 DOI 10.1515/gmj-2014-0053 — Georgian Math. J. 2015; 22 (1):81–104

Research Article

Otar Jokhadze and Sergo Kharibegashvili

On the Cauchy and Cauchy–Darboux problems for semilinear wave equations

Abstract: The Cauchy and Cauchy–Darboux problems for semilinear wave equations in the class of continuous functions are investigated. The questions of existence, uniqueness and nonexistence of global solutions of the problems are considered. The local solvability of the problems is also discussed.

Keywords: Cauchy and Cauchy–Darboux problems, semilinear wave equations, nonexistence, local and globally solvability

MSC 2010: 35L15, 35L70

Otar Jokhadze, Sergo Kharibegashvili: A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia, e-mail: kharibegashvili@yahoo.com, ojokhadze@yahoo.com

1 Problems statement

In the plane of independent variables x and t consider the semilinear wave equation

$$Lu := \square u + g(u) = f(x,t), \tag{1.1}$$

where g is a given nonlinear continuous on $\mathbb{R} := (-\infty, +\infty)$ function, while u is an unknown real function; here

$$\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}.$$

Let $P_0 := P_0(x_0, t_0)$ be an arbitrary point of the domain $\Omega := \{(x,t) : x \in \mathbb{R},\ t > 0\}$ and let

$$D_{P_0} := \{(x,t) : t + x_0 - t_0 < x < -t + x_0 + t_0,\ t > 0\}$$

be the triangular domain bounded by the characteristic segments

$$\gamma_{1,P_0}: \quad x = t + x_0 - t_0, \quad 0 \le t \le t_0,$$
$$\gamma_{2,P_0}: \quad x = -t + x_0 + t_0, \quad 0 \le t \le t_0,$$

of equation (1.1), and by the segment $\gamma_{P_0} : t = 0,\ x_0 - t_0 \le x \le x_0 + t_0$.

For equation (1.1), in the domain D_{P_0} consider the Cauchy problem of finding a solution $u(x,t)$ by the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \gamma_{P_0}, \tag{1.2}$$

where φ and ψ are given real functions on $\mathbb{R}$.

Definition 1.1. Let

$$f \in C(\overline{D}_{P_0}), \quad g \in C(\mathbb{R}), \quad \varphi \in C^1(\gamma_{P_0}), \quad \psi \in C(\gamma_{P_0}). \tag{1.3}$$

We say that a function u is a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0} if $u \in C(\overline{D}_{P_0})$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$ such that $u_n \to u$ and $Lu_n \to f$ in the space $C(\overline{D}_{P_0})$, while $u_n(\,\cdot\,,0) \to \varphi$ and $u_{nt}(\,\cdot\,,0) \to \psi$ for $n \to \infty$ in the spaces $C^1(\gamma_{P_0})$ and $C(\gamma_{P_0})$, respectively.

Remark 1.1. It is obvious that the classical solution of problem (1.1), (1.2) of class $C^2(\overline{D}_{P_0})$ is a strong generalized solution of this problem of class C in the domain D_{P_0}. Conversely, if a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0} belongs to the space $C^2(\overline{D}_{P_0})$, then it will also be a classical solution of this problem.

Definition 1.2. Let

$$f \in C(\overline{\Omega}), \quad g \in C(\mathbb{R}), \quad \varphi \in C^1(\mathbb{R}), \quad \psi \in C(\mathbb{R}). \tag{1.4}$$

We say that problem (1.1), (1.2) is globally solvable in the class C if for any point $P_0 \in \Omega$ this problem has a strong generalized solution of class C in the domain D_{P_0} in the sense of Definition 1.1.

Definition 1.3. Let condition (1.4) be fulfilled. We say that a function $u \in C(\overline{\Omega})$ is a global strong generalized solution of problem (1.1), (1.2) of class C if for any point $P_0 \in \Omega$ it is a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0} in the sense of Definition 1.1.

Remark 1.2. Note that when the theorem of existence and uniqueness of a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0} is valid for any $P_0 \in \Omega$, then we obtain the existence of the unique global strong generalized solution of problem (1.1), (1.2) of class C in the sense of Definition 1.3.

Note that the questions of existence, uniqueness and nonexistence of a global solutions of the Cauchy problem posed for wave equations with nonlinear source term have been studied in numerous works (see e.g. [3, 7, 14] and the references therein). In the present work, for the function g from sufficiently wide class of nonlinear functions, the Cauchy problem will be studied by the methods of a priori estimates and test functions [12] in the class of continuous functions.

