Proc. A. Razmadze Math. Inst. 162(2013), 141–143

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THE CAUCHY-GOURSAT MULTIDIMENSIONAL PROBLEM FOR ONE CLASS OF NONLINEAR HYPERBOLIC SYSTEMS OF SECOND ORDER

In the Euclidean space $\mathbb{R}^{n+1}$ of independent variables $x=(x_1,\dots,x_n)$ and t we consider a semilinear hyperbolic system of the type

$$(Lu)_i : \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial^2 u_i}{\partial x_1^2} - \frac{\partial^2 u_i}{\partial x_2^2} + f_i(u_1,\dots,u_N) = F_i(x_1,x_2,t), \quad i=1,\dots,N, \tag{1}$$

where $f=(f_1,\dots,f_N)$, $F=(F_1,\dots,F_N)$ are the given and $u=(u_1,\dots,u_N)$ is an unknown real vector-functions, $N\geq 2$.

By $D: t>|x|$, $x_2>0$ we denote a half of the light cone of the future which is bounded by a part $S^0 : \partial D\cap\{x_2=0\}$ of the plane $x_2=0$ and by a half $S: t=|x|$, $x_2\geq 0$ of the characteristic conoid $C: t=|x|$ of the system (1). Assume $D_T := \{(x,t)\in D : t<T\}$, $S_T^0 := \{(x,t)\in S^0 : t\leq T\}$, $S_T := \{(x,t)\in S : t\leq T\}$, $T>0$.

For a system of equations (1), we consider the problem of finding a solution $u(x,t)$ of that system by the boundary conditions

$$\frac{\partial u}{\partial x_2}\Big|_{S_T^0} = 0, \quad u\Big|_{S_T} = 0. \tag{2}$$

In the case if $T=\infty$, we have $D_\infty = D$, $S_\infty^0 = S^0$ and $S_\infty = S$.

The problem (1), (2) is the Cauchy-Goursat multidimensional problem when one part of the problem data is a characteristic manifold and the other one is a time type manifold [1]. Note that in a scalar case, where $N=1$, this problem has been investigated in [2].

Below, to the nonlinear vector-function f from (1) we impose the following restrictions:

$$f\in C(\mathbb{R}^N), \quad |f(u)|\leq M_1 + M_2|u|^\alpha, \quad \alpha=\text{const}\geq 0, \quad u\in\mathbb{R}^N, \tag{3}$$

where $|\cdot|$ is the norm in the space $\mathbb{R}^N$, $M_i=\text{const}\geq 0$, $i=1,2$.

2010 *Mathematics Subject Classification.* 35L51, 35L71.

Key words and phrases. The Cauchy-Goursat multidimensional problem, nonlinear hyperbolic systems, local and global solutions.

Let $\overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T) := \{u \in C^2(\overline{D}_T) : \frac{\partial u}{\partial x_2}\big|_{S_T^0} = 0,\, u\big|_{S_T} = 0\}$. Assume that $\overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) : u\big|_{S_T} = 0\}$, where $W_2^k(D_T)$ is the known Sobolev's space consisting of elements $L_2(D_T)$, having generalized derivatives up to the kth order, inclusive, from $L_2(D_T)$, and the equality $u\big|_{S_T} = 0$ is understood in a sense of the trace theory.

Definition 1. Let $f = (f_1, \dots, f_N)$ satisfy the condition (3), where $0 \le \alpha < 3$; $F = (F_1, \dots, F_N) \in L_2(D_T)$. The vector-function $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ is said to be a strict generalized solution of the problem (1), (2) of the class W_2^1 in the domain D_T, if there exists a sequence of vector-functions $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ such that $u^m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, and $Lu^m \to F$ in the space $L_2(D_T)$.

It can be easily seen that the classical solution $u \in C^2(\overline{D}_T)$ of the problem (1), (2) is likewise a strong generalized solution of that problem of the class W_2^1 in the domain D_T in the sense of Definition 1.

