1.1), when $m = 3$, i.e., for the equation

$$\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} + \sum_{\substack{|\alpha|\le 2\\ \alpha_i \le 1}} A^\alpha \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} = F \tag{1.3}$$

in the domain $D := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_i < 1,\ i = 1, 2, 3\right\}$, let us consider the three-dimensional variant of the Darboux problem with the following conditions:

$$\left.\left(\sum_{j=1}^{3} l_{ij} \frac{\partial u}{\partial x_j} + l_i^1 u\right)\right|_{S_i} = f_i, \quad i = 1, 2, 3, \tag{1.4}$$

where $S_1 : x_1 = \lambda_1(x_2, x_3)$, $S_2 : x_2 = \lambda_2(x_1, x_3)$, $S_3 : x_3 = \lambda_3(x_1, x_2)$ are given smooth surfaces, located in $\bar{D}$, i.e., $\lambda_i \in C^1$, $0 \le \lambda_i \le 1$, $i = 1, 2, 3$, l_{ij}, l_i^1, and f_i are given continuous functions on S_i, and u is an unknown real function in the domain D. Assume that the corresponding compatibility conditions for the data of problem (1.3), (1.4) are satisfied.

The fact that one must use caution from the point of view of the well-posedness during the setting of problem (1.3), (1.4) is clearly shown for the following special case of the problem:

$$\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = F, \tag{1.5}$$

$$u|_{S_1:x_1=0} = f_1, \quad l \cdot \nabla u|_{S_2:x_2=kx_3} = 0, \quad u|_{S_3:x_3=0} = f_3, \tag{1.6}$$

where

$$l = (l_1, l_2, l_3) \ne 0, \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad l \cdot \nabla u = \sum_{i=1}^{3} l_i \frac{\partial u}{\partial x_i}, \quad 0 < k < 1.$$

Let us consider the case where the vector l is parallel to the plane $S_2 : x_2 = kx_3$, i.e., where the operator $l \cdot \nabla$ is an internal differential operator in S_2. It is easy to verify that in appropriate classes of smoothness, problem (1.5), (1.6) is well posed only in the case where the condition $l_1(kl_2 + l_3) > 0$ holds, which is equivalent to the condition $l_1 l_3 > 0$, since $l_2 = kl_3$ (l and S_2 are parallel).

2. Reduction of Problem (1.3), (1.4) to a System of Functional Equations in Some Special Case

We consider the reduction of problem (1.3), (1.4) to a system of functional equations in the case where Eq. (1.3) does not have low-order terms and conditions (1.4) do not have principal terms.

In the case where Eq. (1.3) does not have low-order terms in Eq. (1.3) and conditions (1.4) do not have principal terms, under the natural assumption that $l_i^1 = 1$, $i = 1, 2, 3$, problem (1.3), (1.4) takes the form

$$\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = F, \tag{2.1}$$

$$u|_{S_i} = f_i, \quad i = 1, 2, 3. \tag{2.2}$$

We consider problem (2.1), (2.2) in the functional space

$$C^{1,1,1}\left(\overline{D}\right) := \left\{ u \in C\left(\overline{D}\right) : \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \in C\left(\overline{D}\right), \ \alpha_i = 0, 1, \ i = 1, 2, 3 \right\},$$

requiring that $F \in C\left(\overline{D}\right)$ and $f_i \in C^{1,1}(S_i)$, $i = 1, 2, 3$. Here

$$C^{1,1}(S_1) := \left\{ f \in C(S_1) : \ f|_{S_1} = f\left(\lambda_1(z_1, z_2), z_1, z_2\right) \in C^{1,1}\left(\overline{\Omega}\right) \right\},$$

where $\Omega : 0 < z_i < 1$, $i = 1, 2$, and

$$C^{1,1}\left(\overline{\Omega}\right) := \left\{ g \in C\left(\overline{\Omega}\right) : \ \frac{\partial^{|\alpha|} g}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2}} \in C\left(\overline{\Omega}\right), \ \alpha_i = 0, 1, \ i = 1, 2 \right\}.$$

Similarly, we define the spaces $C^{1,1}(S_2)$ and $C^{1,1}(S_3)$.

As is known, the solution u of Eq. (2.1) in the space $C^{1,1,1}\left(\overline{D}\right)$ is given by the formula (see [5])

$$u(x_1, x_2, x_3) = \varphi_0 + \varphi_{12}(x_1, x_2) + \varphi_{13}(x_1, x_3) + \varphi_{23}(x_2, x_3) - \varphi_{12}(0, x_2)$$
$$-\varphi_{13}(x_1, 0) - \varphi_{23}(0, x_3) + \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} F(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma. \tag{2.3}$$

Here

$$\varphi_0 = u(0, 0, 0), \quad \varphi_{12}(x_1, x_2) = u(x_1, x_2, 0), \quad \varphi_{13}(x_1, x_3) = u(x_1, 0, x_3),$$
$$\varphi_{23}(x_2, x_3) = u(0, x_2, x_3), \quad \varphi_{ij} \in C^{1,1}\left(\overline{\Omega}\right), \quad 1 \le i, j \le 3, \quad i < j. \tag{2.4}$$

In this case, obviously, the following natural compatibility conditions must hold:

$$\varphi_{12}(x_1, 0) = \varphi_{13}(x_1, 0), \quad \varphi_{12}(0, x_2) = \varphi_{23}(x_2, 0),$$
$$\varphi_{13}(0, x_3) = \varphi_{23}(0, x_3). \tag{2.5}$$

Substituting expression (2.3) for the solution $u \in C^{1,1,1}\left(\overline{D}\right)$ of Eq. (2.1) in conditions (2.2), we obtain

$$\varphi_{12}(x_1, x_2) + \varphi_{13}(x_1, \lambda_3(x_1, x_2)) + \varphi_{23}(x_2, \lambda_3(x_1, x_2)) + \varphi_0 - \varphi_{12}(0, x_2)$$
$$-\varphi_{13}(x_1, 0) - \varphi_{23}(0, \lambda_3(x_1, x_2)) + \int_0^{x_1} \int_0^{x_2} \int_0^{\lambda_3(x_1, x_2)} F(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma = f_3(x_1, x_2), \tag{2.6}$$

$$\varphi_{13}(x_1, x_3) + \varphi_{12}(x_1, \lambda_2(x_1, x_3)) + \varphi_{23}(\lambda_2(x_1, x_3), x_3) + \varphi_0 - \varphi_{12}(0, \lambda_2(x_1, x_3))$$
$$-\varphi_{13}(x_1, 0) - \varphi_{23}(0, x_3) + \int_0^{x_1} \int_0^{\lambda_2(x_1, x_3)} \int_0^{x_3} F(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma = f_2(x_1, x_3), \tag{2.7}$$

$$\varphi_{23}(x_2, x_3) + \varphi_{12}(\lambda_1(x_2, x_3), x_2) + \varphi_{13}(\lambda_1(x_2, x_3), x_3) + \varphi_0 - \varphi_{12}(0, x_2)$$
$$-\varphi_{13}(\lambda_1(x_2, x_3), 0) - \varphi_{23}(0, x_3) + \int_0^{\lambda_1(x_2, x_3)} \int_0^{x_2} \int_0^{x_3} F(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma = f_1(x_2, x_3). \tag{2.8}$$

For simplicity, we will seek the solution u of problem (2.1), (2.2) in the space

$$\overset{0}{C}{}^{1,1,1}\left(\overline{D}\right) := \left\{ u \in C^{1,1,1}\left(\overline{D}\right): \ u|_{x_1=x_2=0} = u|_{x_1=x_3=0} = u|_{x_2=x_3=0} = 0 \right\}. \tag{2.9}$$

In this case, relative to functions f_i in (2.2), one should require that

$$f_i \in \overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right), \quad i = 1, 2, 3, \tag{2.10}$$

where

$$\overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right) := \left\{ g \in C^{1,1}\left(\overline{\Omega}\right): \ g|_{z_1=0} = g|_{z_2=0} = 0 \right\}. \tag{2.11}$$

If $u \in \overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$, then from (2.9) and (2.3)–(2.5) we obtain

$$\begin{aligned} \varphi_{12}\left(0, x_2\right) = \varphi_{13}\left(x_1, 0\right) = \varphi_{23}\left(0, x_3\right) = 0, \\ \varphi_{12}\left(x_1, 0\right) = \varphi_{13}\left(0, x_2\right) = \varphi_{23}\left(x_2, 0\right) = 0. \end{aligned} \tag{2.12}$$

Therefore, using the notation

$$\psi_1(z_1, z_2) = \varphi_{12}\left(z_1, z_2\right), \quad \psi_2(z_1, z_2) = \varphi_{13}\left(z_1, z_2\right), \quad \psi_3(z_1, z_2) = \varphi_{23}\left(z_1, z_2\right), \tag{2.13}$$

we can rewrite the system of functional equations (2.6)–(2.8) relative to the unknown functions $\psi_i \in \overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right)$, $i = 1, 2, 3$, in the following form:

$$\psi_1\left(z_1, z_2\right) + \psi_2\left(z_1, \lambda_3\left(z_1, z_2\right)\right) + \psi_3\left(z_2, \lambda_3\left(z_1, z_2\right)\right) = g_1\left(z_1, z_2\right), \tag{2.14}$$

$$\psi_2\left(z_1, z_2\right) + \psi_1\left(z_1, \lambda_2\left(z_1, z_2\right)\right) + \psi_3\left(\lambda_2\left(z_1, z_2\right), z_2\right) = g_2\left(z_1, z_2\right), \tag{2.15}$$

$$\psi_3\left(z_1, z_2\right) + \psi_1\left(\lambda_1\left(z_1, z_2\right), z_1\right) + \psi_2\left(\lambda_1\left(z_1, z_2\right), z_2\right) = g_3\left(z_1, z_2\right), \tag{2.16}$$

$$z = (z_1, z_2) \in \overline{\Omega}.$$

Here

$$\begin{aligned} g_1\left(z_1, z_2\right) &= f_3\left(z_1, z_2\right) - \int_0^{z_1}\int_0^{z_2}\int_0^{\lambda_3(z_1,z_2)} F\left(\xi, \eta, \varsigma\right) d\xi d\eta d\varsigma, \\ g_2\left(z_1, z_2\right) &= f_2\left(z_1, z_2\right) - \int_0^{z_1}\int_0^{\lambda_2(z_1,z_2)}\int_0^{z_2} F\left(\xi, \eta, \varsigma\right) d\xi d\eta d\varsigma, \\ g_3\left(z_1, z_2\right) &= f_1\left(z_1, z_2\right) - \int_0^{\lambda_1(z_1,z_2)}\int_0^{z_2}\int_0^{z_3} F\left(\xi, \eta, \varsigma\right) d\xi d\eta d\varsigma. \end{aligned} \tag{2.17}$$

Since $F \in C\left(\overline{D}\right)$ and according to (2.10), (2.11), and (2.17), it is easy to see that

$$g_i \in \overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right), \quad i = 1, 2, 3. \tag{2.18}$$

Remark 1. It follows from the reasoning given above that problem (2.1), (2.2) with $F \in C\left(\overline{D}\right)$, $f_i \in \overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right)$, $i = 1, 2, 3$, in the space $\overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$ is equivalent to the system of functional equations (2.14), (2.15), (2.16) in the space $\overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right)$.

