

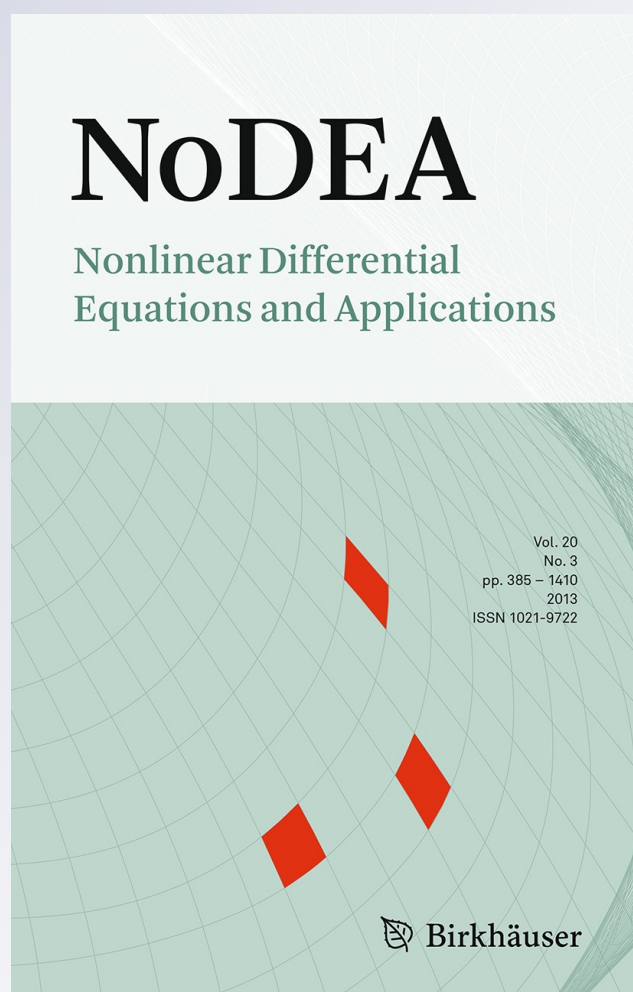
*One multidimensional version of the
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One multidimensional version of the Darboux first problem for one class of semilinear second order hyperbolic systems

S. Kharibegashvili and B. Midodashvili

Abstract. One multidimensional version of the Darboux first problem for one class of semilinear second order hyperbolic systems is investigated. The questions on local and global solvability and nonexistence of a global solution of this problem are considered.

Mathematics Subject Classification. 35L51, 35L71.

Keywords. Multidimensional version of the Darboux first problem, Semilinear second order hyperbolic systems, Local and global solvability, Nonexistence of a global solution.

1. Introduction

In the Euclidean space \mathbb{R}^{n+1} of independent variables $x = (x_1, x_2, \dots, x_n)$ and t consider a second order semilinear hyperbolic system of the form

$$\square u_i + f_i(u_1, \dots, u_N) = F_i, \quad i = 1, \dots, N, \quad (1.1)$$

where $f = (f_1, \dots, f_N)$, $F = (F_1, \dots, F_N)$ are given, and $u = (u_1, \dots, u_N)$ is an unknown real vector-functions, $n \geq 2$, $N \geq 2$, $\square := \frac{\partial^2}{\partial t^2} - \Delta$, $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Denote by $D : t > |x|, x_n > 0$ the half of a light cone of the future bounded by the part $S^0 : \partial D \cap \{x_n = 0\}$ of hyperplane $x_n = 0$ and the half $S : t = |x|, x_n \geq 0$ of the characteristic conoid $C : t = |x|$ of the system (1.1). Let $D_T := \{(x, t) \in D : t < T\}$, $S_T^0 := \{(x, t) \in S^0 : t \leq T\}$, $S_T := \{(x, t) \in S : t \leq T\}$, $T > 0$.

For the system of equations (1.1) consider a problem on finding a solution $u(x, t)$ of this system by the following boundary conditions

$$\frac{\partial u}{\partial x_n} \Big|_{S_T^0} = 0, \quad u \Big|_{S_T} = g, \quad (1.2)$$

where $g = (g_1, \dots, g_N)$ is a given vector-function on S_T . In the case when $T = \infty$ we have $D_\infty = D, S_\infty^0 = S^0$ and $S_\infty = S$.

The problem (1.1), (1.2) represents a multidimensional version of the Darboux first problem for the system (1.1), when one part of the problem data support represents a characteristic manifold, while another part is of time type manifold [1, p. 228, 233].

Questions on the existence and nonexistence of global solution of the Cauchy problem for semilinear scalar equations of the form (1.1) with initial conditions $u|_{t=0} = u_0, \frac{\partial u}{\partial t}|_{t=0} = u_1$ have been considered by many authors (see, e.g. [2–11]). As it is known for second order scalar linear hyperbolic equations multidimensional versions of the Darboux first problem are well-posed and they are globally solvable in suitable function spaces [12–17]. In regard to the multidimensional problem (1.1), (1.2) for scalar case, i.e. when $N = 1$, in the case of nonlinearity of the form $f(u) = \lambda|u|^p u$, as it is shown in the paper [18], depending on the sign of the parameter λ and the values of power exponent p , in some cases the problem (1.1), (1.2) is globally solvable, while in other cases it is not globally solvable. Another multidimensional version of the Darboux first problem for scalar semilinear equation of the form (1.1), where instead of the boundary condition $\frac{\partial u}{\partial x_n}|_{S_T^0} = 0$ in (1.2) is taken $u|_{S_T^0} = 0$, is considered in the work [19]. Worthy to note also that the multidimensional version for the Darboux second problem for scalar semilinear equation of the form (1.1) is studied in the work [20].

In the present work we give certain conditions for the nonlinear vector-function $f = f(u)$ from (1.1), which fulfilment ensures local or global solvability of the problem (1.1), (1.2), while in some cases it will not have global solution, though it will be locally solvable.

The paper is organized in the following way. In the Sect. 2 we define a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T and a global solution of this problem of the same class in the domain D_∞ . In the Sect. 3, with some restrictions on the power of nonlinearity of the vector function f , we consider the cases of local and global solvability of the problem (1.1), (1.2) of the class W_2^1 . In Sect. 4 we give conditions for the vector-function f , which provide the existence and uniqueness of a global solution of the class W_2^1 to this problem. Finally, in the Sect. 5, at some additional conditions on the vector-functions f, F and g , we prove the nonexistence of a global solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_∞ .

2. Definition of a generalized solution of the problem (1.1), (1.2) in D_T and D_∞

Let $C^0(\overline{D}_T, S_T^0, S_T) := \left\{ u \in C^2(\overline{D}_T) : \frac{\partial u}{\partial x_n}|_{S_T^0} = 0, u|_{S_T} = 0 \right\}$.

Let also $W_2^1(D_T, S_T) := \{ u \in W_2^1(D_T) : u|_{S_T} = 0 \}$, where $W_2^k(\Omega)$ is the Sobolev space consisting of the elements of $L_2(\Omega)$ having up to k -th order

generalized derivatives from $L_2(\Omega)$, inclusively. Here the equality $u|_{S_T} = 0$ must be understood in the sense of the trace theory [21, p. 71].

Below under belonging of the vector $v = (v_1, \dots, v_N)$ to some space X we mean belonging of each component $v_i, 1 \leq i \leq N$, of this vector to the same space X . In accordance with this, for simplicity of record, where this will not cause misunderstanding, instead of $v = (v_1, v_2, \dots, v_N) \in [X]^N$ we use the record $v \in X$.

Rewrite the system of equations (1.1) in the form of one vector equation

$$Lu := \square u + f(u) = F. \tag{2.1}$$

Together with the boundary conditions (1.2) we consider corresponding homogeneous boundary conditions

$$\frac{\partial u}{\partial x_n} \Big|_{S_T^0} = 0, \quad u \Big|_{S_T} = 0. \tag{2.2}$$

Below, on the nonlinear vector-function $f = (f_1, \dots, f_N)$ in (1.1) we impose the following requirement

$$f \in C(\mathbb{R}^N), \quad |f(u)| \leq M_1 + M_2|u|^\alpha, \quad \alpha = \text{const} \geq 0, \quad u \in \mathbb{R}^N, \tag{2.3}$$

where $|\cdot|$ is a norm in the space $\mathbb{R}^N, M_i = \text{const} \geq 0, i = 1, 2$.

Remark 2.1. The embedding operator $I : [W_2^1(D_T)]^N \rightarrow [L_q(D_T)]^N$ represents a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [21, p. 86]. At the same time the Nemitski operator $K : [L_q(D_T)]^N \rightarrow [L_2(D_T)]^N$, acting by the formula $Ku = f(u)$, where $u = (u_1, \dots, u_N) \in [L_q(D_T)]^N$, and the vector-function $f = (f_1, \dots, f_N)$ satisfies the condition (2.3).0 is continuous and bounded for $q \geq 2\alpha$ [22, p. 349], [23, pp. 66, 67]. Thus, if $\alpha < \frac{n+1}{n-1}$, i.e. $2\alpha < \frac{2(n+1)}{n-1}$, then there exists number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \geq 2\alpha$. Therefore, in this case the operator

$$K_0 = KI : [W_2^1(D_T)]^N \rightarrow [L_2(D_T)]^N \tag{2.4}$$

will be continuous and compact. It is clear that from $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \rightarrow u$ in the space $W_2^1(D_T)$, then $f(u^m) \rightarrow f(u)$ in the space $L_2(D_T)$.