For the nonlinear equation (1.1), together with the Cauchy problem (1.1), (1.2) we consider the Cauchy–Darboux problem in the angular domains with non-characteristic boundary. Aiming at that, denote by

$$\Lambda := \{(x,t) \in \mathbb{R}^2 : \gamma_2(t) < x < 0,\ t > 0\}$$

the angular domain lying within the characteristic angle $\{(x,t) \in \mathbb{R}^2 : t > |x|\}$, and bounded by the straight beam $\gamma_1 : x = 0, t \geq 0$ and smooth non-characteristic curve $\gamma_2 : x = \gamma_2(t), t \geq 0$, i.e. $|\gamma_2'(t)| \neq 1, t \geq 0$, which go out from the origin $O(0,0)$. In these suppositions, it is obvious that

$$-t < \gamma_2(t) < 0, \quad t > 0, \quad |\gamma_2'(t)| < 1, \quad t \geq 0, \quad \gamma_2(0) = 0. \tag{1.5}$$

Let $\Lambda_T := \Lambda \cap \{t < T\}$, $T := \text{const} > 0$ and $\gamma_{i,T} := \gamma_i \cap \{t \leq T\}$, $i = 1,2$. It is obvious that for $T = \infty$ one has $\Lambda_\infty = \Lambda$ and $\gamma_{i,\infty} = \gamma_i$ for $i = 1,2$.

Below we require that

$$\gamma_2'(t) < 0, \quad t \geq 0. \tag{1.6}$$

For the nonlinear equation (1.1), together with the Cauchy problem (1.1), (1.2) in the domain Λ consider the Cauchy–Darboux problem in the following statement: find in the domain Λ_T a solution $u = u(x,t)$ of this equation by the boundary conditions

$$u_x|_{\gamma_{1,T}} = 0, \quad u|_{\gamma_{2,T}} = 0. \tag{1.7}$$

Note that in the linear case, i.e. when in equation (1.1) the function g is linear, and the boundary conditions (1.7) are replaced by the conditions

$$(\alpha_i u_x + \beta_i u_t)|_{\gamma_{i,T}} = 0, \quad i = 1,2, \quad u(0,0) = 0, \tag{1.8}$$

problem (1.1), (1.8) in the domain Λ_T was studied in [6, 10, 11, 16]. Note also that problem (1.1), (1.7) is equivalent to problem (1.1), (1.8) when the direction (α_2, β_2) coincides with the tangent direction to the curve $\gamma_{2,T}$ at an arbitrary point. In the case of the nonlinear equation (1.1), when on γ_1 and γ_2 we have the homogeneous Dirichlet conditions $u|_{\gamma_{i,T}} = 0$, $i = 1,2$, and one of the curves γ_1 and γ_2 is characteristic, this problem is studied in [1, 9], and when $\gamma_{2,T} : x = t$, $0 \leq t \leq T$ is the characteristic of equation (1.1), is studied in [8]. As shown in [6, 16], such problems arise in the mathematical modeling of small harmonic oscillations of a wedge in a supersonic stream, and also oscillations of a string in the cylinder filled with a viscous liquid.

Definition 1.4. Let $f \in C(\overline{\Lambda}_T)$. We call the function u a strong generalized solution of problem (1.1), (1.7) of class C in the domain Λ_T if $u \in C(\overline{\Lambda}_T)$ and there exists a sequence of functions

$$u_n \in \overset{\circ}{C}{}^2(\overline{\Lambda}_T, \gamma_T) := \{v \in C^2(\overline{\Lambda}_T) : v_x|_{\gamma_{1,T}} = 0,\ v|_{\gamma_{2,T}} = 0\}$$

such that $u_n \to u$ and $Lu_n \to f$ in the space $C(\overline{\Lambda}_T)$ for $n \to \infty$, $\gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Remark 1.3. It is obvious that a classical solution of problem (1.1), (1.7) from the space $\overset{\circ}{C}{}^2(\overline{\Lambda}_T, \gamma_T)$ is a strong generalized solution of the same problem of class C in the domain Λ_T. In turn, if a strong generalized solution of problem (1.1), (1.7) of class C in the domain Λ_T belongs to the space $C^2(\overline{\Lambda}_T)$, then it will also be a classical solution of the same problem.

Definition 1.5. Let $f \in C(\overline{\Lambda}_\infty)$. We say that problem (1.1), (1.7) is globally solvable in the class C if for any finite $T > 0$ this problem has a strong generalized solution of class C in the domain Λ_T.