Before investigating problem (2.1), (2.2) in the space $\overset{0}{C}{}^{1,1}\left(\overline{\Omega}\right)$, we introduce the concept of a strong generalized solution of Eq. (2.1) of class $\overset{0}{C}\left(\overline{D}\right)$, where

$$\overset{0}{C}\left(\overline{D}\right) := \left\{ u \in C\left(\overline{D}\right): u|_{x_1=x_2=0} = u|_{x_1=x_3=0} = u|_{x_2=x_3=0} = 0 \right\}. \tag{2.19}$$

Definition 1. Let $F \in C\left(\overline{D}\right)$. A function u is called a strong generalized solution of Eq. (2.1) of class $\overset{0}{C}\left(\overline{D}\right)$ if $u \in \overset{0}{C}\left(\overline{D}\right)$ and there exists a sequence of functions $u_n \in \overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D})} = 0, \quad \lim_{n\to\infty} \left\| \frac{\partial^3 u_n}{\partial x_1 \partial x_2 \partial x_3} - F \right\|_{C(\overline{D})} = 0. \tag{2.20}$$

It is easy to see that the solution of Eq. (2.1) in the space $\overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$ is also a strong generalized solution of this equation of class $\overset{0}{C}\left(\overline{D}\right)$. Similarly to $\overset{0}{C}\left(\overline{D}\right)$, we introduce the space

$$\overset{0}{C}\left(\overline{\Omega}\right) := \left\{ g \in C\left(\overline{\Omega}\right) : g|_{z_1=0} = g|_{z_2=0} = 0 \right\}. \tag{2.21}$$

The following assertion holds.

Lemma 1. *Let $F \in C\left(\overline{D}\right)$. A function u is a strong generalized solution of Eq.* (2.1) *of class $\overset{0}{C}\left(\overline{D}\right)$ if and only if $u \in \overset{0}{C}\left(\overline{D}\right)$ and the following representation holds*:

$$u(x_1, x_2, x_3) = \varphi_{12}(x_1, x_2) + \varphi_{13}(x_1, x_3) + \varphi_{23}(x_2, x_3) + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} F(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma, \tag{2.22}$$

where

$$\begin{gathered} \varphi_{12}(x_1, x_2) = u(x_1, x_2, 0), \quad \varphi_{13}(x_1, x_3) = u(x_1, 0, x_3), \\ \varphi_{23}(x_2, x_3) = u(0, x_2, x_3), \quad \varphi_{ij} \in \overset{0}{C}\left(\overline{\Omega}\right), \quad 1 \le i, j \le 3, \quad i < j. \end{gathered} \tag{2.23}$$

Proof. Indeed, let $F \in C\left(\overline{D}\right)$ and u be a strong generalized solution of Eq. (2.1) of class $\overset{0}{C}\left(\overline{D}\right)$, i.e., there exists a sequence of functions $u_n \in \overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$ such that conditions (2.20) hold. According to Eq. (2.3), for the function $u_n \in \overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$, by (2.12) with $F = F_n = \dfrac{\partial^3 u_n}{\partial x_1 \partial x_2 \partial x_3}$, the following representation holds:

$$u_n(x_1, x_2, x_3) = \varphi_{12}^n(x_1, x_2) + \varphi_{13}^n(x_1, x_3) + \varphi_{23}^n(x_2, x_3) + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} F_n(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma, \tag{2.24}$$

where

$$\begin{gathered} \varphi_{12}^n(x_1, x_2) = u_n(x_1, x_2, 0), \quad \varphi_{13}^n(x_1, x_3) = u_n(x_1, 0, x_3), \\ \varphi_{23}^n(x_2, x_3) = u_n(0, x_2, x_3), \quad \varphi_{ij}^n \in \overset{0}{C}\left(\overline{\Omega}\right), \quad 1 \le i, j \le 3, \quad i < j. \end{gathered} \tag{2.25}$$

According to (2.20), passing in equalities (2.24) and (2.25) to the limit as $n \to \infty$, we obtain (2.22) and (2.23).

Conversely, let $u \in \overset{0}{C}\left(\overline{D}\right)$ and representation (2.22) and equalities (2.23) hold. Let us denote by $h_\varepsilon(x)$ an arbitrary function continuous in the interval $0 \le x < +\infty$ which satisfies the following conditions:

$$0 \le h_\varepsilon(x) \le 1, \; h_\varepsilon(x) = \begin{cases} 1 & \text{for } x \ge 2\varepsilon, \\ 0 & \text{for } 0 \le x \le \varepsilon, \end{cases} \tag{2.26}$$
$$0 < \varepsilon < 1.$$

It is clear that this function exists and it is easy to construct it. Let us introduce the function

$$\varphi_{ij}^\varepsilon(z_1, z_2) = h_\varepsilon(z_1) h_\varepsilon(z_2) \varphi_{ij}(z_1, z_2), \quad 1 \le i, j \le 3, \quad i < j. \tag{2.27}$$

Since the functions φ_{ij} satisfy conditions (2.12), then by (2.26) and (2.27), it is easy to verify that

$$\lim_{\varepsilon\to\infty}\left\|\varphi_{ij}^{\varepsilon}-\varphi_{ij}\right\|_{C(\overline{\Omega})}=0,\quad 1\le i,j\le 3,\quad i<j. \tag{2.28}$$

Let us denote by $\omega_\delta(z_1,z_2)=\delta^{-2}\omega^0\left(\frac{z_1}{\delta},\frac{z_2}{\delta}\right)$, $\delta>0$, an averaging function, where $\omega^0\in C_0^\infty\left(\mathbb{R}^2\right)$,

$$\int\omega^0 dz_1dz_2=1,\quad \omega^0\ge 0,$$

$\operatorname{supp}\omega^0=\{(z_1,z_2)\in\mathbb{R}^2: z_1^2+z_2^2\le 1\}$ (see [6, p. 9]). According to (2.26)–(2.28), taking into account the property of the convolution operation (see [6, pp. 9, 23]), we easily find that $\varphi_{ij}^{\varepsilon,\delta}=\varphi_{ij}^{\varepsilon}*\omega_\delta\in C^\infty\left(\overline{\Omega}\right)\cap\overset{0}{C}(\Omega)$, $\delta<\varepsilon$, and

$$\lim_{\delta\to 0}\left\|\varphi_{ij}^{\varepsilon,\delta}-\varphi_{ij}^{\varepsilon}\right\|_{C(\overline{\Omega})}=0,\quad 1\le i,j\le 3,\quad i<j. \tag{2.29}$$

Now, assuming that $\varepsilon=1/n$ and $\delta=2/n$, by (2.26)–(2.29) we obtain that the sequence of functions

$$u_n(x_1,x_2,x_3)=\varphi_{12}^{\frac{1}{n},\frac{1}{2n}}(x_1,x_2)+\varphi_{13}^{\frac{1}{n},\frac{1}{2n}}(x_1,x_3)+\varphi_{23}^{\frac{1}{n},\frac{1}{2n}}(x_2,x_3)+\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}F(\xi,\eta,\varsigma)\,d\xi d\eta d\varsigma,$$

$n=1,2,\dots$, belongs to the space $\overset{0}{C}{}^{1,1,1}\left(\overline{D}\right)$ and satisfies conditions (2.20) in view of (2.22) and equalities (2.23). Lemma 1 is proved. □

Let us introduce the weight-function space

$$\overset{0}{C}_{\alpha}\left(\bar{\Omega}\right)=\left\{g\in\overset{0}{C}\left(\bar{\Omega}\right):\ \|g\|_{\overset{0}{C}_{\alpha}(\bar{\Omega})}=\sup_{(z_1,z_2)\in\Omega}\left|z_1^{-\alpha}z_2^{-\alpha}g(z_1,z_2)\right|<+\infty\right\},\quad \alpha=\text{const}>0.$$

It is clear that $\overset{0}{C}_{\alpha}\left(\bar{\Omega}\right)\subset\overset{0}{C}\left(\bar{\Omega}\right)$ and $\overset{0}{C}_{0}\left(\bar{\Omega}\right)=\overset{0}{C}\left(\bar{\Omega}\right)$.

Lemma 2. *Assume that for $z=(z_1,z_2)\in\bar{\Omega}$,*

$$0\le\lambda_i(z_1,z_2)\le M_i\, z_1^{\beta_{i1}}z_2^{\beta_{i2}}, \tag{2.30}$$

$$M_i=\text{const}\ge 0,\ \beta_{ij}=\text{const}\ge 1,\quad i,j=1,2.$$

Then for any $F\in C\left(\bar{D}\right)$ and $f_i\in\overset{0}{C}_{\alpha}\left(\bar{\Omega}\right)$, $i=1,2,3$, if the condition

$$M_i^{\alpha_0}<\frac{1}{2},\quad i=1,2,3, \tag{2.31}$$

holds with $\alpha_0=\min(\alpha,\beta_0+1)$, $\beta_0=\min\limits_{i,j}\beta_{i,j}$, the system of functional equations (2.14)–(2.16) *has at least one solution $\psi=(\psi_1,\psi_2,\psi_3)$ in the space $\overset{0}{C}\left(\bar{\Omega}\right)$. If, in addition, $\lambda_1(z_1,z_2)\equiv 0$, then the solution is unique in $\overset{0}{C}\left(\bar{\Omega}\right)$.*

Proof. Let us rewrite system (2.14)–(2.16) in the form of one vector functional equation

$$\psi(z_1,z_2)+(T\psi)(z_1,z_2)=g(z_1,z_2),\ z=(z_1,z_2)\in\overline{\Omega}, \tag{2.32}$$

where $\psi=(\psi_1,\psi_2,\psi_3)$, $g=(g_1,g_2,g_3)$, and the operator $T:\overset{0}{C}\left(\bar{\Omega}\right)\to\overset{0}{C}\left(\bar{\Omega}\right)$ acts by formula

$$\begin{aligned}(T\psi)(z_1,z_2)=(&\psi_2(z_1,\lambda_3(z_1,z_2))+\psi_3(z_2,\lambda_3(z_1,z_2)),\\ &\psi_1(z_1,\lambda_2(z_1,z_2))+\psi_3(\lambda_2(z_1,z_2),z_2),\\ &\psi_1(\lambda_1(z_1,z_2),z_1)+\psi_2(\lambda_1(z_1,z_2),z_2)).\end{aligned} \tag{2.33}$$