Definition 2.1. Let $f = (f_1, \dots, f_N)$ satisfy the condition (2.3), where $0 \leq \alpha < \frac{n+1}{n-1}, F = (F_1, \dots, F_N) \in L_2(D_T)$ and $g = (g_1, \dots, g_N) \in W_2^1(S_T)$. We call the vector-function $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if there exists a sequence of vector-functions $u^m \in C^2(\overline{D}_T)$ such that $\frac{\partial u^m}{\partial x_n} \Big|_{S_T^0} = 0, u^m \rightarrow u$ in the space $W_2^1(D_T), Lu^m \rightarrow F$ in the space $L_2(D_T)$ and $u^m \Big|_{S_T} \rightarrow g$ in the space $W_2^1(S_T)$. Convergence of the sequence $\{f(u^m)\}$ to $f(u)$ in the space $L_2(D_T)$ when $u^m \rightarrow u$ in the space $W_2^1(D_T)$ follows from the Remark 2.1 When $g = 0$, i.e. in the case of homogeneous boundary conditions (2.2), we assume that $u^m \in \overset{0}{C}{}^2(\overline{D}_T, S_T^0, S_T)$. Then, it is clear that $u \in \overset{0}{W}{}^1_2(D_T, S_T)$.

It is obvious, that the classical solution $u \in C^2(\overline{D_T})$ of the problem (1.1), (1.2) is a strong generalized solution of the class W_2^1 in the domain D_T in the sense of the Definition 2.1.

Remark 2.2. It is easy to verify that if $u \in C^2(\overline{D_T})$ is a classical solution of the problem (1.1), (1.2), then scalarly multiplying the both parts of the system (2.1) by any test vector-function $\varphi = (\varphi_1, \dots, \varphi_N) \in C^1(\overline{D_T})$, satisfying the condition $\varphi|_{t=T} = 0$, after integration by parts, we receive the following equality

$$\int_{D_T} [-u_t \varphi_t + \nabla u \nabla \varphi] dxdt = - \int_{D_T} f(u) \varphi dxdt + \int_{D_T} F \varphi dxdt - \int_{S_T^0 \cup S_T} \frac{\partial u}{\partial \mathbb{N}} \varphi ds, \tag{2.5}$$

where $\frac{\partial}{\partial \mathbb{N}} = \nu_{n+1} \frac{\partial}{\partial t} - \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to ∂D_T , and $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$. Taking into account that $\frac{\partial}{\partial \mathbb{N}}|_{S_T^0} = \frac{\partial}{\partial x_n}$ and S_T represents a characteristic manifold, on which the operator $\frac{\partial}{\partial \mathbb{N}}$ is a inner differential operator, from (1.2) we have $\frac{\partial u}{\partial \mathbb{N}}|_{S_T^0} = 0$, $\frac{\partial u}{\partial \mathbb{N}}|_{S_T} = \frac{\partial g}{\partial \mathbb{N}}|_{S_T}$. Therefore, the equality (2.5) takes the form

$$\int_{D_T} [-u_t \varphi_t + \nabla u \nabla \varphi] dxdt = - \int_{D_T} f(u) \varphi dxdt + \int_{D_T} F \varphi dxdt - \int_{S_T} \frac{\partial g}{\partial \mathbb{N}} \varphi ds. \tag{2.6}$$

It is easy to see that the equality (2.6) is valid also for any vector-function $\varphi = (\varphi_1, \dots, \varphi_N) \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$ in the sense of the trace theory. Note that the equality (2.6) is valid also for a strong generalized solution $u \in W_2^1(D_T)$ of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T in the sense of the Definition 2.1. Indeed, if $u^m \in C^2(D_T)$ is a sequence of vector-functions from the Definition 2.1, then writing equality (2.6) for $u = u^m$ and passing to the limit for $m \rightarrow \infty$, we receive (2.6). Note also that the equality (2.6), valid for any test vector-function $\varphi \in W_2^1(D_T)$, satisfying the condition $\varphi|_{t=T} = 0$, can be put in the basis of the definition of a weak generalized solution $u \in W_2^1(D_T)$ of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T .

Definition 2.2. Let f satisfy the condition (2.3), where $0 \leq \alpha < \frac{n+1}{n-1}$; $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is locally solvable in the class W_2^1 if there exists a number $T_0 = T_0(F, g) > 0$ such that for any $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of the Definition 2.1.

Definition 2.3. Let f satisfy the condition (2.3), where $0 \leq \alpha < \frac{n+1}{n-1}$; $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is globally solvable in the class W_2^1 if for any $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of the Definition 2.1.

Definition 2.4. Let f satisfy the condition (2.3), where $0 \leq \alpha < \frac{n+1}{n-1}$; $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. Vector-function $u = (u_1, \dots, u_N) \in W_{2,loc}^1(D_\infty)$ is called a global strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_∞ if for any $T > 0$ the vector-function $u|_{D_T}$ belongs to the space $W_2^1(D_T)$ and represents a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T in the sense of the Definition 2.1.

Remark 2.3. Reasoning used in the proof of the equality (2.6) makes it possible to conclude that the global strong generalized solution $u = (u_1, \dots, u_N)$ of the problem (1.1), (1.2) of the class W_2^1 in the domain D_∞ in the sense of the Definition 2.4 satisfies the following integral equality

$$\int_{D_\infty} [-u_t \varphi_t + \nabla u \nabla \varphi] dx dt = - \int_{D_\infty} f(u) \varphi dx dt + \int_{D_\infty} F \varphi dx dt - \int_{S_\infty} \frac{\partial q}{\partial \mathfrak{N}} \varphi ds$$

for any test vector-function $\varphi = (\varphi_1, \dots, \varphi_N) \in C^1(D_\infty)$, which is finite with respect to the variable $r = (t^2 + |x|^2)^{1/2}$, i.e. $\varphi = 0$ for $r > r_0 = const > 0$.

3. Some cases of local and global solvability of the problem (1.1), (1.2) in the class W_2^1

For simplicity we consider the case when the boundary conditions (1.2) are homogeneous. In this case the problem (1.1), (1.2) can be written in the form (2.1), (2.2).

Remark 3.1. Before we consider the solvability of the problem (2.1), (2.2) let us consider the same question for a linear case, when vector-function $f = 0$ in (2.1), i.e. for the problem

$$L_0 u := \square u = F(x, t), \quad (x, t) \in D_T, \tag{3.1}$$

$$\frac{\partial u}{\partial x_n} \Big|_{S_T^0} = 0, \quad u|_{S_T} = 0. \tag{3.2}$$

For the problem (3.1), (3.2), by analogy to that in the Definition 2.1 for the problem (1.1), (1.2), we introduce a notion of the strong generalized solution $u = (u_1, \dots, u_N)$ of the class W_2^1 in the domain D_T for $F = (F_1, \dots, F_N) \in L_2(D_T)$, i.e. for vector-function $u = (u_1, \dots, u_N) \in W_2^1(D_T, S_T)$, for which

there exists a sequence of vector-functions $u^m = (u_1^m, \dots, u_N^m) \in \overset{0}{C}^2(\overline{D}_T, S_T^0, S_T)$, such that

$$\lim_{m \rightarrow \infty} \|u^m - u\|_{\overset{0}{W}^{\frac{1}{2}}(D_T, S_T)} = 0, \quad \lim_{m \rightarrow \infty} \|L_0 u^m - F\|_{L_2(D_T)} = 0. \quad (3.3)$$

For the solution $u \in \overset{0}{C}^2(\overline{D}_T, S_T^0, S_T)$ of the problem (3.1), (3.2) the following estimate

$$\|u\|_{\overset{0}{W}^{\frac{1}{2}}(D_T, S_T)} \leq c(T) \|F\|_{L_2(D_T)}, \quad c(T) = \sqrt{T} \exp \frac{1}{2} (T + T^2), \quad (3.4)$$

is valid. Indeed, after scalar multiplication of the both parts of the vector equation (3.1) by $2 \frac{\partial u}{\partial t}$ and integration in the domain D_τ , $0 < \tau \leq T$, and simple transformations by help of the equalities (3.2) and integration by parts we receive the equality [18], [24, p. 116]

$$\int_{\Omega_\tau} \left[\left(\frac{\partial u}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 \right] dx = 2 \int_{D_\tau} F \frac{\partial u}{\partial t} dx dt, \quad (3.5)$$

where $\Omega_\tau := D_T \cap \{t = \tau\}$. Since $S_\tau : t = |x|$, $x_n \geq 0$, $t \leq T$, then due to (3.2) we have

$$u(x, \tau) = \int_{|x|}^\tau \frac{\partial}{\partial t} u(x, s) ds, \quad (x, \tau) \in \Omega_\tau.$$

Squaring scalarly the both parts of the received equality, integrating it in the domain Ω_τ and using the Schwartz inequality, we obtain

$$\begin{aligned} \int_{\Omega_\tau} u^2 dx &= \int_{\Omega_\tau} \left(\int_{|x|}^\tau \frac{\partial}{\partial t} u(x, s) \right)^2 dx \leq \int_{\Omega_\tau} (\tau - |x|) \left[\int_{|x|}^\tau \left(\frac{\partial u}{\partial t} \right)^2 ds \right] dx \\ &\leq T \int_{\Omega_\tau} \left[\int_{|x|}^\tau \left(\frac{\partial u}{\partial t} \right)^2 ds \right] dx = T \int_{D_\tau} \left(\frac{\partial u}{\partial t} \right)^2 dx dt. \end{aligned} \quad (3.6)$$

Let

$$w(\tau) := \int_{\Omega_\tau} \left[u^2 + \left(\frac{\partial u}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 \right] dx.$$

Taking into account the inequality $2F \frac{\partial u}{\partial t} \leq \left(\frac{\partial u}{\partial t} \right)^2 + F^2$, due to (3.5) and (3.6) we have

$$\begin{aligned}
 w(\tau) &\leq (1 + T) \int_{D_T} \left(\frac{\partial u}{\partial t}\right)^2 dxdt + \int_{D_\tau} F^2 dxdt \\
 &\leq (1 + T) \int_{D_T} \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 \right] dxdt + \|F\|_{L_2(D_T)}^2 \\
 &= (1 + T) \int_0^\tau w(s) ds + \|F\|_{L_2(D_T)}^2, \quad 0 < \tau \leq T.
 \end{aligned}
 \tag{3.7}$$

According to the Gronwall's Lemma [25, p.13] from (3.7) it follows that

$$w(\tau) \leq \|F\|_{L_2(D_T)}^2 \exp(1 + T)T, \quad 0 < \tau \leq T.
 \tag{3.8}$$

By help of (3.8) we have

$$\begin{aligned}
 \|u\|_{W_{\frac{1}{2}}^1(D_T, S_T)}^2 &= \int_{D_T} \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 \right] dxdt \\
 &= \int_0^T w(\tau) d\tau \leq T \|F\|_{L_2(D_T)}^2 \exp(1 + T)T,
 \end{aligned}$$

whence it follows the estimate (3.4).