Definition 1.6. Let $g \in C(\mathbb{R})$ and $f \in C(\overline{\Lambda}_\infty)$. We call a function $u \in C(\overline{\Lambda}_\infty)$ a global strong generalized solution of problem (1.1), (1.7) of class C in the domain Λ_∞ if for any finite $T > 0$ the function $u|_{\Lambda_T}$ is a strong generalized solution of this problem of class C in the domain λ_T in the sense of Definition 1.4.

Definition 1.7. Let $g \in C(\mathbb{R})$ and $f \in C(\overline{\Lambda}_\infty)$. We say that problem (1.1), (1.7) is locally solvable in the class C if there exists a positive number $T_0 = T_0(f)$ such that for $T \leq T_0$ this problem has at least one strong generalized solution of class C in the domain Λ_T in the sense of Definition 1.4.

Below we show that for certain conditions of a nonlinear function g, problem (1.1), (1.2) is locally solvable; the global solvability conditions are obtained, the violation of which, generally speaking, may cause the nonexistence of a solution at a finite moment of time.

The paper is organized as follows. In Section 2, the condition is given on a nonlinear function g, which allows us to prove the a priori estimate of a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_T. In Section 3, problem (1.1), (1.2) is equivalently reduced to the Volterra nonlinear integral equation. In Section 4, we consider the question of the global solvability of problem (1.1), (1.2) in the class of continuous functions C. In Section 5, we study the questions of smoothness, uniqueness and existence of a global solution of the Cauchy problem in Ω. In Section 6, we study the question of a local solvability. In Section 7, we consider the case of the nonexistence of global solvability. In Section 8, we obtain an a priori estimate for the solution of the Cauchy–Darboux problem (1.1), (1.7). In Section 9, the cases of global solvability are studied, and in Section 10, the smoothness of solutions of this problem is investigated. Finally, in Section 11, we consider the question of uniqueness, existence and nonexistence of global solutions, as well as the local solvability of problem (1.1), (1.7).

2 A priori estimate of a strong generalized solution of Cauchy problem (1.1), (1.2) of class C in the domain D_{P_0}

Denote

$$G(s) := \int_0^s g(\sigma)d\sigma, \quad s \in \mathbb{R}. \tag{2.1}$$

Lemma 2.1. *Let condition* (1.4) *be fulfilled and*

$$G(s) \geq -M_1 - M_2 s^2, \quad s \in \mathbb{R}, \quad M_i := \text{const} \geq 0, \quad i = 1, 2. \tag{2.2}$$

Then, if u is a strong generalized solution of problem (1.1), (1.2) *of class C in the domain D_{P_0}, for any point $P_0 \in \Omega$ the a priori estimate*

$$\|u\|_{C(\overline{D}_{P_0})} \leq c_1\big(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|G(\varphi)\|^{\frac{1}{2}}_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}\big) + c_2 \tag{2.3}$$

is valid with positive constants $c_1 = c_1(t_0, M_2)$ and $c_2 = c_2(t_0, M_1, M_2)$ not depending on u and f, φ, ψ.

Proof. Let u be a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0} and $P_0 \in \Omega$. Then due to Definition 1.1 there exists a consequence of functions $u_n \in C^2(\overline{D}_{P_0})$ such that

$$\begin{aligned} &\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_{P_0})} = 0, \qquad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_{P_0})} = 0, \\ &\lim_{n\to\infty} \|u_n(\,\cdot\,,0) - \varphi\|_{C^1(\gamma_{P_0})} = 0, \quad \lim_{n\to\infty} \|u_{nt}(\,\cdot\,,0) - \psi\|_{C(\gamma_{P_0})} = 0, \end{aligned} \tag{2.4}$$

and therefore in view of $g \in C(\mathbb{R})$

$$\lim_{n\to\infty} \|g(u_n) - g(u)\|_{C(\overline{D}_{P_0})} = 0. \tag{2.5}$$

Consider a function $u_n \in C^2(\overline{D}_{P_0})$ as a solution of the Cauchy problem

$$Lu_n = f_n, \tag{2.6}$$

$$u_n(x,0) = \varphi_n(x), \quad u_{nt}(x,0) = \psi_n(x), \quad x \in \gamma_{P_0}. \tag{2.7}$$

Here

$$f_n := Lu_n, \quad \varphi_n := u_n(\cdot\,,0), \quad \psi_n := u_{nt}(\cdot\,,0). \tag{2.8}$$