Since $F \in C\left(\bar{D}\right)$ and $f_i \in \overset{0}{C}_\alpha\left(\bar{\Omega}\right)$, $i = 1, 2, 3$, and due to (2.17) and (2.30), it is easy to see that

$$
\begin{aligned}
\left|g_1\left(z_1, z_2\right)\right| &\leq z_1^\alpha z_2^\alpha \left\|f_3\right\|_{\overset{0}{C}_\alpha\left(\bar{\Omega}\right)} + z_1 z_2 \lambda_3\left(z_1, z_2\right)\|F\|_{C\left(\bar{D}\right)} \\
&\leq z_1^\alpha z_2^\alpha \left\|f_3\right\|_{\overset{0}{C}_\alpha\left(\bar{\Omega}\right)} + M_3 z_1^{1+\beta_{31}} z_2^{1+\beta_{32}}\|F\|_{C\left(\bar{D}\right)} \\
&\leq z_1^{\alpha_0} z_2^{\alpha_0}\left[z_1^{\alpha-\alpha_0} z_2^{\alpha-\alpha_0}\left\|f_3\right\|_{\overset{0}{C}_\alpha\left(\bar{\Omega}\right)} + M_3 z_1^{1+\beta_{31}-\alpha_0} z_2^{1+\beta_{32}-\alpha_0}\|F\|_{C\left(\bar{D}\right)}\right],
\end{aligned}
$$

whence by the fact that $\alpha_0 = \min\left(\alpha, \beta_0 + 1\right)$, we have $g_1 \in \overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)$. Similarly, we obtain that

$$
g_i \in \overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right), \quad 1 \leq i \leq 3. \tag{2.34}
$$

According to (2.31), there exists a positive number q such that

$$
\max_{1 \leq i \leq 3} 2 M_i^{\alpha_0} < q, \quad 0 < q = \text{const} < 1. \tag{2.35}
$$

It is easy to see that the operator T in (2.33) is a linear continuous operator acting in the space $\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)$. Now let us consider the norm of the operator $T : \overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right) \to \overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)$. In view of (2.30) and (2.31), with $z = \left(z_1, z_2\right) \in \Omega$, i.e., when $0 < z_i < 1$, $i = 1, 2$, we have

$$
\begin{aligned}
\left|(T\psi)_1\left(z_1, z_2\right)\right| &= \left|\psi_2\left(z_1, \lambda_3\left(z_1, z_2\right)\right) + \psi_3\left(z_2, \lambda_3\left(z_1, z_2\right)\right)\right| \\
&\leq z_1^{\alpha_0} \lambda_3^{\alpha_0}\left(z_1, z_2\right)\left\|\psi_2\right\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)} + z_2^{\alpha_0} \lambda_3^{\alpha_0}\left(z_1, z_2\right)\left\|\psi_3\right\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)} \\
&\leq z_1^{\alpha_0} M_3^{\alpha_0} z_1^{\alpha_0\beta_{31}} z_2^{\alpha_0\beta_{32}}\left\|\psi_2\right\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)} + z_2^{\alpha_0} M_3^{\alpha_0} z_1^{\alpha_0\beta_{31}} z_2^{\alpha_0\beta_{32}}\left\|\psi_3\right\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)} \\
&\leq z_1^{\alpha_0} z_2^{\alpha_0}\left(\max_{1 \leq i \leq 3} M_i^{\alpha_0}\right)\left[z_1^{\alpha_0\beta_{31}} z_2^{\alpha_0\left(\beta_{32}-1\right)} + z_1^{\alpha_0\left(\beta_{31}-1\right)} z_2^{\alpha_0\beta_{32}}\right]\left(\max_{1 \leq i \leq 3}\left\|\psi_i\right\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)}\right) \\
&< q z_1^{\alpha_0} z_2^{\alpha_0} \max_{1 \leq i \leq 3}\left\|\psi_i\right\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)} = q z_1^{\alpha_0} z_2^{\alpha_0}\|\psi\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)}.
\end{aligned} \tag{2.36}
$$

Similarly, we obtain

$$
\left|(T\psi)_i\left(z_1, z_2\right)\right| < q z_1^{\alpha_0} z_2^{\alpha_0}\|\psi\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)}, \ \left(z_1, z_2\right) \in \Omega, \quad i = 2, 3. \tag{2.37}
$$

It follows from (2.36) and (2.37) that

$$
\|T\|_{\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right) \to \overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)} \leq q < 1. \tag{2.38}
$$

Due to (2.38), the operator $(I + T)$ is invertible in the space $\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)$ and the inverse operator $(I + T)^{-1}$ is represented in the form of the operator series converging in $\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)$:

$$
(I + T)^{-1} = \sum_{n=0}^{\infty}(-T)^n,
$$

and in view of (2.34), in the space $\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right)$, there exists a unique solution of Eq. (2.32) representable in the form

$$
\psi = \sum_{n=0}^{\infty}(-T)^n g. \tag{2.39}
$$

Since $\overset{0}{C}_{\alpha_0}\left(\bar{\Omega}\right) \subset \overset{0}{C}\left(\bar{\Omega}\right)$, the function ψ in (2.39) is also a solution of the system of functional equations (2.14)–(2.16) in the space $\overset{0}{C}\left(\bar{\Omega}\right)$.

Now let us show that if the additional requirement $\lambda_1(z_1,z_2)\equiv 0$ holds, then the system of functional equations (2.14)–(2.16) cannot have more than one solution in the space $\overset{0}{C}(\bar\Omega)$. Actually, in this case, with $g_i=0$, $i=1,2,3$, in view of (2.12) and (2.13), from (2.16) it follows that $\psi_3=0$, and from (2.14) and (2.15) we obtain the following system of homogeneous equations with respect to unknown functions ψ_1 and ψ_2:

$$\begin{aligned}\psi_1(z_1,z_2)+\psi_2(z_1,\lambda_3(z_1,z_2))=0,\ z\in\overline{\Omega},\\ \psi_2(z_1,z_2)+\psi_1(z_1,\lambda_2(z_1,z_2))=0,\ z\in\overline{\Omega}.\end{aligned}\tag{2.40}$$

Eliminating function ψ_2 from system (2.40), we obtain the following functional equation with respect to ψ_1 in the space $\overset{0}{C}(\bar\Omega)$:

$$\psi_1(z_1,z_2)-\psi_1(z_1,\lambda_2(z_1,\lambda_3(z_1,z_2)))=0,\quad z\in\overline{\Omega}.\tag{2.41}$$

Let us consider the mapping $J:\bar\Omega\to\bar\Omega$ acting by the formula

$$J(z_1,z_2)=(z_1,\lambda_2(z_1,\lambda_3(z_1,z_2))),\ z\in\overline{\Omega}.\tag{2.42}$$

It follows from (2.41) and (2.42) that

$$\psi_1(z)=\psi_1(J^n(z)),\quad z=(z_1,z_2)\in\overline{\Omega},\quad n=1,2,\dots.\tag{2.43}$$

Since $0\le z_i\le 1$, $i=1,2$, for $z=(z_1,z_2)\in\overline{\Omega}$, we obtain from (2.30)

$$0\le\lambda_i(z_1,z_2)\le M_i z_2,\quad (z_1,z_2)\in\bar\Omega,\quad i=2,3,\tag{2.44}$$

where, according to (2.31),

$$0\le\max(M_2,M_3)=k<1.\tag{2.45}$$

From (2.42), (2.44), and (2.45) we have

$$J^n(z)=(z_1,J_n(z)),\quad 0\le J_n(z)\le k^{2n}z_2,\quad z\in\bar\Omega,\quad n=1,2,\dots.\tag{2.46}$$

In turn, from (2.46) it follows that

$$\lim_{n\to\infty}J_n(z)=0,\quad z\in\bar\Omega,\quad n=1,2,\dots.\tag{2.47}$$

Since by (2.12) and (2.13) we have $\psi_1(z_1,0)=\varphi_{12}(z_1,0)=0$, taking into account the continuity of the function $\psi_1(z_1,z_2)$, from (2.43), (2.46), and (2.47) we have

$$\begin{aligned}\psi_1(z)&=\lim_{n\to\infty}\psi_1(z)=\lim_{n\to\infty}\psi_1(J^n(z))=\lim_{n\to\infty}\psi_1(z_1,J_n(z))\\ &=\psi_1\Big(z_1,\lim_{n\to\infty}J_n(z)\Big)=\psi_1(z_1,0)=0,\quad z\in\bar\Omega.\end{aligned}\tag{2.48}$$

The second equation of system (2.40) and Eq. (2.48) imply that $\psi_2(z)=0$, $z\in\bar\Omega$. Thus, if $g_i=0$, $i=1,2,3$, then system (2.14)–(2.16) can have only a trivial solution in the space $\overset{0}{C}(\bar\Omega)$. The second part of Lemma 2 is proved. □

Remark 2. Similarly to Remark 1, by Lemma 1 and the reasoning above, we see that for $F\in C(\bar D)$ and $f_i\in\overset{0}{C}(\bar\Omega)$, $i=1,2,3$, problem (2.1), (2.2) has a strong generalized solution in the class $\overset{0}{C}(\bar D)$ if and only if the system of functional equations (2.14)–(2.16) has a solution in the space $\overset{0}{C}(\bar\Omega)$.

Lemma 2 and Remark 2 imply the following assertion.

Theorem 1. *Let conditions* (2.30) *and* (2.31) *be satisfied. Then for any* $F\in C(\bar D)$ *and* $f_i\in\overset{0}{C}_\alpha(\bar\Omega)$, $\alpha=\mathrm{const}>0$, $i=1,2,3$, *problem* (2.1), (2.2) *has a strong generalized solution in* $\overset{0}{C}(\bar D)$, *which is unique in this class if the additional condition* $\lambda_1=0$ *holds.*

Below we consider the question on the classical solvability of problem (2.1), (2.3) in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ for the case where for any $z=(z_1,z_2)\in\overline{\Omega}$,

$$\lambda_1(z_1,z_2)\equiv 0,\quad \lambda_i(z_1,z_2)=\lambda_i(z_2)\in C^1([0,1]),\tag{2.49}$$
$$i=2,3;\quad \lambda_i(0)=0,\quad i=2,3,$$

where, as for the setting of problem (1.3), (1.4), we assume that $0\le\lambda_i\le 1$, $i=2,3$.

Under the conditions of (2.49) and due to (2.12) and (2.13), the function ψ_3 in (2.16) is defined uniquely:

$$\psi_3(z_1,z_2)=g_3(z_1,z_2),\quad z=(z_1,z_2)\in\bar{\Omega},\tag{2.50}$$

and with respect to the unknown functions ψ_1 and ψ_2, system (2.14), (2.15) has the form

$$\psi_1(z_1,z_2)+\psi_2(z_1,\lambda_3(z_2))=\tilde{g}_1(z_1,z_2),\quad z=(z_1,z_2)\in\overline{\Omega},\tag{2.51}$$

$$\psi_2(z_1,z_2)+\psi_1(z_1,\lambda_2(z_2))=\tilde{g}_2(z_1,z_2),\quad z=(z_1,z_2)\in\overline{\Omega}.\tag{2.52}$$

Here

$$\begin{aligned}\tilde{g}_1(z_1,z_2)&=g_1(z_1,z_2)-g_3(z_2,\lambda_3(z_2)),\\ \tilde{g}_2(z_1,z_2)&=g_2(z_1,z_2)-g_3(\lambda_2(z_2),z_2).\end{aligned}\tag{2.53}$$

Below, we assume that $\tilde{g}_i\in\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, $i=1,2$.