Remark 3.2. Due to (3.3) a priori estimate (3.4) is also valid for a strong generalized solution of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T .

Since the space $C_0^\infty(D_T)$ of finite infinitely differentiable in the D_T functions is dense in the $L_2(D_T)$, then for given $F = (F_1, \dots, F_N) \in L_2(D_T)$ there exists the sequence of vector-functions $F^m = (F_1^m, \dots, F_N^m) \in C_0^\infty(D_T)$ such that $\lim_{m \rightarrow \infty} \|F^m - F\|_{L_2(D_T)} = 0$. For fixed m , extending F^m with even manner with respect to the variable x_n in the domain $D_T^- := \{(x, t) \in \mathbb{R}^{n+1} : x_n < 0, |x| < t < T\}$ and then with zero beyond the domain $D_T \cup D_T^-$ and leaving the same notation for it, we shall have $F^m \in C^\infty(\mathbb{R}_+^{n+1})$, for which the support $supp F^m \subset D_\infty \cup D_\infty^-$, where $\mathbb{R}_+^{n+1} := \mathbb{R}^{n+1} \cap \{t \geq 0\}$. Denote by u^m the solution of the Cauchy problem:

$$L_0 u^m := \square u^m = F^m, \quad u^m|_{t=0} = 0, \quad \frac{\partial u^m}{\partial t} \Big|_{t=0} = 0,
 \tag{3.9}$$

which, as it is well-known [26, p. 192], exists, is unique and belongs to the space $C^\infty(\mathbb{R}_+^{n+1})$. Since $supp F^m \subset D_\infty \cup D_\infty^- \subset \{(x, t) \in \mathbb{R}^{n+1} : t > |x|\}$, $u^m|_{t=0} = 0$ and $\frac{\partial u^m}{\partial t} \Big|_{t=0} = 0$, then taking into account geometry of the domain of dependence of the solution of linear wave equation $L_0 u^m = F^m$, we shall have $supp u^m \subset \{(x, t) \in \mathbb{R}^{n+1} : t > |x|\}$ [26, p. 191] and, in particular, $u^m|_{S_T} = 0$. On the other hand the vector-function $\tilde{u}^m(x_1, \dots, x_n, t) = u^m(x_1, \dots, -x_n, t)$ is also a solution of the same Cauchy problem (3.9), because the vector-function F^m is even with respect to the variable x_n . Therefore, due to the uniqueness of the solution of the Cauchy problem we have $\tilde{u}^m = u^m$, i.e. $u^m(x_1, \dots, -x_n, t) = u^m(x_1, \dots, x_n, t)$, and thereby the vector-function u^m is also an even function with respect the variable x_n . Whence, in turn, it follows that $\frac{\partial u^m}{\partial x_n} \Big|_{x_n=0} = 0$, which with the condition $u^m|_{S_T} = 0$ means that if we leave the same notation

for the restriction of vector-function u^m in the domain D_T , then it is obvious that $u^m \in \overset{0}{C}^2(\overline{D}_T, S_T^0, S_T)$. Further, due to (3.4) and (3.9) the following inequality

$$\lim_{m \rightarrow \infty} \|u^m - u^k\|_{\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)} \leq c(T) \|F^m - F^k\|_{L_2(D_T)} \tag{3.10}$$

is valid.

Since the sequence $\{F^m\}$ is fundamental in $L_2(D_T)$, then due to (3.10) the sequence $\{u^m\}$ is also fundamental in the complete space $\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$. Therefore, there exists a vector-function $u \in \overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$ such that $\lim_{m \rightarrow \infty} \|u^m - u\|_{\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)} = 0$, and since $L_0 u^m = F^m \rightarrow F$ in the space $L_2(D_T)$, then this vector-function according to the Remark 3.1 will be a strong generalized solution of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T . The uniqueness of this solution from the space $\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$ follows, due to the Remark 3.2, from the a priori estimate (3.4). Therefore, for the solution u of the problem (3.1), (3.2) we can write $u = L_0^{-1}F$, where $L_0^{-1} : [L_2(D_T)]^N \rightarrow \left[\overset{0}{W}_{\frac{1}{2}}(D_T, S_T) \right]^N$ is a linear continuous operator with a norm admitting in view of (3.4) the following estimate

$$\|L_0^{-1}\|_{[L_2(D_T)]^N \rightarrow [\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)]^N} \leq \sqrt{T} \exp \frac{1}{2}(T^2 + T). \tag{3.11}$$

Remark 3.3. Taking into account (3.11), when the condition (2.3) is fulfilled, where $0 \leq \alpha < \frac{n+1}{n-1}$ and $F \in L_2(D_T)$, due to the Remark 2.1, it is easy to see that the vector-function $u = (u_1, \dots, u_N) \in \overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$ represents a strong generalized solution of the problem (2.1), (2.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the following functional equation

$$u = L_0^{-1}(-f(u) + F) \tag{3.12}$$

in the space $\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$.

Remark 3.4. Let the condition (2.3) be fulfilled and $0 \leq \alpha < \frac{n+1}{n-1}$. Rewrite the equation (3.12) in the form

$$u = Au := L_0^{-1}(-K_0 u + F), \tag{3.13}$$

where the operator $K_0 : [\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)]^N \rightarrow [L_2(D_T)]^N$ from (2.4) due to the Remark 2.1 is a continuous and compact operator. Therefore, according to (3.11) and (3.13) the operator $A : [\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)]^N \rightarrow [\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)]^N$ is also continuous and compact. Denote by $B(0, r_0) := \{u = (u_1, \dots, u_N) \in \overset{0}{W}_{\frac{1}{2}}(D_T, S_T) : \|u\|_{\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)} \leq r_0\}$ a closed (convex) ball with radius $r_0 > 0$ and center in null element in the Hilbert space $\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$. Since the operator A from (3.13), acting in the space $\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$, is a compact continuous

operator, then, according to the Schauder principle, for solvability of the equation (3.13) in the space $\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$ it suffices to prove that the operator A maps the ball $B(0, r_0)$ into itself for some $r_0 > 0$ [27, p. 370].

Theorem 3.1. *Let f satisfy the condition (2.3), where $1 \leq \alpha < \frac{n+1}{n-1}$; $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem (1.1), (1.2) is locally solvable in the class $W_{\frac{1}{2}}^1$, i.e. there exists a number $T_0 = T_0(F) > 0$ such that for any $T < T_0$ this problem has a strong generalized solution of the class $W_{\frac{1}{2}}^1$ in the domain D_T in the sense of the Definition 2.1.*

Proof. Due to Remark 3.4 it suffices to prove the existence of positive numbers $T_0 = T_0(F)$ and $r_0 = r_0(T, F)$ such that for $T < T_0$ the operator A from (3.13) maps the ball $B(0, r_0)$ into itself. For this purpose let us evaluate $\|Au\|_{\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)}$ for $u \in \overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$. If $u = (u_1, \dots, u_N) \in \overset{0}{W}_{\frac{1}{2}}(D_T, S_T)$, then let us denote by \tilde{u} the vector-function which represents an even extension of u through planes $x_n = 0$ and $t = T$. It is obvious that $\tilde{u} \in \overset{0}{W}_{\frac{1}{2}}(D_T^*) := \{v \in W_{\frac{1}{2}}^1(D_T^*) : v|_{\partial D_T^*} = 0\}$, where $D_T^* : |x| < t < 2T - |x|$.

Using inequality [28, p. 258]

$$\int_{\Omega} |v| d\Omega \leq (mes \Omega)^{1-1/p} \|v\|_{p, \Omega}, \quad p \geq 1,$$

and taking into account equalities

$$\|\tilde{u}\|_{L_p(D_T^*)}^p = 4\|u\|_{L_p(D_T)}^p, \quad \|\tilde{u}\|_{\overset{0}{W}_{\frac{1}{2}}(D_T^*)}^2 = 4\|u\|_{\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)}^2,$$

from known multiplicative inequality [21, p. 78]

$$\|v\|_{p, \Omega} \leq \beta \|\nabla_{x,t} v\|_{m, \Omega}^{\tilde{\alpha}} \|v\|_{r, \Omega}^{1-\tilde{\alpha}} \quad \forall v \in \overset{0}{W}_{\frac{1}{2}}(\Omega), \quad \Omega \subset \mathbb{R}^{n+1},$$

$$\nabla_{x,t} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t} \right), \quad \tilde{\alpha} = \left(\frac{1}{r} - \frac{1}{p} \right) \left(\frac{1}{r} - \frac{1}{\tilde{m}} \right)^{-1}, \quad \tilde{m} = \frac{(n+1)m}{n+1-m}$$

for $\Omega = D_T^* \subset \mathbb{R}^{n+1}$, $v = \tilde{u}$, $r = 1$, $m = 2$ and $1 < p \leq \frac{2(n+1)}{n+1-m}$, where $\beta = const > 0$ does not depend on v and T , we obtain the following inequality:

$$\|u\|_{L_p(D_T)} \leq c_0 (mes D_T)^{\frac{1}{p} + \frac{1}{n+1} - \frac{1}{2}} \|u\|_{\overset{0}{W}_{\frac{1}{2}}(D_T, S_T)} \quad \forall u \in \overset{0}{W}_{\frac{1}{2}}(D_T, S_T), \quad (3.14)$$

where $c_0 = const > 0$ does not depend on u and T .