Multiplying both sides of (2.6) by $2u_{nt}$ and integrating the obtained equality in the domain

$$D_{P_0,\tau} := \{(x,t) \in D_{P_0} : 0 < t < \tau\}, \quad 0 < \tau < t_0,$$

due to (2.1) we have

$$\int\limits_{D_{P_0,\tau}} (u_{nt}^2)_t dxdt - 2\int\limits_{D_{P_0,\tau}} u_{nxx}u_{nt} dxdt + 2\int\limits_{D_{P_0,\tau}} [G(u_n)]_t dxdt = 2\int\limits_{D_{P_0,\tau}} f_n u_{nt} dxdt.$$

Let us set $\Omega_{P_0,\tau} := \overline{D}_{P_0} \cap \{t = \tau\}$, $0 < \tau < t_0$. Then in view of (2.7), integrating by parts the left-hand side of the last equality we have

$$\begin{aligned} 2\int\limits_{D_{P_0,\tau}} f_n u_{nt} dxdt = \sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0,\tau}} \nu_t^{-1}[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2) + 2G(u_n)\nu_t^2]ds \\ - \int\limits_{x_0-t_0}^{x_0+t_0} [\varphi_n'^2(x) + \psi_n^2(x) + 2G(\varphi_n)]dx + \int\limits_{\Omega_{P_0,\tau}} [u_{nx}^2 + u_{nt}^2 + 2G(u_n)]dx, \end{aligned} \tag{2.9}$$

where $\nu := (\nu_x, \nu_t)$ is the unit vector of the outer normal to $\partial D_{P_0,\tau}$ and $\gamma_{i,P_0,\tau} := \gamma_{i,P_0} \cap \{t \le \tau\}$, $i = 1,2$.

Taking into account the fact that everywhere on the characteristics γ_{i,P_0}, $i = 1,2$, of equation (1.1) we have the relations

$$\nu_t|_{\gamma_{i,P_0}} > 0, \quad (\nu_t^2 - \nu_x^2)|_{\gamma_{i,P_0}} = 0, \quad i = 1,2,$$

using (2.2), from (2.9) we have

$$\begin{aligned} w_n(\tau) \le 2\int\limits_{D_{P_0,\tau}} f_n u_{nt} dxdt + \sqrt{2}\sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0,\tau}} (M_1 + M_2 u_n^2)ds \\ + 2\int\limits_{\Omega_{P_0,\tau}} (M_1 + M_2 u_n^2)dx + \int\limits_{x_0-t_0}^{x_0+t_0} [\varphi_n'^2(x) + \psi_n^2(x) + 2|G(\varphi_n)|]dx. \end{aligned} \tag{2.10}$$

Here

$$w_n(\tau) := \int\limits_{\Omega_{P_0,\tau}} (u_{nx}^2 + u_{nt}^2)dx + \sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0,\tau}} \nu_t^{-1}(u_{nx}\nu_t - u_{nt}\nu_x)^2 ds. \tag{2.11}$$

Due to (2.7) it is easy to see that

$$u_n(x,t) = \varphi_n(x) + \int\limits_0^t u_{nt}(x,\sigma)d\sigma, \quad (x,t) \in \overline{D}_{P_0,\tau}.$$

Squaring both sides of this equality and applying the Cauchy and Schwarz inequalities, we get

$$u_n^2(x,t) \le 2\varphi_n^2(x) + 2t\int\limits_0^t u_{nt}^2(x,\sigma)d\sigma, \quad (x,t) \in \overline{D}_{P_0,\tau}. \tag{2.12}$$

It follows from (2.12) that

$$\sum_{i=1}^{2}\int\limits_{\gamma_{i,P_0,\tau}} u_n^2 ds = \sqrt{2}\int\limits_{x_0-t_0}^{x_0-t_0+\tau} u_n^2(x,x+t_0-x_0)dx + \sqrt{2}\int\limits_{x_0+t_0-\tau}^{x_0+t_0} u_n^2(x,-x+t_0+x_0)dx$$
$$\le 2\sqrt{2}\int\limits_{x_0-t_0}^{x_0-t_0+\tau}\varphi_n^2(x)dx + 2\sqrt{2}\int\limits_{x_0+t_0-\tau}^{x_0+t_0}\varphi_n^2(x)dx + 2\sqrt{2}\tau\int\limits_{x_0-t_0}^{x_0-t_0+\tau}dx\int\limits_0^{x+t_0-x_0}u_{nt}^2(x,t)dt$$
$$+ 2\sqrt{2}\tau\int\limits_{x_0+t_0-\tau}^{x_0+t_0}dx\int\limits_0^{-x+t_0+x_0}u_{nt}^2(x,t)dt$$
$$\le 2\sqrt{2}\int\limits_{x_0-t_0}^{x_0+t_0}\varphi_n^2(x)dx + 2\sqrt{2}\tau\int\limits_{D_{P_0,\tau}}u_{nt}^2(x,t)dxdt. \tag{2.13}$$