Remark 3. It is easy to verify that the function g from the space $\overset{0}{C}{}^{1,1,1}\left(\bar{\Omega}\right)$ is uniquely defined by the mixed derivative $\dfrac{\partial^2 g}{\partial z_1\partial z_2}\in C\left(\bar{\Omega}\right)$ as follows:

$$g(z_1,z_2)=\int_0^{z_1}\int_0^{z_2}\frac{\partial^2 g(\xi_1,\xi_2)}{\partial z_1\partial z_2}d\xi_1 d\xi_2,\ \ z\in(z_1,z_2)\in\bar{\Omega}.\tag{2.54}$$

Remark 4. Taking into account Remark 3 and Eq. (2.54) and introducing the notation

$$\omega=\frac{\partial^2\psi_1}{\partial z_1\partial z_2},\quad \omega_1=\frac{\partial^2\psi_2}{\partial z_1\partial z_2},\tag{2.55}$$

we see that system (2.51), (2.52) in the space $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$ is equivalent to the following system:

$$\omega(z_1,z_2)+\lambda_3'(z_2)\,\omega_1(z_1,\lambda_3(z_2))=\frac{\partial^2\tilde{g}_1(z_1,z_2)}{\partial z_1\partial z_2},\tag{2.56}$$

$$\omega_1(z_1,z_2)+\lambda_2'(z_2)\,\omega(z_1,\lambda_2(z_2))=\frac{\partial^2\tilde{g}_2(z_1,z_2)}{\partial z_1\partial z_2}\tag{2.57}$$

for the unknown functions ω and ω_1 in the space $C\left(\bar{\Omega}\right)$.

Excluding the function ω_1, from system (2.56), (2.57), relative to the unknown function $\omega\in C\left(\bar{\Omega}\right)$, we obtain the functional equation

$$\omega(z_1,z_2)-a(z_2)\,\omega(z_1,\lambda(z_2))=g(z_1,z_2),\ \ z=(z_1,z_2)\in\bar{\Omega},\tag{2.58}$$

where

$$a(z_2)=\lambda_3'(z_2)\,\lambda_2'(\lambda_3(z_2)),\quad g(z_1,z_2)=\frac{\partial^2\tilde{g}_1(z_1,z_2)}{\partial z_1\partial z_2},\quad -\lambda_2'(z_2)\,\frac{\partial^2\tilde{g}_2(z_1,\lambda_3(z_2))}{\partial z_1\partial z_2}\tag{2.59}$$

and

$$\lambda(z_2)=\lambda_2(\lambda_3(z_2)).\tag{2.60}$$

Remark 5. If conditions (2.49) hold, taking into account Remarks 3 and 4, we easily see that problem (2.1), (2.2) in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ for $F \in C(\bar{D})$ and $f_i \in \overset{0}{C}{}^{1,1}(\bar{\Omega})$, $i = 1, 2, 3$, is equivalent to the functional equation (2.58) in the space $C(\bar{\Omega})$, where a, g, and λ are given by Eqs. (2.59) and (2.60). On the other hand, considering the variable z_1 as a parameter, for fixed $z_1 \in [0, 1]$, the functional equation (2.58) relative to the variable z_2 in the space of continuous functions is investigated in [2].

The results of [2], as is obvious from the proofs, are valid also for Eq. (2.58). Therefore, below we will formulate these results without proof, assuming that a and λ are arbitrary functions of the class $C([0, 1])$, having no relation to Eqs. (2.59) and (2.60).

Lemma 3. *If the condition*

$$|a(z_2)| < 1, \quad 0 \le z_2 \le 1$$

holds, then Eq. (2.58) *is uniquely solvable in the space* $C\left(\bar{\Omega}\right)$*, i.e., for any function* $g \in C\left(\bar{\Omega}\right)$*, Eq.* (2.58) *has a unique solution* $\omega \in C\left(\bar{\Omega}\right)$*. In this case, the estimate*

$$\|\omega\|_{C(\bar{\Omega})} \le \frac{1}{1-q} \|g\|_{C(\bar{\Omega})},$$

holds, where $q = \max\limits_{0 \le z_2 \le 1} |a(z_2)| < 1$.

Lemma 4. *Let a continuous mapping* $\lambda : [0, 1] \to [0, 1]$ *in* (2.60) *strictly monotonically increase. Denote by* $I := \{z_2 \in [0, 1] : \lambda(z_2) = z_2\}$ *the set of fixed points of the mapping. If the condition*

$$|a(z_2)| < 1, \quad z_2 \in I,$$

holds, then for any function $g \in C\left(\bar{\Omega}\right)$*, Eq.* (2.58) *has a unique solution* $\omega \in C\left(\bar{\Omega}\right)$*, for which the following estimate holds*:

$$\|\omega\|_{C(\bar{\Omega})} \le c \|g\|_{C(\bar{\Omega})}, \tag{2.61}$$

where c is a positive constant independent of g.

Lemma 5. *Let a continuous mapping* $\lambda : [0, 1] \to [0, 1]$ *in* (2.60) *strictly monotonically increase and*

$$\lambda(0) = 0, \quad \lambda(z_2) < z_2$$

for $0 < z_2 \le 1$*. If* $|a(0)| < 1$*, then for any function* $g \in C\left(\bar{\Omega}\right)$*, Eq.* (2.58) *has a unique solution* $\omega \in C\left(\bar{\Omega}\right)$*, for which estimate* (2.61) *holds. But, if* $|a(0)| > 1$*, then Eq.* (2.58) *for any function* $g \in C\left(\bar{\Omega}\right)$ *has a solution* $\omega \in C\left(\bar{\Omega}\right)$*, although the homogeneous equation corresponding to* (2.58) *has an infinite set of linearly independent solutions in the class of continuous functions* $C\left(\bar{\Omega}\right)$*.*

Lemma 6. *If* $\lambda : [0, 1] \to [0, 1]$ *is a continuous homeomorphism and the condition*

$$|a(z_2)| \ne 1, \quad 0 \le z_2 \le 1,$$

holds, then Eq. (2.58) *is uniquely solvable in the class* $C\left(\bar{\Omega}\right)$.

Lemma 7. *Let* $\lambda : [0, 1] \to [0, 1]$ *be a continuous homeomorphism that leaves the endpoints of the segment fixed. If the condition*

$$|a(z_2)| > 1 \quad \forall z_2 \in I := \{z_2 \in [0, 1] : \lambda(z_2) = z_2\}$$

holds, then for any function $g \in C\left(\bar{\Omega}\right)$*, Eq.* (2.58) *has a unique solution* $\omega \in C\left(\bar{\Omega}\right)$ *with estimate* (2.61)*.*

Lemma 8. *Let a continuous mapping* $\lambda : [0, 1] \to [0, 1]$ *strictly monotonically increase,* $\lambda\left(z_2^0\right) = z_2^0$ *for a certain number* $z_2^0 \in (0, 1)$*, and* $\lambda(z_2) < z_2$ *with* $z_2^0 < z_2 \le 1$*. Let the condition*

$$|a(z_2)| \ne 1, \quad 0 \le z_2 \le z_2^0,$$

hold. If

$$\left|a\left(z_2^0\right)\right| < 1$$

for any function $g \in C\left(\bar{\Omega}\right)$*, then Eq.* (2.58) *has a unique solution* $\omega \in C\left(\bar{\Omega}\right)$ *with estimate* (2.61)*. But if*

$$\left|a\left(z_2^0\right)\right| > 1,$$

then for any function $g \in C\left(\bar{\Omega}\right)$*, Eq.* (2.58) *has a solution* $\omega \in C\left(\bar{\Omega}\right)$*, although the homogeneous equation corresponding to* (2.58) *has an infinite set of linearly independent solutions in the class of continuous functions* $C\left(\bar{\Omega}\right)$*.*

Remark 6. If $\lambda(z_2) \le z_2$ for all $z_2 \in [0,1]$, then under the conditions of Lemmas 3 and 4, and also Lemma 5 with $|a(0)| < 1$, the unique solution ω of Eq. (2.58), together with estimate (2.61), also satisfies an estimate of the form

$$|\omega(z_1, z_2)| \le c \max_{0 \le \xi \le z_2} |g(z_1, \xi)| \quad \forall z \in (z_1, z_2) \in \bar{\Omega} \tag{2.62}$$

with a positive constant c independent of g and $z = (z_1, z_2) \in \bar{\Omega}$.

Remarks 5 and 6 and Lemmas 3–8, where $a(z_2) = \lambda_3'(z_2)\,\lambda_2'(\lambda_3(z_2)) = [\lambda_2(\lambda_3(z_2))]'$, imply the following theorems.

Theorem 2. *Let conditions* (2.49) *and* $|\lambda'(z_2)| < 1$, $0 \le z_2 \le 1$*, hold, where* $\lambda(z_2) = \lambda_2(\lambda_3(z_2))$*. Then for any* $F \in C\left(\bar{D}\right)$ *and* $f_i \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$*,* $i = 1,2,3$*, problem* (2.1)*,* (2.2) *has a unique solution* u *in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$*. If, in addition,* $\lambda_i(z_2) \le z_2$*,* $z_2 \in [0,1]$*,* $i = 2,3$*, then for a solution* $u \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ *of problem* (2.1)*,* (2.2) *we have the estimate*

$$\|u\|_{\overset{0}{C}{}^{1,1,1}\left(\bar{D}_P\right)} \le c\left[\|F\|_{C\left(\bar{D}_P\right)} + \sum_{i=1}^{3} \|f_i\|_{\overset{0}{C}{}^{1,1}\left(\bar{\Omega}_{P_i}\right)}\right] \quad \forall P \in D \tag{2.63}$$

with a positive constant c *independent of* F*,* f_i*,* $i = 1,2,3$*, and the point* $P = P\left(x_1^0, x_2^0, x_3^0\right) \in D$*, where*

$$D_P := \left\{(x_1,x_2,x_3) \in D : x_i < x_i^0,\ i = 1,2,3\right\}, \quad \Omega_{P_1} := \left\{(z_2,z_3) \in \Omega : z_i < x_i^0,\ i = 2,3\right\},$$

$$\Omega_{P_2} := \left\{(z_1,z_3) \in \Omega : z_i < x_i^0,\ i = 1,3\right\}, \quad \Omega_{P_3} := \left\{(z_1,z_2) \in \Omega : z_i < x_i^0,\ i = 1,2\right\}.$$

Theorem 3. *Let conditions* (2.49) *hold and let a continuous mapping* $\lambda : [0,] \to [0,1]$ *strictly monotonically increase with* $\lambda(z_2) < z_2$ *for* $0 < z_2 \le 1$*. If* $|\lambda'(0)| < 1$*, then for any* $F \in C\left(\bar{D}\right)$ *and* $f_i \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$*,* $i = 1,2,3$*, problem* (2.1)*,* (2.2) *has a unique solution* u *in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$*. Under the additional conditions* $\lambda_i(z_2) \le z_2$*,* $z_2 \in [0,1]$*,* $i = 2,3$*, the solution* $u \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ *of problem* (2.1)*,* (2.2) *satisfies estimate* (2.63)*. If* $|\lambda'(0)| > 1$*, then for any* $F \in C\left(\bar{D}\right)$ *and* $f_i \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$*,* $i = 1,2,3$*, problem* (2.1)*,* (2.2) *has a solution* $u \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$*, although the homogeneous problem corresponding to* (2.1)*,* (2.2) *has an infinite set of linearly independent solutions in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$*.*

Remark 7. Note that if conditions (2.49) hold, then an analog of Theorem 1 is also valid. If the conditions $\lambda_i(0) = 0$, $i = 1,2$; $\lambda(z_2) = \lambda_2(\lambda_3(z_2)) \le M z_2^{\beta}$, $0 \le z_2 \le 1$; $0 \le M = \text{const} < 1$, $\beta = \text{const} \ge 1$, hold together with (2.49), then for any $F \in C\left(\bar{D}\right)$ and $f_i \in \overset{0}{C}{}_{\alpha}\left(\bar{\Omega}\right)$, $\alpha = \text{const} > 0$, $i = 1,2,3$, problem (2.1), (2.2) has a unique strong generalized solution in the class $\overset{0}{C}\left(\bar{D}\right)$.