Since $mes D_T = \frac{\omega_n}{n+1} T^{n+1}$, where ω_n is the volume of a unit ball in \mathbb{R}^n , then for $p = 2\alpha$ from (3.14) we have

$$\|u\|_{L_{2\alpha}(D_T)} \leq C_T \|u\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)} \quad \forall u \in \dot{W}^{\frac{0}{2}}(D_T, S_T), \quad (3.15)$$

where $C_T = c_0 \left(\frac{\omega_n}{n+1}\right)^{\alpha_1} T^{(n+1)\alpha_1}$, $\alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}$.

Note that $\alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $\alpha < \frac{n+1}{n-1}$, and therefore

$$\lim_{T \rightarrow 0} C_T = 0. \quad (3.16)$$

For $\|K_0 u\|_{L_2(D_T)}$, where $u \in \dot{W}^{\frac{0}{2}}(D_T, S_T)$ and operator K_0 acts according to the formula (2.4), due to (2.3) and (3.15) we have the following estimate

$$\begin{aligned} \|K_0 u\|_{L_2(D_T)}^2 &\leq \int_{D_T} (M_1 + M_2 |u|^\alpha)^2 dx dt \leq 2M_1^2 mes D_T \\ &\quad + 2M_2^2 \int_{D_T} |u|^{2\alpha} dx dt = 2M_1^2 mes D_T + 2M_2^2 \|u\|_{L_{2\alpha}(D_T)}^{2\alpha} \\ &\leq 2M_1^2 mes D_T + 2M_2^2 C_T^{2\alpha} \|u\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)}^{2\alpha}, \end{aligned}$$

whence we have

$$\|K_0 u\|_{L_2(D_T)} \leq M_1 (2mes D_T)^{\frac{1}{2}} + \sqrt{2} M_2 C_T^\alpha \|u\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)}^\alpha. \quad (3.17)$$

From (3.11), (3.13) and (3.17) it follows that

$$\begin{aligned} \|Au\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)} &= \|L_0^{-1}(-K_0 u + F)\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)} \\ &\leq \|L_0^{-1}\|_{[L_2(D_T)]^N \rightarrow [\dot{W}^{\frac{0}{2}}(D_T, S_T)]^N} \|(-K_0 u + F)\|_{L_2(D_T)} \\ &\leq \left[\sqrt{T} \exp \frac{1}{2}(T + T^2) \right] (\|K_0 u\|_{L_2(D_T)} + \|F\|_{L_2(D_T)}) \\ &\leq \left[\sqrt{T} \exp \frac{1}{2}(T + T^2) \right] \left(M_1 (2mes D_T)^{\frac{1}{2}} \right. \\ &\quad \left. + \sqrt{2} M_2 C_T^\alpha \|u\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)}^\alpha \right. \\ &\quad \left. + \|F\|_{L_2(D_T)} \right) = a(T) \|u\|_{\dot{W}^{\frac{0}{2}}(D_T, S_T)}^\alpha + b(T). \quad (3.18) \end{aligned}$$

Here

$$a(T) = \sqrt{2} M_2 C_T^\alpha \sqrt{T} \exp \frac{1}{2}(T + T^2), \quad (3.19)$$

$$b(T) = \left[\sqrt{T} \exp \frac{1}{2}(T + T^2) \right] \left(M_1 (2mes D_T)^{1/2} + \|F\|_{L_2(D_T)} \right). \quad (3.20)$$

For fixed $T > 0$ consider equation

$$az^\alpha + b = z \quad (3.21)$$

with respect to unknown $z \in \mathbb{R}$, where $a = a(T)$ and $b = b(T)$ are defined by (3.19) and (3.20).

First consider the case when $\alpha > 1$. Simple analysis, analogous to that for $\alpha = 3$ performed in the work [27, pp. 373, 374], shows that:

1. for $b = 0$, together with trivial root $z_{1=0}$, equation (3.21) has a unique positive root $z_2 = a^{-1/(\alpha-1)}$;
2. if $b > 0$, then for $0 < b < b_0$, where

$$b_0 = b_0(T) = \left[\alpha^{-1/(\alpha-1)} - \alpha^{-\alpha/(\alpha-1)} \right] a^{-1/(\alpha-1)}, \tag{3.22}$$

equation (3.21) has two positive roots z_1 and z_2 , $0 < z_1 < z_2$, and for $b = b_0$ these roots merge and we have one positive root $z_1 = z_2 = z_0 = (\alpha a)^{-1/(\alpha-1)}$;

3. for $b > b_0$ equation (3.21) does not have nonnegative roots. Note that for $0 < b < b_0$ the inequality $z_1 < z_0 = (\alpha a)^{-1/(\alpha-1)} < z_2$ is valid.

In view of absolute continuity of the Lebesgue integral we have $\lim_{T \rightarrow 0} \|F\|_{L_2(D_T)} = 0$. Therefore, taking into account that $mes D_T = \frac{\omega_n}{n+1} T^{n+1}$, then from (3.20) it follows that $\lim_{T \rightarrow 0} b(T) = 0$. At the same time, since $-\frac{1}{\alpha-1} < 0$ for $\alpha > 1$, due to (3.16) from (3.19) and (3.22) we get $\lim_{T \rightarrow 0} b_0(T) = \infty$. Therefore, there exists a number $T_0 = T_0(F) > 0$ such that for $0 < T < T_0$ in view of (3.19)–(3.22) it will hold condition $0 < b < b_0$, and thereby equation (3.21) will have at least one positive root, we denote it by $r_0 = r_0(T, F)$.

In the case when $\alpha = 1$ equation (3.21) represents a linear equation, where $\lim_{T \rightarrow 0} a(T) = 0$. Therefore, for $0 < T < T_0$, where $T_0 = T_0(F)$ is a sufficiently small positive number, this equation will have a unique positive root $z(T, F) = b(1 - a)^{-1}$, which we also denote by $r_0 = r_0(T, F)$.

Now we will show that the operator A from (3.13) maps the ball $B(0, r_0) \subset W^{0,1/2}(D_T, S_T)$ into itself. Indeed, due to (3.18) and equality $ar_0^\alpha + b = r_0$ for any $u \in B(0, r_0)$ we have

$$\|Au\|_{W^{0,1/2}(D_T, S_T)} \leq a \|u\|_{W^{0,1/2}(D_T, S_T)}^\alpha + b \leq ar_0^\alpha + b = r_0. \tag{3.23}$$

In view of the Remark 3.4 this proves the Theorem 3.1.

Theorem 3.2. *Let f satisfy the condition (2.3), where $0 \leq \alpha < 1$; $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem (1.1), (1.2) is globally solvable in the class $W^{1,1/2}_2$, i.e. for any $T > 0$ this problem has a strong generalized solution of the class $W^{1,1/2}_2$ in the domain D_T in the sense of the Definition 2.1.*

Proof. According to the Remark 3.4 for the proof of the Theorem 3.2 it suffices to show that for any $T > 0$ there exists a number $r_0 = r_0(T, F) > 0$ such that the operator A from (3.13) maps the ball $B(0, r_0) \subset W^{0,1/2}(D_T, S_T)$ into itself. Let $1/2 < \alpha < 1$, then, since $2\alpha > 1$ the inequality (3.15) is valid, and therefore, the estimate (3.18) is also valid. For fixed $T > 0$, because $\alpha < 1$ there exists a number $r_0 = r_0(T, F) > 0$ such that

$$a(T)s^\alpha + b(T) \leq r_0 \quad \forall s \in [0, r_0]. \tag{3.24}$$

Indeed, the function $\frac{\lambda(s)}{s}$, where $\lambda(s) = a(T)s^\alpha + b(T)$, is a continuous decreasing function and $\lim_{s \rightarrow +0} \frac{\lambda(s)}{s} = +\infty$, $\lim_{s \rightarrow +\infty} \frac{\lambda(s)}{s} = 0$. Therefore there exists a number $s = r_0(T, F) > 0$ such that $\frac{\lambda(s)}{s}|_{s=r_0} = 1$. Whence, since the function

$\lambda(s)$ for $s \geq 0$ is a monotonic increasing function, (3.24) follows immediately. Now, in view of (3.18) and (3.24), for any $u \in B(0, r_0)$ it is valid the inequality (3.23), i.e. $A(B(0, r_0)) \subset B(0, r_0)$.

The case $0 \leq \alpha \leq \frac{1}{2}$ can be reduced to the previous case $\frac{1}{2} < \alpha < 1$ because the vector-function f , satisfying the condition (2.3) for $0 \leq \alpha \leq \frac{1}{2}$, satisfies the same condition (2.3) for a certain fixed $\alpha = \alpha_1 \in (\frac{1}{2}, 1)$ with other positive constants M_1 and M_2 (it is easy to see that $M_1 + M_2|u|^\alpha \leq (M_1 + M_2) + M_2|u|^{\alpha_1} \forall u \in \mathbb{R}, \alpha < \alpha_1$). This proves the Theorem 3.2 completely.

Remark 3.5. The global solvability of the problem (1.1), (1.2) in the Theorem 3.2 is proved for the case when the function f satisfies the condition (2.3), where $0 \leq \alpha < 1$. In the case $1 \leq \alpha < \frac{n+1}{n-1}$ in the Theorem 3.1 it is proved the local solvability of this problem, although in this case for additional conditions posed on f , the problem (1.1), (1.2) will be globally solvable as it is shown in the following theorem.