Definition 2. Let $f = (f_1, \dots, f_N)$ satisfy the condition (3), where $0 \le \alpha < 3$; $F = (F_1, \dots, F_N) \in L_{2,\mathrm{loc}}(D_\infty)$ and $F\big|_{D_T} \in L_2(D_T)$ for any $T > 0$. We say that the problem (1), (2) is locally solvable in the class W_2^1, if there exists a number $T_0 = T_0(F) > 0$ such that for $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 1.

Definition 3. Let $f = (f_1, \dots, f_N)$ satisfy the condition (3), where $0 \le \alpha < 3$, $F = (F_1, \dots, F_N) \in L_{2,\mathrm{loc}}(D_\infty)$ and $F\big|_{D_T} \in L_2(D_T)$ for any $T > 0$. We say that the problem (1), (2) is globally solvable in the class W_2^1, if for $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 1.

When investigating the problem (1), (2) in a sense of the above-given definitions of local and global solvability, it turned out that for $0 \le \alpha \le 1$, where α is the growth exponent of power nonlinearity in the condition (3), the problem (1), (2) is globally solvable. In the case, where $1 < \alpha < 3$, for the global solvability of the problem (1), (2) it is not enough to have only one restriction (3) to the nonlinearity growth of the vector-function f. For this problem to be globally solvable for $1 < \alpha < 3$, one needs additional, of structural character, restrictions to the nonlinear vector-function f. According to what has been said, we have the following theorems.

Theorem 1. *Let $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. Let $0 \le \alpha \le 1$ and the vector-function f satisfy the condition* (3). *Then the problem* (1), (2) *is globally solvable in the class W_2^1, i.e. for any $T > 0$, this* *problem has at least one strong generalized solution of the class* W_2^1 *in the domain* D_T *in the sense of Definition* 1.

Theorem 2. *Let* $F \in L_{2,\mathrm{loc}}(D_\infty)$ *and* $F \in L_2(D_T)$ *for any* $T > 0$*. Let the vector-function* f *satisfy the condition* (3)*, where* $1 < \alpha < 3$*. Then the problem* (1)*,* (2) *is locally solvable in the class* W_2^1*, i.e. there exists the number* $T_0 = T_0(F) > 0$ *such that for* $T < T_0$ *this problem has at least one strong generalized solution of the class* W_2^1 *in the domain* D_T *in the sense of Definition* 1.

Theorem 3. *Let* f *satisfy the condition* (3)*, where* $1 < \alpha < 3$ *and* $f = \nabla G$*, i.e.* $f_i(u) = \frac{\partial}{\partial u_i} G(u)$*,* $u \in \mathbb{R}^N$*,* $i = 1, \dots, N$*, where* $G = G(u) \in C^1(\mathbb{R}^N)$ *is the scalar function satisfying the conditions* $G(0) = 0$ *and* $G(u) \geq 0\ \forall\, u \in \mathbb{R}^N$*. Let* $F \in L_{2,\mathrm{loc}}(D_\infty)$ *and* $F\big|_{D_T} \in L_2(D_T)$ *for any* $T > 0$*. Then the problem* (1)*,* (2) *is globally solvable in the class* W_2^1 *in the sense of Definition* 3.

Theorem 4. *Let* $f_i = \sum\limits_{j=1}^{N} a_{ij} |u_j|^{\alpha_j}$*,* $i = 1, \dots, N$*;* $a_{ij} = \mathrm{const}$*,* $\det(a_{ij})_{i,j=1}^N \neq 0$*,* $1 < \alpha_j = \mathrm{const} < 3$*,* $i, j = 1, \dots, N$*. Then there exists the vector-function* $F \in L_{2,\mathrm{loc}}(D_\infty)$*,* $F\big|_{D_T} \in L_2(D_T)\, \forall\, T > 0$ *such that the problem* (1)*,* (2) *is not globally solvable in the class* W_2^1*, i.e. there exists the number* $T_1 = T_1(F) > 0$ *such that for* $T > T_1$ *the problem* (1)*,* (2) *has no strong generalized solution of the class* W_2^1 *in the domain* D_T *in the sense of Definition* 1.

References

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