3. Investigation of Problem (1.3), (1.4) in the Case Where Equation (1.3) and Boundary Conditions (1.4) Do Not Have Low-Order Terms

In the domain $D := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_i < 1,\ i = 1, 2, 3\}$, let us consider the following version of problem (1.3), (1.4):

$$\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = F, \tag{3.1}$$

$$\left.\left(\sum_{j=1}^{3} l_{ij} \frac{\partial u}{\partial x_j}\right)\right|_{S_i} = f_i, \quad i = 1, 2, 3, \tag{3.2}$$

where $\sum\limits_{j=1}^{3} |l_{ij}| \neq 0$, $i = 1, 2, 3$, with the assumption $l_{ij} = \text{const}$, $1 \leq i, j \leq 3$.

Considering problem (3.1), (3.2) in the class $\overset{0}{C}{}^{1,1,1}(\bar{D})$, substituting representation (2.3) of a solution of Eq. (3.1) in conditions (3.2), and taking into account Eqs. (2.12), with respect to the unknown functions $\varphi_{ij} \in \overset{0}{C}{}^{1,1}(\bar{\Omega})$, $1 \leq i, j \leq 3$, $i < j$, we obtain the following system of functional-differential equations:

$$\begin{aligned} &l_{11} \frac{\partial \varphi_{12}}{\partial x_1}(\lambda_1(x_2, x_3), x_2) + l_{12} \frac{\partial \varphi_{12}}{\partial x_2}(\lambda_1(x_2, x_3), x_2) + l_{11} \frac{\partial \varphi_{13}}{\partial x_1}(\lambda_1(x_2, x_3), x_3) \\ &+ l_{13} \frac{\partial \varphi_{13}}{\partial x_3}(\lambda_1(x_2, x_3), x_3) + l_{12} \frac{\partial \varphi_{23}}{\partial x_2}(x_2, x_3) + l_{13} \frac{\partial \varphi_{23}}{\partial x_3}(x_2, x_3) = \tilde{f}_1(x_2, x_3), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &l_{21} \frac{\partial \varphi_{12}}{\partial x_1}(x_1, \lambda_2(x_1, x_3)) + l_{22} \frac{\partial \varphi_{12}}{\partial x_2}(x_1, \lambda_2(x_1, x_3)) + l_{21} \frac{\partial \varphi_{13}}{\partial x_1}(x_1, x_3) + l_{23} \frac{\partial \varphi_{13}}{\partial x_3}(x_1, x_3) \\ &+ l_{22} \frac{\partial \varphi_{23}}{\partial x_2}(\lambda_2(x_1, x_3), x_3) + l_{23} \frac{\partial \varphi_{23}}{\partial x_3}(\lambda_2(x_1, x_3), x_3) = \tilde{f}_2(x_1, x_3), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &l_{31} \frac{\partial \varphi_{12}}{\partial x_1}(x_1, x_2) + l_{32} \frac{\partial \varphi_{12}}{\partial x_2}(x_1, x_2) + l_{31} \frac{\partial \varphi_{13}}{\partial x_1}(x_1, \lambda_3(x_1, x_2)) + l_{33} \frac{\partial \varphi_{13}}{\partial x_3}(x_1, \lambda_3(x_1, x_2)) \\ &+ l_{32} \frac{\partial \varphi_{23}}{\partial x_2}(x_2, \lambda_3(x_1, x_2)) + l_{33} \frac{\partial \varphi_{23}}{\partial x_3}(x_2, \lambda_3(x_2, x_3)) = \tilde{f}_3(x_1, x_2), \end{aligned} \tag{3.5}$$

where $\tilde{f}_i$, $i = 1, 2, 3$, are certain functions expressed as functions of F and f_i, $i = 1, 2, 3$.

Using notation (2.13), let us rewrite system (3.3)–(3.5) in the form

$$\begin{aligned} &l_{31} \frac{\partial \psi_1}{\partial z_1}(z_1, z_2) + l_{32} \frac{\partial \psi_1}{\partial z_2}(z_1, z_2) + l_{31} \frac{\partial \psi_2}{\partial z_1}(z_1, \lambda_3(z_1, z_2)) + l_{33} \frac{\partial \psi_2}{\partial z_2}(z_1, \lambda_3(z_1, z_2)) \\ &+ l_{32} \frac{\partial \psi_3}{\partial z_1}(z_2, \lambda_3(z_1, z_2)) + l_{33} \frac{\partial \psi_3}{\partial z_2}(z_2, \lambda_3(z_1, z_2)) = \tilde{f}_3(z_1, z_2), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &l_{21} \frac{\partial \psi_1}{\partial z_1}(z_1, \lambda_2(z_1, z_2)) + l_{22} \frac{\partial \psi_1}{\partial z_2}(z_1, \lambda_2(z_1, z_2)) + l_{21} \frac{\partial \psi_2}{\partial z_1}(z_1, z_2) + l_{23} \frac{\partial \psi_2}{\partial z_2}(z_1, z_2) \\ &+ l_{22} \frac{\partial \psi_3}{\partial z_1}(\lambda_2(z_1, z_2), z_2) + l_{23} \frac{\partial \psi_3}{\partial z_2}(\lambda_2(z_1, z_2), z_2) = \tilde{f}_2(z_1, z_2), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &l_{11} \frac{\partial \psi_1}{\partial z_1}(\lambda_1(z_1, z_2), z_1) + l_{12} \frac{\partial \psi_1}{\partial z_2}(\lambda_1(z_1, z_2), z_1) + l_{11} \frac{\partial \psi_2}{\partial z_1}(\lambda_1(z_1, z_2), z_2) \\ &+ l_{13} \frac{\partial \psi_2}{\partial z_2}(\lambda_1(z_1, z_2), z_2) + l_{12} \frac{\partial \psi_3}{\partial z_1}(z_1, z_2) + l_{13} \frac{\partial \psi_3}{\partial z_2}(z_1, z_2) = \tilde{f}_1(z_1, z_2), \end{aligned} \tag{3.8}$$

where $z = (z_1, z_2) \in \bar{\Omega}$.

Below we assume that conditions (2.49) hold and

$$l_{i1} = 0, \quad i = 1, 2, 3, \tag{3.9}$$

$$f_1 \in C^2\left(\bar{\Omega}\right); \quad f_i,\ \frac{\partial f_i}{\partial x_1} \in C\left(\bar{\Omega}\right), \quad i = 2, 3; \quad F,\ \frac{\partial F}{\partial x_2},\ \frac{\partial F}{\partial x_3} \in C\left(\bar{D}\right). \tag{3.10}$$

If conditions (2.49) and (3.9) hold, taking into account (2.12) and (2.13), we see that Eq. (3.8) takes the form

$$l_{12}\frac{\partial \psi_3}{\partial z_1}(z) + l_{13}\frac{\partial \psi_3}{\partial z_2}(z) = \tilde{f}_1(z), \quad z = (z_1, z_2) \in \bar{\Omega}, \tag{3.11}$$

where, as is easy to verify,

$$\tilde{f}_1(z_1, z_2) = f_1(z_1, z_2) - l_{11}\int\limits_0^{z_1}\int\limits_0^{z_2} F(0, \eta, \varsigma)\, d\eta\, d\varsigma. \tag{3.12}$$

As is known (see [14, p. 255]), for the well-posedness of the Cauchy problem for Eq. (3.11) with zero initial conditions, for $z_1 = 0$ and $z_2 = 0$, it is necessary to require that

$$l_{12} \cdot l_{13} > 0. \tag{3.13}$$

It is easy to verify that if conditions (3.10) hold, the function $\tilde{f}$ in (3.12) belongs to class $C^2\left(\bar{\Omega}\right)$, and then in the case of (3.13), Eq. (3.11) has a unique solution ψ_3 in the space $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, where one must take into account the zero Cauchy initial conditions with $z_1 = 0$, $z_2 = 0$, and, of course, if necessary, compatibility conditions hold at the point $(0, 0) \in \bar{\Omega}$:

$$f_1(0, 0) = \frac{\partial f_1}{\partial z_1}(0, 0) = \frac{\partial f_1}{\partial z_2}(0, 0) = 0. \tag{3.14}$$

This solution can easily be constructed in quadratures. After the unknown function ψ_3 is found, system (3.6), (3.7), under conditions (2.49) and (3.9), relative to the unknown functions ψ_1 and ψ_2, can be rewritten as follows:

$$\begin{aligned} &l_{32}\frac{\partial \psi_1}{\partial z_2}(z_1, z_2) + l_{33}\frac{\partial \psi_2}{\partial z_2}(z_1, \lambda_3(z_2)) = f_4(z_1, z_2), \\ &l_{23}\frac{\partial \psi_2}{\partial z_2}(z_1, z_2) + l_{22}\frac{\partial \psi_1}{\partial z_2}(z_1, \lambda_2(z_2)) = f_5(z_1, z_2), \end{aligned} \tag{3.15}$$

where $f_4, f_5 \in C^1\left(\bar{\Omega}\right)$ are expressed in a certain form by the known functions F and f_i, $i = 1, 2, 3$. In this case, we assume that $f_4(0, z_2) = f_5(0, z_2) = 0$, $0 \le z_2 \le 1$.