Theorem 3.3. *Let f satisfy the condition (2.3), where $1 \leq \alpha < \frac{n+1}{n-1}$, and $f = \nabla G$, i.e. $f_i(u) = \frac{\partial}{\partial u_i} G(u)$, $u \in \mathbb{R}^N$, $i = 1, \dots, N$, where $G = G(u) \in C^1(\mathbb{R}^N)$ is a scalar function satisfying conditions $G(0) = 0$ and $G(u) \geq 0 \forall u \in \mathbb{R}^N$. Let $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem (1.1), (1.2) is globally solvable in the class W_2^1 , i.e. for any $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of the Definition 2.1.*

Proof. First let us show that for any fixed $T > 0$, when the conditions of the Theorem 3.3 are fulfilled, for a strong generalized solution u of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T the a priori estimate (3.4) is valid. Indeed, due to the Definition 2.1 there exists a sequence of the vector-functions $u^m \in C^2(\overline{D}_T, S_T^0, S_T)$ such that

$$\lim_{m \rightarrow \infty} \|u^m - u\|_{W_2^1(D_T)} = 0, \quad \lim_{m \rightarrow \infty} \|Lu^m - F\|_{L_2(D_T)} = 0. \quad (3.25)$$

Let

$$F^m := Lu^m, \quad (3.26)$$

then due to the equality (3.5) we have

$$\int_{\Omega_\tau} \left[\left(\frac{\partial u^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial u^m}{\partial x_i} \right)^2 \right] dx = 2 \int_{D_\tau} (F^m - f(u^m)) \frac{\partial u^m}{\partial t} dx dt. \quad (3.27)$$

Since $f = \nabla G$, then $f(u^m) \frac{\partial u^m}{\partial t} = \frac{\partial}{\partial t} G(u^m)$ and taking into account that $u^m|_{S_T} = 0$, $\nu_{n+1}|_{S_\tau^0} = 0$, $\nu_{n+1}|_{\Omega_\tau} = 1$, $G(0) = 0$, by integration by parts we have

$$\begin{aligned} \int_{D_\tau} f(u^m) \frac{\partial u^m}{\partial t} dxdt &= \int_{D_\tau} \frac{\partial}{\partial t} G(u^m) dxdt = \int_{\partial D_\tau} G(u^m) \nu_{n+1} ds \\ &= \int_{S_\tau^0 \cup S_\tau \cup \Omega_\tau} G(u^m) \nu_{n+1} ds = \int_{\Omega_\tau} G(u^m) dx. \end{aligned} \tag{3.28}$$

In view of (3.28) and $G \geq 0$ from (3.27) we have

$$\begin{aligned} \int_{\Omega_\tau} \left[\left(\frac{\partial u^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial u^m}{\partial x_i} \right)^2 \right] dx &= 2 \int_{D_\tau} F^m \frac{\partial u^m}{\partial t} dxdt \\ &\quad - 2 \int_{\Omega_\tau} G(u^m) dx \leq 2 \int_{D_\tau} F^m \frac{\partial u^m}{\partial t} dxdt. \end{aligned} \tag{3.29}$$

Using the same reasonings as those used for receiving the estimate (3.4), from (3.29) we get the following inequality

$$\|u^m\|_{\overset{0}{W} \frac{1}{2}(D_T, S_T)} \leq c(T) \|F^m\|_{L_2(D_T)}, \quad c(T) = \sqrt{T} \exp \frac{1}{2}(T + T^2),$$

whence, due to (3.25) and (3.26), it follows (3.4).

According to the Remarks 3.3 and 3.4, at fulfilment of the conditions of the Theorem 3.3, the vector-function $u \in \overset{0}{W} \frac{1}{2}(D_T, S_T)$ represents a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 then and only then, when u represents a solution to the functional equation $u = Au$ from (3.13) in the space $\overset{0}{W} \frac{1}{2}(D_T, S_T)$, where the operator $A : [\overset{0}{W} \frac{1}{2}(D_T, S_T)]^N \rightarrow [\overset{0}{W} \frac{1}{2}(D_T, S_T)]^N$ is continuous and compact. At the same time, as it was shown above, for any $\mu \in [0, 1]$ and for any solution of the equation $u = \mu Au$ with the parameter μ in the space $\overset{0}{W} \frac{1}{2}(D_T, S_T)$ it is valid the following a priori estimate $\|u\|_{\overset{0}{W} \frac{1}{2}(D_T, S_T)} \leq \mu c(T) \|F\|_{L_2(D_T)} \leq c(T) \|F\|_{L_2(D_T)}$ with positive constant $c(T)$, not depending on u, μ and F . Therefore, according to the Leray-Schauder theorem [27, p. 375] the equation (3.13), and therefore the problem (1.1), (1.2) has at least one strong generalized solution of the class W_2^1 in the domain D_T for any $T > 0$. The Theorem 3.3 is proved.

4. The uniqueness and existence of a global solution of the problem (1.1), (1.2) in the class W_2^1

Below we impose on the nonlinear vector-function $f = (f_1, \dots, f_N)$ from (1.1) additional requirements

$$f \in C^1(\mathbb{R}^N), \quad \left| \frac{\partial f_i(u)}{\partial u_j} \right| \leq M_3 + M_4 |u|^\gamma \quad \forall u \in \mathbb{R}^n, \quad 1 \leq i, j \leq N, \tag{4.1}$$

where $M_3, M_4, \gamma = const \geq 0$. For simplicity we assume that the vector-function $g = 0$ in the boundary condition (1.2).

Remark 4.1. It is obvious that from (4.1) it follows the condition (2.3) for $\gamma = \alpha - 1$, and in the case $\gamma < \frac{2}{n-1}$ we have $1 \leq \alpha = \gamma + 1 < \frac{n+1}{n-1}$.

Theorem 4.1. Let the condition (4.1) be fulfilled, where $0 \leq \gamma < \frac{2}{n-1}$; $F \in L_2(D_T)$ and $g = 0$. Then the problem (1.1), (1.2) cannot have more than one strong generalized solution of the class W_2^1 in the domain D_T in the sense of the Definition 2.1.

Proof. Let $F \in L_2(D_T)$, $g = 0$, and assume that the problem (1.1), (1.2) has two strong generalized solutions u^1 and u^2 of the class W_2^1 in the domain D_T in the sense of Definition 2.1, i.e. there exist two sequences of vector-functions $u^{im} \in C^0(\overline{D}_T, S_T^0, S_T)$, $i = 1, 2$; $m = 1, 2, \dots$, such that

$$\lim_{m \rightarrow \infty} \|u^{im} - u^i\|_{W_{\frac{1}{2}}^0(D_T, S_T)} = 0, \quad \lim_{m \rightarrow \infty} \|Lu^{im} - F\|_{L_2(D_T)} = 0, \quad i = 1, 2. \tag{4.2}$$

Let

$$w = u^2 - u^1, \quad w^m = u^{2m} - u^{1m}, \quad F^m = Lu^{2m} - Lu^{1m}. \tag{4.3}$$

In view of (4.2) and (4.3) we have

$$\lim_{m \rightarrow \infty} \|w^m - w\|_{W_{\frac{1}{2}}^0(D_T, S_T)} = 0, \quad \lim_{m \rightarrow \infty} \|F^m\|_{L_2(D_T)} = 0. \tag{4.4}$$

In accordance with (4.3) consider the vector-function $w^m \in C^0(\overline{D}_T, S_T^0, S_T)$ as a solution of the following problem

$$\square w^m = -[f(u^{2m}) - f(u^{1m})] + F^m, \tag{4.5}$$

$$\left. \frac{\partial w^m}{\partial x_n} \right|_{S_T^0} = 0, \quad w^m|_{S_T} = 0. \tag{4.6}$$

From (4.5), (4.6) and in view of the equality (3.5) it follows

$$\begin{aligned} \int_{\Omega_\tau} \left[\left(\frac{\partial w^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dx &= 2 \int_{D_\tau} F^m \frac{\partial w^m}{\partial t} dx dt \\ &- 2 \int_{D_\tau} [f(u^{2m}) - f(u^{1m})] \frac{\partial w^m}{\partial t} dx dt, \quad 0 < \tau \leq T. \end{aligned} \tag{4.7}$$

Taking into account the equality

$$f_i(u^{2m}) - f_i(u^{1m}) = \sum_{j=1}^N \int_0^1 \frac{\partial}{\partial u_j} f_i(u^{1m} + s(u^{2m} - u^{1m})) ds (u_j^{2m} - u_j^{1m}),$$

we receive

$$\begin{aligned} &[f(u^{2m}) - f(u^{1m})] \frac{\partial w^m}{\partial t} \\ &= \sum_{i,j=1}^N \left[\int_0^1 \frac{\partial}{\partial u_j} f_i(u^{1m} + s(u^{2m} - u^{1m})) ds \right] (u_j^{2m} - u_j^{1m}) \frac{\partial w_i^m}{\partial t}. \end{aligned} \tag{4.8}$$

From (4.1) and obvious inequality $|d_1 + d_2|^\gamma \leq 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \leq 2^\gamma (|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma \geq 0, d_i \in \mathbb{R}$, we have

$$\left| \int_0^1 \frac{\partial}{\partial u_j} f_i(u^{1m} + s(u^{2m} - u^{1m})) ds \right| \leq \int_0^1 [M_3 + M_4|(1-s)u^{1m} + su^{2m}|^\gamma] ds \leq M_3 + 2^\gamma M_4 (|u^{1m}|^\gamma + |u^{2m}|^\gamma). \quad (4.9)$$