Below we assume that

$$l_{32} \neq 0, \quad l_{23} \neq 0. \tag{3.16}$$

Differentiating the equations of system (3.15) with respect to z_1, in notation (2.55), we obtain

$$\begin{aligned} &l_{32}\omega(z_1, z_2) + l_{33}\omega_1(z_1, \lambda_3(z_2)) = \frac{\partial f_4(z_1, z_2)}{\partial z_1}, \\ &l_{23}\omega_1(z_1, z_2) + l_{22}\omega(z_1, \lambda_2(z_2)) = \frac{\partial f_5(z_1, z_2)}{\partial z_1}. \end{aligned} \tag{3.17}$$

Under assumption (3.16), excluding the unknown function ω_1 from system (3.17), relative to the unknown function ω we obtain the equation

$$\omega(z_1, z_2) - \frac{l_{33}l_{22}}{l_{32}l_{23}}\omega(z_1, \lambda(z_2)) = f(z_1, z_2), \ \ z = (z_1, z_2) \in \bar{\Omega}, \tag{3.18}$$

where $\lambda(z_2) = \lambda_2(\lambda_3(z_2))$ and $f \in C\left(\bar{\Omega}\right)$ are expressed by known functions F and f_i, $i = 1, 2, 3$.

Remark 8. Taking into account Remark 4 and the assumptions made above, we see that problem (3.1), (3.2) in the class $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ is equivalent to Eq. (3.18) in the class $C\left(\bar{\Omega}\right)$. Equation (3.18) is a special case of Eq. (2.58) with $a(z_2) = \frac{l_{33}l_{22}}{l_{32}l_{23}} = \text{const}$ for which Lemmas 3–8 were formulated. Therefore, for problem (3.1), (3.2) the following theorems hold.

Theorem 4. *Let conditions* (2.49), (3.9), (3.13), *and* (3.16) *hold. If* $|l_{33}l_{22}| < |l_{32}l_{23}|$, *then for any* F *and* f_i, $i = 1, 2, 3$, *which satisfy conditions* (3.10), *problem* (3.1), (3.2) *has a unique solution* u *in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$. *If, in addition,* $\lambda_i(z_2) \le z_2$, $z_2 \in [0, 1]$, $i = 2, 3$, *then for the solution* $u \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ *of problem (3.1), (3.2), we have the estimate*

$$\|u\|_{\overset{0}{C}{}^{1,1,1}(\bar{D}_P)} \le c\left[\|F\|_{C(\bar{D}_P)} + \sum_{i=2}^{3}\left\|\frac{\partial F}{\partial x_i}\right\|_{C(\bar{D}_P)}\right.$$

$$\left. + \|f_1\|_{C^2(\bar{\Omega}_{P_1})} + \sum_{i=1}^{2}\left(\|f_i\|_{C(\bar{\Omega}_{P_i})} + \left\|\frac{\partial f_i}{\partial x_1}\right\|_{C(\bar{\Omega}_{P_i})}\right)\right] \quad \forall P \in D \tag{3.19}$$

with a positive constant c *independent of* F, f_i, $i = 1, 2, 3$, *and the point* $P(x_1^0, x_2^0, x_3^0) \in D$, *where*

$$D_p := \left\{(x_1, x_2, x_3) \in D : x_i < x_i^0,\ i = 1, 2, 3\right\}, \quad \Omega_{p_1} := \{(z_2, z_3) \in \Omega : z_i < x_i^0,\ i = 2, 3\},$$

$$\Omega_{p_2} := \left\{(z_1, z_3) \in \Omega : z_i < x_i^0,\ , i = 1, 3\right\}, \quad \Omega_{p_3} := \{(z_1, z_2) \in \Omega : z_i < x_i^0,\ i = 1, 2\}.$$

If $|l_{33}l_{22}| > |l_{32}l_{23}|$, *the continuous mapping* $\lambda : [0, 1] \to [0, 1]$ *in* (3.18) *strictly monotonically increases, and* $\lambda(z_2) < z_2$, $0 < z_2 \le 1$, *then problem* (3.1), (3.2) *for any* F *and* f_i, $i = 1, 2, 3$, *which satisfy conditions* (3.10), *has a solution* $u \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$, *although the homogeneous problem corresponding to* (3.1), (3.2), *has an infinite set of linearly independent solutions in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$.

Theorem 5. *Let conditions* (2.49), (3.9), (3.13), *and* (3.16) *hold, a continuous mapping* $\lambda : [0, 1] \to [0, 1]$ *be a homeomorphism, and the condition* $|l_{33}l_{22}| \ne |l_{32}l_{23}|$ *hold. Then problem* (3.1), (3.2) *with any* F *and* f_i, $i = 1, 2, 3$, *which satisfy conditions* (3.10), *has a unique solution* u *in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$.

Finally, for Eq. (3.1) in the domain $D : 0 < x_i < 1$, $i = 1, 2, 3$, consider the problem in which, instead of boundary conditions (3.2), the following conditions are posed:

$$u|_{S_i} = f_i, \quad i = 1, 3; \qquad \left.\left(\sum_{j=1}^{3} l_{2j}\frac{\partial u}{\partial x_j}\right)\right|_{S_2} = f_2. \tag{3.20}$$

In this case, in notation (2.13), problem (3.1), (3.20) in the class $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ is reduced to the system of Eqs. (2.14), (2.16), and (3.7) for the unknown functions $\psi_i \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, $i = 1, 2, 3$. Due to (2.49), this system, in view of (2.12) and (2.13), takes the form

$$\psi_1(z_1, z_2) + \psi_2(z_1, \lambda_3(z_2)) = g_4(z_1, z_2), \tag{3.21}$$

$$l_{21}\frac{\partial \psi_1}{\partial z_1}(z_1, \lambda_2(z_2)) + l_{22}\frac{\partial \psi_1}{\partial z_2}(z_1, \lambda_2(z_2)) + l_{21}\frac{\partial \psi_2}{\partial z_1}(z_1, z_2) + l_{23}\frac{\partial \psi_2}{\partial z_2}(z_1, z_2) = g_5(z_1, z_2). \tag{3.22}$$

where $z = (z_1, z_2) \in \bar{\Omega}$. According to (2.17) and (3.7)

$$\psi_3(z_1, z_2) = g_3(z_1, z_2) = f_1(z_1, z_2),$$

$$g_4(z_1, z_2) = g_1(z_1, z_2) - \psi_3(z_2, \lambda_3(z_2)) = f_3(z_1, z_2) - f_1(z_2, \lambda_3(z_2)) - \int_0^{z_1}\int_0^{z_2}\int_0^{\lambda_3(z_2)} F(\xi, \eta, \varsigma)\, d\xi d\eta d\varsigma,$$

$$g_5(z_1, z_2) = \tilde{f}_2(z_1, z_2) - l_{22}\frac{\partial \psi_3}{\partial z_1}(\lambda_2(z_2), z_2) - l_{23}\frac{\partial \psi_3}{\partial z_2}(\lambda_2(z_2), z_2). \tag{3.23}$$

We assume that $g_4 \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$.

Excluding the function ψ_1 from system (3.21), (3.22), for the unknown function ψ_2 we obtain the following functional-differential equation:

$$l_{21}\frac{\partial\psi_2}{\partial z_1}(z_1,z_2)+l_{23}\frac{\partial\psi_2}{\partial z_2}(z_1,z_2)-l_{21}\frac{\partial\psi_2}{\partial z_1}(z_1,\lambda_3(\lambda_2(z_2)))$$

$$-l_{22}\lambda_3'(\lambda_2(z_2))\frac{\partial\psi_2}{\partial z_2}(z_1,\lambda_3(\lambda_2(z_2)))=g_6(z_1,z_2),\ \ z=(z_1,z_2)\in\bar{\Omega}, \tag{3.24}$$

where

$$g_6(z_1,z_2)=g_5(z_1,z_2)-l_{21}\frac{\partial g_4}{\partial z_1}(z_1,\lambda_2(z_2))-l_{22}\frac{\partial g_4}{\partial z_2}(z_1,\lambda_2(z_2)). \tag{3.25}$$

Consider separately the following cases:

$$\text{(i)}\quad \lambda_3(0)=0,\quad \lambda_2(z_2)=kz_2,\quad 0<k=\text{const}\le 1,\quad l_{22}=kl_{23}; \tag{3.26}$$

$$\text{(ii)}\quad \lambda_i(0)=0,\quad i=2,3;\quad l_{21}=0. \tag{3.27}$$

In the case (3.26), relative to the new unknown function $\omega_1(z_1,z_2)=\psi_1(z_1,kz_2)+\psi_2(z_1,z_2)$, Eq. (3.22) can be written in the form

$$l_{21}\frac{\partial\omega_1}{\partial z_1}(z_1,z_2)+l_{23}\frac{\partial\omega_1}{\partial z_2}(z_1,z_2)=g_5(z_1,z_2),\ z=(z_1,z_2)\in\bar{\Omega}. \tag{3.28}$$

Due to (2.22), (3.20), and (3.23), it is easy to verify that for $\lambda_2(z_2)=kz_2$,

$$g_5(z_1,z_2)=f_2(z_1,z_2)-l_{21}\int_0^{\lambda_2(z_2)}\int_0^{z_2}F(z_1,\eta,\varsigma)d\eta d\varsigma-l_{22}\int_0^{z_1}\int_0^{z_2}F(\xi,\lambda_2(z_2),\varsigma)d\xi d\varsigma$$

$$-l_{23}\int_0^{z_1}\int_0^{\lambda_2(z_2)}F(\xi,\eta,z_2)d\xi d\eta-l_{22}\frac{\partial f_1}{\partial z_1}(\lambda_2(z_2),z_2)-l_{23}\frac{\partial f_1}{\partial z_2}(\lambda_2(z_2),z_2),\quad z=(z_1,z_2)\in\bar{\Omega}. \tag{3.29}$$

It is clear from formula (3.29) that the condition $g_5\in C^2\left(\bar{\Omega}\right)$ holds if

$$f_3\in\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right),\quad f_1\in C^3\left(\bar{\Omega}\right),\quad f_2\in C^2\left(\bar{\Omega}\right),\quad F\in C^2\left(\bar{D}\right). \tag{3.30}$$

By analogy to Eq. (3.11) and conditions (3.13) and (3.14), in order to ensure the unique solvability of Eq. (3.28) in the space $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, it suffices to require the smoothness conditions (3.30) and the conditions

$$F|_{x_1=0}=\left.\frac{\partial F}{\partial x_1}\right|_{x_1=0}=0,\quad \frac{\partial f_1}{\partial z_i}(0,0)=\frac{\partial^2 f_1}{\partial z_i\partial z_j}(0,0)=0,\quad i,j=1,2,$$

$$f_2(0,0)=\frac{\partial f_2}{\partial z_1}(0,0)=\frac{\partial f_2}{\partial z_2}(0,0)=0, \tag{3.31}$$

$$l_{21}\cdot l_{23}>0. \tag{3.32}$$

Therefore, if when conditions (3.30)–(3.32) hold, the function ω_1 is a unique solution of Eq. (3.28) in the space $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, then since $\omega_1(z_1,z_2)=\psi_1(z_1,k\,z_2)+\psi_2(z_1,\,z_2)$, due to (3.21) with respect to ψ_1 and ψ_2 in the space $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, we obtain the following system:

$$\psi_1(z_1,z_2)+\psi_2(z_1,\lambda_3(z_2))=g_4(z_1,z_2),\quad z=(z_1,z_2)\in\bar{\Omega}, \tag{3.33}$$

$$\psi_2(z_1,z_2)+\psi_1(z_1,k\,z_2)=\omega_1(z_1,z_2),\quad z=(z_1,z_2)\in\bar{\Omega}. \tag{3.34}$$

Excluding the function ψ_2 from system (3.33), (3.34), with respect to ψ_1 we obtain the equation

$$\psi_1(z_1,z_2)-\psi_1(z_1,k\,\lambda_3(z_2))=\omega_2(z_1,z_2),\quad z=(z_1,z_2)\in\bar{\Omega}, \tag{3.35}$$

where $\omega_2(z_1,z_2)=g_4(z_1,z_2)-\omega_1(z_1,\lambda_3(z_2))$.