From (4.8) and (4.9), taking into account (4.3), it follows

$$\begin{aligned} \left| [f(u^{2m}) - f(u^{1m})] \frac{\partial w^m}{\partial t} \right| &\leq \sum_{i,j=1}^N [M_3 + 2^\gamma M_4 (|u^{1m}|^\gamma + |u^{2m}|^\gamma)] |w_j^m| \left| \frac{\partial w_i^m}{\partial t} \right| \\ &\leq N^2 [M_3 + 2^\gamma M_4 (|u^{1m}|^\gamma + |u^{2m}|^\gamma)] |w^m| \left| \frac{\partial w^m}{\partial t} \right| \\ &\leq \frac{1}{2} N^2 M_3 \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t} \right)^2 \right] \\ &\quad + 2^\gamma N^2 M_4 (|u^{1m}|^\gamma + |u^{2m}|^\gamma) |w^m| \left| \frac{\partial w^m}{\partial t} \right|. \end{aligned} \quad (4.10)$$

Due to (4.7) and (4.10) we have

$$\begin{aligned} \int_{\Omega_\tau} \left[\left(\frac{\partial w^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dx &\leq \int_{D_\tau} \left[\left(\frac{\partial w^m}{\partial t} \right)^2 + (F^m)^2 \right] dx dt \\ &\quad + N^2 M_3 \int_{D_\tau} \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t} \right)^2 \right] dx dt \\ &\quad + 2^{\gamma+1} N^2 M_4 \int_{D_\tau} (|u^{1m}|^\gamma + |u^{2m}|^\gamma) |w^m| \left| \frac{\partial w^m}{\partial t} \right| dx dt. \end{aligned} \quad (4.11)$$

The latter integral in the right side of (4.11) can be estimated by Holder's inequality

$$\begin{aligned} \int_{D_\tau} (|u^{1m}|^\gamma + |u^{2m}|^\gamma) |w^m| \left| \frac{\partial w^m}{\partial t} \right| dx dt &\leq \left(\left\| |u^{1m}|^\gamma \right\|_{L_{n+1}(D_\tau)} + \left\| |u^{2m}|^\gamma \right\|_{L_{n+1}(D_\tau)} \right) \|w^m\|_{L_p(D_\tau)} \left\| \frac{\partial w^m}{\partial t} \right\|_{L_2(D_\tau)}. \end{aligned} \quad (4.12)$$

Here $\frac{1}{n+1} + \frac{1}{p} + \frac{1}{2} = 1$, i.e. for

$$p = \frac{2(n+1)}{n-1}. \quad (4.13)$$

In view of (3.14) for $q \leq \frac{2(n+1)}{n-1}$ we have

$$\|v\|_{L_q(D_\tau)} \leq C_q(T) \|v\|_{\overset{0}{W}_{\frac{1}{2}}(D_\tau, S_\tau)} \quad \forall v \in \overset{0}{W}_{\frac{1}{2}}(D_\tau, S_\tau), \quad 0 < \tau \leq T, \quad (4.14)$$

with positive constant $C_q(T)$, not depending on $v \in \overset{0}{W}_{\frac{1}{2}}(D_\tau, S_\tau)$ and $\tau \in [0, T]$.

According to the theorem $\gamma < \frac{2}{n-1}$ and, therefore, $\gamma(n+1) < \frac{2(n+1)}{n-1}$. Thus, from (4.13), (4.14) we receive

$$\begin{aligned} \| |u^{im}|^\gamma \|_{L_{n+1}(D_T)} &= \| |u^{im}| \|_{L_{\gamma(n+1)}^\gamma(D_T)}^\gamma \leq C_{\gamma(n+1)}^\gamma(T) \| |u^{im}| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma, \\ i &= 1, 2; m \geq 1, \end{aligned} \tag{4.15}$$

$$\| |w^m| \|_{L_p(D_\tau)} \leq C_p(T) \| |w^m| \|_{W_{\frac{1}{2}}^1(D_\tau)}, m \geq 1. \tag{4.16}$$

In view of the first equality from (4.2) there exists natural number m_0 such that for $m \geq m_0$ we shall have

$$\| |u^{im}| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma \leq \| |u^i| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma + 1, \quad i = 1, 2; m \geq m_0. \tag{4.17}$$

In view of these inequalities from (4.12)–(4.16) it follows that

$$\begin{aligned} &2\gamma+1 N^2 M_4 \int_{D_\tau} (|u^{1m}|^\gamma + |u^{2m}|^\gamma) |w^m| \left| \frac{\partial w^m}{\partial t} \right| dxdt \\ &\leq 2\gamma+1 N^2 M_4 C_{\gamma(n+1)}^\gamma(T) \left(\| |u^1| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma \right. \\ &\quad \left. + \| |u^2| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma + 2 \right) C_p(T) \| |w^m| \|_{W_{\frac{1}{2}}^0(D_\tau, S_\tau)} \| \left| \frac{\partial w^m}{\partial t} \right| \|_{L_2(D_\tau)} \\ &\leq M_5 \left(\| |w^m| \|_{W_{\frac{1}{2}}^1(D_\tau)}^2 + \left\| \frac{\partial w^m}{\partial t} \right\|_{L_2(D_\tau)}^2 \right) \leq 2M_5 \| |w^m| \|_{W_{\frac{1}{2}}^1(D_\tau)}^2 \\ &= 2M_5 \int_{D_\tau} \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dxdt, \end{aligned} \tag{4.18}$$

where $M_5 = 2\gamma N^2 M_4 C_{\gamma(n+1)}^\gamma(T) \left(\| |u^1| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma + \| |u^2| \|_{W_{\frac{1}{2}}^0(D_T, S_T)}^\gamma + 2 \right) C_p(T)$.

Due to (4.17) from (4.11) we have

$$\begin{aligned} \int_{\Omega_\tau} \left[\left(\frac{\partial w^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dx &\leq M_6 \int_{D_\tau} \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t} \right)^2 \right. \\ &\quad \left. + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dxdt + \int_{D_T} (F^m)^2 dxdt, \quad 0 < \tau \leq T, \end{aligned} \tag{4.19}$$

where $M_6 = 1 + M_3 N^2 + 2M_5$.

Note, that the inequality (3.6) is valid for w^m too, and, therefore,

$$\begin{aligned} \int_{\Omega_\tau} (w^m)^2 dx &\leq T \int_{D_\tau} \left(\frac{\partial w^m}{\partial t} \right)^2 dxdt \\ &\leq T \int_{D_\tau} \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dxdt. \end{aligned} \tag{4.20}$$

Putting

$$\lambda_m(\tau) := \int_{\Omega_\tau} \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i} \right)^2 \right] dx \tag{4.21}$$

and adding (4.18) to (4.19), we receive

$$\lambda_m(\tau) \leq (M_6 + T) \int_0^\tau \lambda_m(s) ds + \|F^m\|_{L_2(D_T)}^2.$$

Whence, by the Gronwall's lemma [25, p. 13], it follows that

$$\lambda_m(\tau) \leq \|F^m\|_{L_2(D_T)}^2 \exp(M_6 + T)\tau. \tag{4.22}$$

From (4.20) and (4.21) we have

$$\|w^m\|_{W_2^1(D_T)}^2 = \int_0^T \lambda_m(\tau) d\tau \leq T \|F^m\|_{L_2(D_T)}^2 \exp(M_6 + T)T. \tag{4.23}$$

In view of (4.3), (4.4) from (4.22) it follows that

$$\begin{aligned} \|w\|_{W_2^1(D_T)} &= \lim_{m \rightarrow \infty} \|w - w^m + w^m\|_{W_2^1(D_T)} \\ &\leq \lim_{m \rightarrow \infty} \|w - w^m\|_{W_2^1(D_T)} + \lim_{m \rightarrow \infty} \|w^m\|_{W_2^1(D_T)} \\ &= \lim_{m \rightarrow \infty} \|w - w^m\|_{W_2^1(D_T)} = \lim_{m \rightarrow \infty} \|w - w^m\|_{\overset{0}{W} \frac{1}{2}(D_T, S_T)} = 0. \end{aligned}$$

Therefore $w = u_2 - u_1 = 0$, i.e. $u_2 = u_1$. The Theorem 4.1 is proved.

From the Theorems 3.2, 3.3, 4.1 and the Remark 4.1 it follows the next theorem of existence and uniqueness.

Theorem 4.2. *Let the vector-function f satisfy the condition (4.1), where $0 < \gamma < \frac{2}{n-1}$, and either f satisfy the condition (2.3) for $\alpha < 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \geq 0 \forall u \in \mathbb{R}^N$. Then for any $F \in L_2(D_T)$ and $g = 0$ the problem (1.1), (1.2) has unique strong generalized solution $u \in \overset{0}{W} \frac{1}{2}(D_T, S_T)$ of the class W_2^1 in the domain D_T in the sense of the Definition 2.1. The following theorem on existence of global solution of this problem follows from the Theorem 4.2.*

Theorem 4.3. *Let the vector-function f satisfy the condition (4.1), where $0 < \gamma < \frac{2}{n-1}$, and either f satisfy the condition (2.3) for $\alpha < 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \geq 0 \forall u \in \mathbb{R}^N$. Let $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem (1.1), (1.2) has unique global strong generalized solution $u \in \overset{0}{W} \frac{1}{2,loc}(D_\infty, S_\infty)$ of the class W_2^1 in the domain D_∞ in the sense of the Definition 2.1.*

Proof. According to Theorem 4.2 when the conditions of Theorem 4.3 are fulfilled for $T = k$, where k is a natural number, there exists unique strong generalized solution $u^k \in \overset{0}{W} \frac{1}{2}(D_T, S_T)$ of the problem (1.1), (1.2) of the class W_2^1 in the domain $D_{T=k}$ in the sense of Definition 2.1. Since $u^{k+1}|_{D_{T=k}}$ is also a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain $D_{T=k}$, then in view of Theorem 4.2 we have $u^k = u^{k+1}|_{D_{T=k}}$. Thus one can construct unique global generalized solution $u \in \overset{0}{W} \frac{1}{2,loc}(D_\infty, S_\infty)$ of the problem (1.1), (1.2) of the class W_2^1 in the domain D_∞ in the sense of the Definition 2.4 in the following way:

$$u(x, t) = u^k(x, t), \quad (x, t) \in D_\infty, \quad k = [t] + 1,$$

where $[t]$ is an integer part of the number t . The Theorem 4.3 is proved.

5. The cases of absence of a global solution of the problem (1.1), (1.2) of the class W_2^1

Theorem 5.1. *Let the vector-function $f = (f_1, \dots, f_N)$ satisfy the condition (2.3), where $1 < \alpha < \frac{n+1}{n-1}$, and there exist numbers l_1, \dots, l_N , $\sum_{i=1}^N |l_i| \neq 0$, such that*

$$\sum_{i=1}^N l_i f_i(u) \leq c_0 - c_1 \left| \sum_{i=1}^N l_i u_i \right|^\beta \quad \forall u \in \mathbb{R}^N, \quad 1 < \beta = \text{const} < \frac{n+1}{n-1}, \quad (5.1)$$

where $c_0, c_1 = \text{const}, c_1 > 0$. Let $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. Let at least one of functions $F_0 = \sum_{i=1}^N l_i F_i - c_0$ or $\frac{\partial g_0}{\partial \mathbb{N}} \Big|_{S_\infty}$, where $g_0 = \sum_{i=1}^N l_i g_i$, is nontrivial (i.e. differs from zero on a subset with positive measure in D_∞ or S_∞ , respectively). Then if

$$g_0 \geq 0, \quad \frac{\partial g_0}{\partial \mathbb{N}} \Big|_{S_\infty} \leq 0, \quad F_0|_{D_\infty} \geq 0, \quad (5.2)$$

there exists finite positive number $T_0 = T_0(F, g)$ such that for $T > T_0$ the problem (1.1), (1.2) does not have a strong generalized solution of the class W_2^1 in the domain D_T in the sense of the Definition 2.1. Here $\frac{\partial}{\partial \mathbb{N}}$ is a derivative with respect to the conormal to S_∞ , i.e. $\frac{\partial}{\partial \mathbb{N}} = \nu_{n+1} \frac{\partial}{\partial t} - \sum_{i=1}^N \nu_i \frac{\partial}{\partial x_i}$, where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is a unit vector of the outer normal to $\partial D_\infty = S_\infty$, which is an inner differential operator on the characteristic manifold S_∞ .

Proof. Let $G_T : |x| < t < T$, $G_T^- = G_T \cap \{x_n < 0\}$, $S_T^- : t = |x|, x_n \leq 0, t \leq T$. It is obvious that $D_T = G_T^+ : G_T \cap \{x_n > 0\}$ and $G_T = G_T^- \cup (S_T^0 \setminus \partial S_T^0) \cup G_T^+$, where $S_T^0 = \partial D_T \cap \{x_n = 0\}$. Let $u = (u_1, \dots, u_N)$ represents a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T in the sense of the Definition 2.1. Let us extend the vector-functions u, F and g evenly with respect to variable x_n in G_T^- and S_T^- , respectively. For simplicity we leave the same notations u, F and g to the extended functions defined in G_T and $S_T^- \cup S_T$. Let us show that the function $u = (u_1, \dots, u_N)$, defined in the domain G_T , satisfies the following integral equality

$$\begin{aligned} \int_{G_T} [-u_t w_t + \nabla u \nabla w] dx dt &= - \int_{G_T} f(u) w dx dt \\ &+ \int_{G_T} F w dx dt - \int_{S_T^- \cup S_T} \frac{\partial g}{\partial \mathbb{N}} w ds \end{aligned} \quad (5.3)$$

for any vector-function $w = (w_1, \dots, w_N) \in W_2^1(G_T)$ such that $w|_{t=T} = 0$ in the sense of the trace theory. Indeed, if $w \in W_2^1(G_T)$ and $w|_{t=T} = 0$, then it is obvious that $w|_{D_T} \in W_2^1(D_T)$ and $\tilde{w} \in W_2^1(D_T)$, where by definition $\tilde{w}(x_1, \dots, x_n, t) = w(x_1, \dots, -x_n, t)$, $(x_1, \dots, x_n, t) \in D_T$, and $\tilde{w}|_{t=T} = 0$.

Therefore, according to the equality (2.6) from the Remark 2.2, for $\varphi = w$ and $\varphi = \tilde{w}$ we have

$$\int_{D_T} [-u_t w_t + \nabla u \nabla w] dxdt = - \int_{D_T} f(u) w dxdt + \int_{D_T} F w dxdt - \int_{S_T} \frac{\partial g}{\partial \mathbb{N}} w ds \quad (5.4)$$

and

$$\int_{D_T} [-u_t \tilde{w}_t + \nabla u \nabla \tilde{w}] dxdt = - \int_{D_T} f(u) \tilde{w} dxdt + \int_{D_T} F \tilde{w} dxdt - \int_{S_T} \frac{\partial g}{\partial \mathbb{N}} \tilde{w} ds, \quad (5.5)$$

respectively. Since u, F and g are even vector-functions with respect to variable x_n and $\tilde{w}(x_1, \dots, x_n, t) = w(x_1, \dots, -x_n, t)$, $(x_1, \dots, x_n, t) \in D_T$, then we have

$$\int_{D_T} [-u_t \tilde{w}_t + \nabla u \nabla \tilde{w}] dxdt = \int_{G_T^-} [-u_t w_t + \nabla u \nabla w] dxdt, \quad (5.6)$$

$$\begin{aligned} & - \int_{D_T} f(u) \tilde{w} dxdt + \int_{D_T} F \tilde{w} dxdt - \int_{S_T} \frac{\partial g}{\partial \mathbb{N}} \tilde{w} ds \\ & = - \int_{G_T^-} f(u) w dxdt + \int_{G_T^-} F w dxdt - \int_{S_T^-} \frac{\partial g}{\partial \mathbb{N}} w ds. \end{aligned} \quad (5.7)$$

From (5.5)–(5.7) it follows that

$$\int_{G_T^-} [-u_t w_t + \nabla u \nabla w] dxdt = - \int_{G_T^-} f(u) w dxdt + \int_{G_T^-} F w dxdt - \int_{S_T^-} \frac{\partial g}{\partial \mathbb{N}} w ds. \quad (5.8)$$

Finally, summing the equalities (5.4) and (5.8) we receive (5.3).

Let us use the method of test functions [9, pp. 10–12].

In the integral equality (5.3) for test vector-function w we choose $w = (l_1 \psi, \dots, l_N \psi)$, where $\psi = \psi_0 [2T^{-2}(t^2 + |x|^2)]$, while a scalar function $\psi_0 \in C^2(\mathbb{R})$ satisfies the following conditions: $\psi_0 \geq 0$, $\psi'_0 \leq 0$; $\psi(\sigma) = 1$ for $0 \leq \sigma \leq 1$ and $\psi(\sigma) = 0$ for $\sigma \geq 2$ [9, p. 22]. For the chosen test function w , using notations $v = \sum_{i=1}^N l_i u_i$, $g_0 = \sum_{i=1}^N l_i g_i$, $F_* = \sum_{i=1}^N l_i F_i$, $f_0 = \sum_{i=1}^N l_i f_i$, the integral equality (5.3) takes the following form

$$\int_{G_T} [-v_t \psi_t + \nabla v \nabla \psi] dxdt = - \int_{G_T} f_0(u) \psi dxdt + \int_{G_T} F_* \psi dxdt - \int_{S_T^- \cup S_T} \frac{\partial g_0}{\partial \mathbb{N}} \psi ds. \quad (5.9)$$

Due to $\psi|_{t \geq T} = 0$ and the equality $v|_{S_T^- \cup S_T} = g_0$ in the sense of the trace theory, integrating by parts the left side of the equality (5.9), we get

$$\begin{aligned} \int_{G_T} [-v_t \psi_t + \nabla v \nabla \psi] dx dt &= \int_{G_T} v \square \psi dx dt - \int_{S_T^- \cup S_T} v \frac{\partial \psi}{\partial \mathbb{N}} ds \\ &= \int_{G_T} v \square \psi dx dt - \int_{S_T^- \cup S_T} g_0 \frac{\partial \psi}{\partial \mathbb{N}} ds. \end{aligned} \tag{5.10}$$

From (5.9) and (5.10), in view of (5.1) and $\psi \geq 0$, we have

$$\begin{aligned} \int_{G_T} v \square \psi dx dt &\geq \int_{G_T} [c_1 |v|^\beta - c_0] \psi dx dt + \int_{G_T} F_* \psi dx dt + \int_{S_T^- \cup S_T} g_0 \frac{\partial \psi}{\partial \mathbb{N}} ds \\ &\quad - \int_{S_T^- \cup S_T} \frac{\partial g_0}{\partial \mathbb{N}} \psi ds = c_1 \int_{G_T} |v|^\beta \psi dx dt + \int_{G_T} (F_* - c_0) \psi dx dt \\ &\quad + \int_{S_T^- \cup S_T} g_0 \frac{\partial \psi}{\partial \mathbb{N}} ds - \int_{S_T^- \cup S_T} \frac{\partial g_0}{\partial \mathbb{N}} \psi ds. \end{aligned} \tag{5.11}$$

In view of the properties of the function ψ and the inequalities (5.2) we have

$$\begin{aligned} \frac{\partial \psi}{\partial \mathbb{N}} \Big|_{S_T^- \cup S_T} &\geq 0, \quad \int_{S_T^- \cup S_T} g_0 \frac{\partial \psi}{\partial \mathbb{N}} ds \geq 0, \\ \int_{S_T^- \cup S_T} \frac{\partial g_0}{\partial \mathbb{N}} \psi ds &\leq 0, \quad \int_{G_T} F_0 \psi dx dt \geq 0, \end{aligned} \tag{5.12}$$

where $F_0 = F_* - c_0 = \sum_{i=1}^N l_i F_i - c_0$. In derivation of the inequality (5.12) we took into account that $\nu_{n+1}|_{S_T^- \cup S_T} < 0$.