Taking into account Remark 3, we see that in the class $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, Eq. (3.35) is equivalent to the equation

$$\mu\left(z_1,z_2\right)-k\,\lambda_3'\left(z_2\right)\mu\left(z_1,k\,\lambda_3\left(z_2\right)\right)=\omega_3\left(z_1,z_2\right),\quad z=\left(z_1,z_2\right)\in\bar{\Omega},\tag{3.36}$$

relative to the new unknown function $\mu\left(z_1,z_2\right)=\dfrac{\partial^2\psi_1}{\partial z_1\partial z_2}\left(z_1,z_2\right)$ in the class $C\left(\bar{\Omega}\right)$. Therefore, taking into account Lemmas 3–8 formulated above for functional equations of the form (3.36), we obtain the following theorems.

Theorem 6. *Let conditions* (2.49), (3.26), *and* (3.32) *hold and* $\lambda_3\in C^2\left([0,1]\right)$. *Let at least one of the following conditions hold*:

(1) $\left|\lambda_3'\left(z_2\right)\right|<1/k$, $0\le z_2\le 1$;

(2) *the mapping* $\lambda_3:[0,1]\to[0,1]$ *strictly monotonically increases and* $\left|\lambda_3'\left(0\right)\right|<1/k$, $\lambda_3\left(z_2\right)<\frac{z_2}{k}$ *for* $0<z_2\le 1$.

Then for any F *and* f_i, $i=1,2,3$, *satisfying conditions* (3.30) *and* (3.31), *problem* (3.1), (3.20) *has a unique solution in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$. *In the case where the mapping* $\lambda_3:[0,1]\to[0,1]$ *strictly monotonically increases,* $\lambda_3\left(z_2\right)<z_2/k$ *with* $0<z_2\le 1$, *but in contrast to condition* (2), *the inequality* $\left|\lambda_3'\left(0\right)\right|>1/k$ *holds, problem* (3.1), (3.20) *for any* F *and* f_i, $i=1,2,3$, *satisfying conditions* (3.30) *and* (3.31), *is solvable in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$, *although the homogeneous problem corresponding to* (3.1), (3.20), *has an infinite set of linearly independent solutions in the space* $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$.

Remark 9. Note that if the condition (2) of Theorem 6 holds, for the solution $u\in\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ of problem (3.1), (3.20), the following estimate holds:

$$\begin{aligned}\|u\|_{\overset{0}{C}{}^{1,1,1}\left(\bar{D}_P\right)}\le c\Big[&\|F\|_{C^2\left(\bar{D}_P\right)}+\|f_1\|_{C^3\left(\bar{\Omega}_{P_1}\right)}\\&+\|f_2\|_{C^2\left(\bar{\Omega}_{P_2}\right)}+\|f_3\|_{\overset{0}{C}{}^{1,1}\left(\bar{\Omega}_{P_3}\right)}\Big],\end{aligned}\tag{3.37}$$

where c is a positive constant independent of F, f_i, $i=1,2,3$, and the point $P=P\left(x_1^0,x_2^0,x_3^0\right)\in D$, and the domains D_P and Ω_{P_i}, $i=1,2,3$, are given in Theorem 2.

Now let us consider case (3.27). In this case, Eq. (3.24) takes the form

$$l_{23}\frac{\partial\psi_2}{\partial z_2}\left(z_1,z_2\right)-l_{22}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\frac{\partial\psi_2}{\partial z_2}\left(z_1,\lambda_3\left(\lambda_2\left(z_2\right)\right)\right)=g_6\left(z_1,z_2\right).\tag{3.38}$$

Differentiating Eq. (3.38) with respect to z_1, relative to the new unknown function $\omega_0\left(z_1,z_2\right)=\dfrac{\partial^2\psi_2}{\partial z_1\partial z_2}\left(z_1,z_2\right)$ we obtain the equation

$$l_{23}\omega_0\left(z_1,z_2\right)-l_{22}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\omega_0\left(z_1,\lambda_3\left(\lambda_2\left(z_2\right)\right)\right)=\frac{\partial g_6}{\partial z_1}\left(z_1,z_2\right).\tag{3.39}$$

Remark 10. According to Remark 3 and the assumption $g_6\left(0,z_2\right)=0$, $0\le z_2\le 1$, we see that Eq. (3.38) in the space $\overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$ is equivalent to Eq. (3.39) in the space $C\left(\bar{\Omega}\right)$. Therefore, assuming that

$$l_{23}\ne 0,\tag{3.40}$$

Eq. (3.39) can be written in the form

$$\omega_0\left(z_1,z_2\right)-a_0\left(z_2\right)\omega_0\left(z_1,\lambda_0\left(z_2\right)\right)=g_7\left(z_1,z_2\right),\tag{3.41}$$

where

$$a_0\left(z_2\right)=\frac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2\right)\right),\ \lambda_0\left(z_2\right)=\lambda_3\left(\lambda_2\left(z_2\right)\right),\ g_7=\frac{1}{l_{23}}\frac{\partial g_6}{\partial z_1}.\tag{3.42}$$

Now using Lemmas 3–8 for Eq. (3.41), on the solvability of problem (3.1), (3.20) in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$, we have the following theorem.

Theorem 7. *Let conditions* (2.49), (3.27), *and* (3.40) *hold and*

$$F \in C\left(\bar{D}\right), \; f_i \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right), \quad i = 1,2,3, \quad g_4 \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right), \quad g_6\big|_{z_1=0} = 0, \quad g_7 \in C\left(\bar{\Omega}\right),$$

where g_4, g_6, and g_7 are expressed by the given functions F, f_1, f_2, f_3 by formulas (3.23), (3.25), *and* (3.42). *Assume that at least one of the following conditions hold*:

(1) $\left|\dfrac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\right| < 1$, $0 \le z_2 \le 1$;

(2) *the continuous mapping $\lambda_0 : [0,1] \to [0,1]$, where, according to* (3.42), $\lambda_0\left(z_2\right) = \lambda_3\left(\lambda_2\left(z_2\right)\right)$, *strictly monotonically increases and*

$$\left|\frac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\right| < 1 \quad \forall z_2 : \lambda_0\left(z_2\right) = z_2;$$

(3) *the continuous mapping $\lambda_0 : [0,1] \to [0,1]$ strictly monotonically increases and*

$$\left|\frac{l_{22}}{l_{23}}\lambda_3'(0)\right| < 1, \quad \lambda_0\left(z_2\right) < z_2, \quad 0 < z_2 \le 1; \tag{3.43}$$

(4) *the continuous mapping $\lambda_0 : [0,1] \to [0,1]$ is a homeomorphism and*

$$\left|\frac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\right| \ne 1, \quad 0 \le z_2 \le 1;$$

(5) *the continuous mapping $\lambda_0 : [0,1] \to [0,1]$ is a homeomorphism and*

$$\left|\frac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\right| > 1 \quad \forall z_2 : \lambda_0\left(z_2\right) = z_2;$$

(6) *the continuous mapping $\lambda_0 : [0,1] \to [0,1]$ strictly monotonically increases, for a certain number $z_2^0 \in (0,1)$, with $\lambda\left(z_2\right) < z_2$ for all $z_2 \in \left(z_2^0, 1\right]$, and*

$$\left|\frac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2\right)\right)\right| \ne 1 \quad \forall z_2 \in \left[0, z_2^0\right], \qquad \left|\frac{l_{22}}{l_{23}}\lambda_3'\left(\lambda_2\left(z_2^0\right)\right)\right| < 1.$$

Then problem (3.1), (3.20) *has a unique solution in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$. In the case where in condition* (3.43) *instead of the inequality $\left|\dfrac{l_{22}}{l_{23}}\lambda_3'(0)\right| < 1$, the opposite inequality holds, $\left|\dfrac{l_{22}}{l_{23}}\lambda_3'(0)\right| > 1$, problem* (3.1), (3.20) *is solvable in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$, although the homogeneous problem corresponding to* (3.1), (3.20), *has an infinite set of linearly independent solutions in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$.*

4. Problem (1.3), (2.2) in the Presence of Low-Order Terms in Equation (1.3)

Below we present a brief scheme of the study of problem (1.3), (2.2) in the presence of low-order terms in Eq. (1.3). Considering problem (1.3), (2.2) in the class $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ relative to coefficients A^α of low-order derivatives, we require that

$$A^\alpha \in C^{\alpha_1,\alpha_2,\alpha_3} := \left\{A \in C\left(\bar{D}\right) : \frac{\partial^{|\beta|}A}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}} \in C\left(\bar{D}\right), \; \beta_i \le \alpha_i, \; i = 1,2,3\right\}.$$

In this case, as is known, there exists the Riemann function $v(y_1, y_2, y_3; x_1, x_2, x_3)$ of Eq. (1.3) and for a solution $u \in \overset{0}{C}{}^{1,1,1}(\bar{D})$ of this equation, due to (2.12), instead of representation (2.3), the following formula holds (see [5]):

$$
\begin{aligned}
u(x_1, x_2, x_3) = \int_0^{x_1}\int_0^{x_2} v(y_1, y_2, 0; x_1, x_2, x_3) \Big[\frac{\partial^2}{\partial x_1 \partial x_2} \varphi_{12}(y_1, y_2) + A_{13}(y_1, y_2, 0) \frac{\partial}{\partial x_1} \varphi_{12}(y_1, y_2) \\
+ A_{23}(y_1, y_2, 0) \frac{\partial}{\partial x_2} \varphi_{12}(y_1, y_2) + A_3(y_1, y_2, 0) \varphi_{12}(y_1, y_2) \Big] dy_1 dy_2 \\
+ \int_0^{x_1}\int_0^{x_3} \mathrm{v}(y_1, 0, y_3; x_1, x_2, x_3) \Big[\frac{\partial^2}{\partial x_1 \partial x_3} \varphi_{13}(y_1, y_3) + A_{12}(y_1, 0, y_3) \frac{\partial}{\partial x_1} \varphi_{13}(y_1, y_3) \\
+ A_{23}(y_1, 0, y_3) \frac{\partial}{\partial x_3} \varphi_{13}(y_1, y_3) + A_2(y_1, 0, y_3) \varphi_{13}(y_1, y_3) \Big] dy_1 dy_3 \\
+ \int_0^{x_2}\int_0^{x_3} \mathrm{v}(0, y_2, y_3; x_1, x_2, x_3) \Big[\frac{\partial^2}{\partial x_2 \partial x_3} \varphi_{23}(y_2, y_3) + A_{12}(0, y_2, y_3) \frac{\partial}{\partial x_2} \varphi_{23}(y_2, y_3) \\
+ A_{13}(0, y_2, y_3) \frac{\partial}{\partial x_3} \varphi_{23}(y_2, y_3) + A_1(0, y_2, y_3) \varphi_{23}(y_2, y_3) \Big] dy_2 dy_3 \\
+ \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} \mathrm{v}(y_1, y_2, y_3; x_1, x_2, x_3) F(y_1, y_2, y_3) \, dy_1 dy_2 dy_3,
\end{aligned}
\tag{4.1}
$$

where $A_{12} = A^{1,1,0}$, $A_{13} = A^{1,0,1}$, $A_{23} = A^{0,1,1}$, $A_1 = A^{1,0,0}$, $A_2 = A^{0,1,0}$, $A_3 = A^{0,0,1}$, $A = A^{0,0,0}$, $\varphi_{12} = u|_{x_3=0}$, $\varphi_{13} = u|_{x_2=0}$, $\varphi_{23} = u|_{x_1=0}$, $\varphi_{ij} \in \overset{0}{C}{}^{1,1}(\bar{\Omega})$, $1 \le i, j \le 3$, $i < j$.

Substituting the expression for $u \in \overset{0}{C}{}^{1,1,1}(\bar{D})$ in (4.1) and in conditions (2.2), we obtain the system of equations relative to the unknown functions $\varphi_{ij} \in \overset{0}{C}{}^{1,1}(\bar{\Omega})$, $1 \le i, j \le 3$, $i < j$, which can be studied on the basis of known properties of the Riemann function of Eq. (1.3) (see [9]). Below, we consider another way to construct solutions of problem (1.3), (2.2).

We solve problem (1.3), (2.2) by the method of sequential approximations, assuming

$$
u_0 = 0, \tag{4.2}
$$

$$
\frac{\partial^3 u_n}{\partial x_1 \partial x_2 \partial x_3} = F - \sum_{\substack{|\alpha| \le 2,\\ \alpha_i \le 1}} A^\alpha \frac{\partial^{|\alpha|} u_{n-1}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}, \quad u_n|_{S_i} = f_i, \quad i = 1, 2, 3, \quad n = 1, 2, \ldots. \tag{4.3}
$$

We solve problem (4.3) with $n = 1, 2, \ldots$ under the conditions of Theorem 2 or Theorem 3 in the case where $|\lambda'(0)| < 1$. In these cases, for the solution $u \in \overset{0}{C}{}^{1,1,1}(\bar{D})$ of problem (2.1), (2.2) together with estimate (2.63) with $f_i = 0$, $i = 1, 2, 3$, and $F \in C(\bar{D})$ such that

$$
|F(x_1, x_2, x_3)| \le M (x_1 + x_2 + x_3)^k, \quad M, k = \mathrm{const} \ge 0, \tag{4.4}
$$

the following estimate holds:

$$
\left| \frac{\partial^{|\alpha|} u(x_1, x_2, x_3)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \right| \le MK \frac{(x_1 + x_2 + x_3)^{k+1}}{k+1}, \tag{4.5}
$$

$$
K = \mathrm{const} > 0, \quad |\alpha| \le 2, \quad \alpha_i \le 1.
$$

From (4.2), (4.3) and estimate (2.63) we have

$$\left|\frac{\partial^{|\alpha|}\left(u_1-u_0\right)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\left(x_1,x_2,x_3\right)\right|=\left|\frac{\partial^{|\alpha|}u_1}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\left(x_1,x_2,x_3\right)\right|$$

$$\leq c\left[\|F\|_{C(\bar{D})}+\sum_{i=1}^{3}\|f_i\|_{\overset{0}{C}{}^{1,1}(\bar{\Omega})}\right],\ |\alpha|\leq 2,\quad \alpha_i\leq 1,\quad i=1,2,3. \tag{4.6}$$

Due to (4.3), the function $(u_2-u_1)\in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ satisfies the following problem:

$$\frac{\partial^3\left(u_2-u_1\right)}{\partial x_1\partial x_2\partial x_3}=-\sum_{\substack{|\alpha|\leq 2\\ \alpha_i\leq 1}}A^{\alpha}\frac{\partial^{|\alpha|}\left(u_1-u_0\right)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}},\quad \left.(u_2-u_1)\right|_{S_i}=0,\quad i=1,2,3. \tag{4.7}$$

For the right-hand side of Eq. (4.7), according to (4.6) we have the estimate

$$\left|-\sum_{\substack{|\alpha|\leq 2\\ \alpha_i\leq 1}}A^{\alpha}\frac{\partial^{|\alpha|}\left(u_1-u_0\right)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\left(x_1,x_2,x_3\right)\right|$$

$$\leq c\left[\sum_{\substack{|\alpha|\leq 2\\ \alpha_i\leq 1}}\|A^{\alpha}\|_{C(\bar{D})}\right]\left[\|F\|_{C(\bar{D})}+\sum_{i=1}^{3}\|f_i\|_{\overset{0}{C}{}^{1,1}(\bar{\Omega})}\right],\quad (x_1,x_2,x_3)\in\bar{D}. \tag{4.8}$$

Due to (4.8), for the right-hand side of Eq. (4.7), estimate (4.4) holds with $k=0$ and

$$M=c\left[\sum_{\substack{|\alpha|\leq 2\\ \alpha_i\leq 1}}\|A^{\alpha}\|_{C(\bar{D})}\right]\left[\|F\|_{C(\bar{D})}+\sum_{i=1}^{3}\|f_i\|_{\overset{0}{C}{}^{1,1}(\bar{\Omega})}\right]. \tag{4.9}$$

But in this case, by (4.5) for the solution (u_2-u_1) of problem (4.7), the estimate

$$\left|\frac{\partial^{|\alpha|}\left(u_2-u_1\right)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\left(x_1,x_2,x_3\right)\right|\leq MK\frac{\left(x_1+x_2+x_3\right)}{1!},\quad |\alpha|\leq 2,\quad \alpha_i\leq 1. \tag{4.10}$$

holds.

Since by (4.2) the function $(u_{n+1}-u_n)\in \overset{0}{C}{}^{1,1,1}$ is a solution of the problem

$$\frac{\partial^3\left(u_{n+1}-u_n\right)}{\partial x_1\partial x_2\partial x_3}=-\sum_{\substack{|\alpha|\leq 2\\ \alpha_i\leq 1}}A^{\alpha}\frac{\partial^{|\alpha|}\left(u_n-u_{n-1}\right)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}},$$

$$\left.(u_{n+1}-u_n)\right|_{S_i}=0,\quad i=1,2,3,$$

using the same reasoning as for estimate (4.10), by induction we obtain

$$\left|\frac{\partial^{|\alpha|}\left(u_{n+1}-u_n\right)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\left(x_1,x_2,x_3\right)\right|\leq MK^n\frac{\left(x_1+x_2+x_3\right)^n}{n!}, \tag{4.11}$$

$$|\alpha|\leq 2,\alpha_i\leq 1,\quad n=1,2,\ldots,$$

where M is given by formula (4.9).

It follows from (4.11) that the series

$$u = \sum_{n=1}^{\infty} (u_n - u_{n-1}) \tag{4.12}$$

converges in the space

$$\overset{0}{C}{}^{\alpha}\left(\bar{D}\right) := \left\{ u \in \overset{0}{C}\left(\bar{D}\right) : \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \partial x_3^{\beta_3}} \in C\left(\bar{D}\right),\ \beta_i \le \alpha_i,\ i = 1,2,3 \right\}$$

for $|\alpha| \le 2$, $\alpha_i \le 1$, $i = 1,2,3$, and, therefore, the sum u of series (4.12) belongs to the space $\overset{0}{C}{}^{\alpha}\left(\bar{D}\right)$ with $|\alpha| \le 2$, $\alpha_i \le 1$, $i = 1,2,3$. Thus, using (4.2) and taking into account the fact that $u_n = \sum\limits_{k=1}^{n} (u_k - u_{k-1})$, we have

$$\lim_{n\to\infty} \|u_n - u\|_{\overset{0}{C}{}^{\alpha}\left(\bar{D}\right)} = 0, \quad |\alpha| \le 2, \quad \alpha_i \le 1, \quad i = 1,2,3. \tag{4.13}$$

Since $u_n \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$, due to (4.13), passing in Eq. (4.3) to the limit as $n \to \infty$, we obtain that the function u from (4.11) belongs to the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ and is a solution of problem (1.3), (2.2). For the proof of the uniqueness of a solution of problem (1.3), (2.2) in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$, it suffices to show that the homogeneous problem corresponding to (1.3), (2.2) has only the zero solution in this space. Indeed, let $\tilde{u} \in \overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$ be a solution of the homogeneous problem. Since the functions $u_n = \tilde{u}$, $n = 1,2,\dots$, satisfy recursion relations (4.3) with $F = 0$ and $f_i = 0$, $i = 1,2,3$, by analogy with how estimate (4.10) was obtained, we have

$$\left| \frac{\partial^{|\alpha|} \tilde{u}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} (x_1, x_2, x_3) \right| \le M K^n \frac{(x_1 + x_2 + x_3)^n}{n!}, \quad |\alpha| \le 2, \quad \alpha_i \le 1,$$

whence for $\alpha = 0$, passing to the limit as $n \to \infty$, we obtain $\tilde{u} = 0$. Thus, under the conditions of Theorem 2 or Theorem 3, i.e., when $F \in C\left(\bar{D}\right)$, $f_i \in \overset{0}{C}{}^{1,1}\left(\bar{\Omega}\right)$, $i = 1,2,3$, conditions from (2.49) hold and either $|\lambda'(z_2)| < 1$, $0 \le z_2 \le 1$, $\lambda(z_2) = \lambda_2(\lambda_3(z_2))$, $\lambda_i(z_2) \le z_2$, $z_2 \in [0,1]$, $i = 2,3$, or the continuous mapping $\lambda : [0,1] \to [0,1]$ strictly monotonically increases, for $\lambda(z_2) < z_2$ for $0 < z_2 \le 1$, $\lambda_i(z_2) \le z_2$, $z_2 \in [0,1]$ and $|\lambda'(0)| < 1$, problem (1.3), (2.2) has a unique solution in the space $\overset{0}{C}{}^{1,1,1}\left(\bar{D}\right)$.

Problems (1.3), (3.2) and (1.3), (3.20) can be studied similarly.

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