Assuming that the functions F, g and ψ are fixed, introduce into consideration a function of one variable

$$\gamma(T) = \int_{G_T} F_0 \psi dx dt + \int_{S_T^- \cup S_T} g_0 \frac{\partial \psi}{\partial \mathbb{N}} ds - \int_{S_T^- \cup S_T} \frac{\partial g_0}{\partial \mathbb{N}} \psi ds, \quad T > 0. \tag{5.13}$$

Because of the absolute continuity of integral and the inequalities (5.12), the function $\gamma(T)$ from (5.13) is nonnegative, continuous and nondecreasing function and

$$\lim_{T \rightarrow 0} \gamma(T) = 0. \tag{5.14}$$

Besides, since according to supposition at least one from the functions $\frac{\partial g_0}{\partial \mathbb{N}} \Big|_{S_\infty^- \cup S_\infty}$ or F_0 is nontrivial, then

$$\lim_{T \rightarrow +\infty} \gamma(T) > 0. \tag{5.15}$$

In view of (5.13) the inequality (5.11) can be rewritten as follows

$$c_1 \int_{G_T} |v|^\beta \psi dx dt \leq \int_{G_T} v \square \psi dx dt - \gamma(T). \tag{5.16}$$

If in Young's inequality with the parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{\beta} a^\beta + (\beta' \varepsilon^{\beta'-1})^{-1} b^{\beta'}$$

where $\beta' = \beta/(\beta - 1)$, we take $a = |v|\psi^{1/\beta}$, $b = |\square\psi|/\psi^{1/\beta}$, then taking into account equality $\frac{\beta'}{\beta} = \beta' - 1$, we have

$$|v\square\psi| = |v|\psi^{1/\beta} \frac{|\square\psi|}{\psi^{1/\beta}} \leq \frac{\varepsilon}{\beta} |v|^\beta \psi + \frac{1}{\beta' \varepsilon^{\beta'-1}} \frac{|\square\psi|^{\beta'}}{\psi^{\beta'-1}}. \tag{5.17}$$

In view of (5.17) from (5.16) we have

$$\left(c_1 - \frac{\varepsilon}{\beta}\right) \int_{G_T} |v|^\beta \psi dxdt \leq \frac{1}{\beta' \varepsilon^{\beta'-1}} \int_{G_T} \frac{|\square\psi|^{\beta'}}{\psi^{\beta'-1}} dxdt - \gamma(T),$$

whence for $\varepsilon < c_1\beta$ we obtain

$$\int_{G_T} |v|^\beta \psi dxdt \leq \frac{\beta}{(c_1\beta - \varepsilon)\beta' \varepsilon^{\beta'-1}} \int_{G_T} \frac{|\square\psi|^{\beta'}}{\psi^{\beta'-1}} dxdt - \frac{\beta}{c_1\beta - \varepsilon} \gamma(T). \tag{5.18}$$

Taking into account inequalities $\beta' = \beta/(\beta - 1)$, $\beta = \beta'/(\beta' - 1)$ and also equality

$$\min_{0 < \varepsilon < c_1\beta} \frac{\beta}{(c_1\beta - \varepsilon)\beta' \varepsilon^{\beta'-1}} = \frac{1}{c_1^{\beta'}},$$

which is reached for $\varepsilon = c_1$, then from (5.18) it follows that

$$\int_{G_T} |v|^\beta \psi dxdt \leq \frac{1}{c_1^{\beta'}} \int_{G_T} \frac{|\square\psi|^{\beta'}}{\psi^{\beta'-1}} dxdt - \frac{\beta'}{c_1} \gamma(T). \tag{5.19}$$

According to the properties of the function ψ_0 , the test function $\psi(x, t) = \psi_0[2T^{-2}(t^2 + |x|^2)] = 0$ for $r = (t^2 + |x|^2)^{1/2} > T$. Therefore after substitution of variables $t = \frac{1}{\sqrt{2}}T\xi_0$, $x = \frac{1}{\sqrt{2}}T\xi$ we have

$$\int_{G_T} \frac{|\square\psi|^{\beta'}}{\psi^{\beta'-1}} dxdt = \int_{\substack{r = (t^2 + |x|^2)^{1/2} < T \\ t > |x|}} \frac{|\square\psi|^{\beta'}}{\psi^{\beta'-1}} dxdt = \left(\frac{1}{\sqrt{2}}T\right)^{n+1-2\beta'} \varkappa_0. \tag{5.20}$$

Here

$$\varkappa_0 := \int_{\substack{1 < |\xi_0|^2 + |\xi|^2 < 2 \\ \xi_0 > |\xi|}} \frac{|2(1-n)\psi'_0 + 4(\xi_0^2 - |\xi|^2)\psi''_0|^{\beta'}}{\psi_0^{\beta'-1}} d\xi d\xi_0 < +\infty. \tag{5.21}$$

As it is known the test function $\psi(x, t) = \psi_0[2T^{-2}(t^2 + |x|^2)]$ with properties mentioned above, for which the condition (5.21) is valid, exists [9, p. 22].

Due to (5.20), from the inequality (5.19) and equality $\psi_0(\sigma) = 1$, for $0 \leq \sigma \leq 1$, we have

$$\int_{r \leq T/\sqrt{2}} |v|^\beta dxdt \leq \int_{D_T} |v|^\beta \psi dxdt \leq \frac{\left(\frac{1}{\sqrt{2}}T\right)^{n+1-2\beta'}}{c_1^{\beta'}} \alpha_0 - \frac{\beta'}{c_1} \gamma(T). \quad (5.22)$$

When $\beta < (n + 1)/(n - 1)$, i.e. when $n + 1 - 2\beta' < 0$, the equation

$$\lambda(T) = \frac{\left(\frac{1}{\sqrt{2}}T\right)^{n+1-2\beta'}}{c_1^{\beta'}} \alpha_0 - \frac{\beta'}{c_1} \gamma(T) = 0 \quad (5.23)$$

has a unique positive root $T = T_0(F, g)$, since the function

$$\lambda_1(T) = \left(\left(\frac{1}{\sqrt{2}}T \right)^{n+1-2\beta'} / c_1^{\beta'} \right) \alpha_0$$

is a positive, continuous, strictly decreasing function on the interval $(0, +\infty)$, besides, $\lim_{T \rightarrow 0} \lambda_1(T) = +\infty$ and $\lim_{T \rightarrow +\infty} \lambda_1(T) = 0$, and function $\gamma(T)$, as it is stated above, is a nonnegative, continuous and nondecreasing function, satisfying the conditions (5.14) and (5.15). Besides, $\lambda(T) < 0$ for $T > T_0$ and $\lambda(T) > 0$ for $0 < T < T_0$. Therefore, for $T > T_0$ the right side of the inequality (5.22) is negative value, which is impossible. This contradiction proves the Theorem 5.1

Remark 5.1. Let us consider one class of vector-functions f , satisfying the condition (5.1):

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^N a_{ij} |u_j|^{\beta_{ij}} + b_i, \quad i = 1, \dots, N, \quad (5.24)$$

where $a_{ij} = const > 0$, $b_i = const$, $1 < \beta_{ij} = const < \frac{n+1}{n-1}$; $i, j = 1, \dots, N$. In this case we can assume that $l_1 = l_2 = \dots = l_N = -1$. Indeed, let us choose $\beta = const$ in such a way that $1 < \beta < \beta_{ij}$; $i, j = 1, \dots, N$. Then it is easy to verify that $|s|^{\beta_{ij}} \geq |s|^\beta - 1 \forall s \in (-\infty, +\infty)$. Using the inequality [29, p. 302]

$$\sum_{i=1}^N |y_i|^\beta \geq N^{1-\beta} \left| \sum_{i=1}^N y_i \right|^\beta \forall y = (y_1, \dots, y_N) \in \mathbb{R}^N, \beta = const > 1,$$

we have

$$\begin{aligned} \sum_{i=1}^N f_i(u_1, \dots, u_N) &\geq a_0 \sum_{i,j=1}^N |u_j|^{\beta_{ij}} + \sum_{i=1}^N b_i \geq a_0 \sum_{i,j=1}^N (|u_j|^\beta - 1) + \sum_{i=1}^N b_i \\ &= a_0 N \sum_{j=1}^N |u_j|^\beta - a_0 N^2 + \sum_{i=1}^N b_i \geq a_0 N^{2-\beta} \left| \sum_{j=1}^N u_j \right|^\beta + \sum_{i=1}^N b_i - a_0 N^2, \end{aligned}$$

$$a_0 = \min_{i,j} a_{ij} > 0.$$

Whence it follows the inequality (5.1), where

$$l_1 = l_2 = \dots = l_N = -1, c_0 = a_0 N^2 - \sum_{i=1}^N b_i, c_1 = a_0 N^{2-\beta} > 0.$$

Note that the vector-function f , represented by the equalities (5.24), also satisfies the condition (5.1) for $l_1 = l_2 = \dots = l_N = -1$ for less restrictive conditions, when $a_{ij} = \text{const} \geq 0$, but $a_{ik_i} > 0$, where k_1, \dots, k_N is any arbitrary fixed permutation of the numbers $1, 2, \dots, N$; $i, j = 1, \dots, N$.

Remark 5.2. From the Theorem 5.1 it follows that when its conditions are fulfilled, then the problem (1.1), (1.2) can not have a global strong generalized solution of the class W_2^1 in the domain D_∞ in the sense of the Definition 2.4.

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