Analogously,

$$\int\limits_{\Omega_{P_0,\tau}} u_n^2 dx \le 2\int\limits_{\Omega_{P_0,\tau}}\varphi_n^2(x)dx + 2\tau\int\limits_{x_0-t_0+\tau}^{x_0+t_0-\tau}dx\int\limits_0^{\tau}u_{nt}^2(x,t)dt$$
$$\le 2\int\limits_{x_0-t_0}^{x_0+t_0}\varphi_n^2(x)dx + 2\tau\int\limits_{D_{P_0,\tau}}u_{nt}^2(x,t)dxdt. \tag{2.14}$$

Taking into account the inequality

$$2\int\limits_{D_{P_0,\tau}} f_n u_{nt}dxdt \le \int\limits_{D_{P_0,\tau}} u_{nt}^2 dxdt + \|f_n\|^2_{L_2(D_{P_0,\tau})},$$

due to (2.13) and (2.14) from (2.10) it follows that

$$w_n(\tau) \le (1+8\tau M_2)\int\limits_{D_{P_0,\tau}} u_{nt}^2 dxdt + 8t_0M_1 + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \int\limits_{x_0-t_0}^{x_0+t_0}[8M_2\varphi_n^2(x) + \varphi_n'^2(x) + \psi_n^2(x) + 2|G(\varphi_n)|]dx.$$

Whence in view of (2.11) we have

$$w_n(\tau) \le \alpha\int\limits_0^{\tau} w_n(\sigma)d\sigma + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \beta_n, \quad 0<\tau\le t_0,$$

where

$$\alpha := 1 + 8t_0M_2, \quad \beta_n := 8t_0M_1 + \int\limits_{x_0-t_0}^{x_0+t_0}[8M_2\varphi_n^2(x) + \varphi_n'^2(x) + \psi_n^2(x) + 2|G(\varphi_n)|]dx.$$

From the last inequality, taking into account that the value $\|f_n\|^2_{L_2(D_{P_0,\tau})}$ as a function of τ is nondecreasing, by the Gronwall Lemma we obtain

$$w_n(\tau) \le \exp(\tau\alpha)(\|f_n\|^2_{L_2(D_{P_0,\tau})} + \beta_n). \tag{2.15}$$

It is easy to see that $\nu_t\frac{\partial}{\partial x} - \nu_x\frac{\partial}{\partial t}$ is the inner differentiation operator along the direction of the unit tangential vector to γ_{1,P_0}. Therefore the integration along the segment γ_{1,P_0} gives

$$u_n(x_0,t_0) = \varphi_n(x_0-t_0) + \int\limits_{\gamma_{1,P_0}}(\nu_t u_{nx} - \nu_x u_{nt})ds.$$

Hence, squaring both sides of this equality and applying the Cauchy and Schwarz inequalities, we obtain

$$u_n^2(x_0,t_0) \le 2\varphi_n^2(x_0-t_0) + 2\int\limits_{\gamma_{1,P_0}}ds\int\limits_{\gamma_{1,P_0}}(\nu_t u_{nx} - \nu_x u_{nt})^2 ds \le 2\varphi_n^2(x_0-t_0) + 2\sqrt{2}t_0\int\limits_{\gamma_{1,P_0}}(\nu_t u_{nx} - \nu_x u_{nt})^2 ds.$$

Whence, due to (2.11) and (2.15) we get

$$
\begin{aligned}
u_n^2(x_0,t_0) &\le 2\varphi_n^2(x_0-t_0)+4t_0\exp(t_0\alpha)(\|f_n\|^2_{L_2(D_{P_0})}+\beta_n)\\
&\le 2\varphi_n^2(x_0-t_0)+4t_0\exp(t_0\alpha)\big(\|f_n\|^2_{C(\overline{D}_{P_0})}\operatorname{mes}D_{P_0}+8t_0M_1+16t_0M_2\|\varphi_n\|^2_{C(\gamma_{P_0})}+2t_0\|\varphi_n'\|^2_{C(\gamma_{P_0})}\\
&\qquad+2t_0\|\psi_n\|^2_{C(\gamma_{P_0})}+4t_0\|G(\varphi_n)\|_{C(\gamma_{P_0})}\big)\\
&= 2\varphi_n^2(x_0-t_0)+\exp(t_0\alpha)\big(4t_0^3\|f_n\|^2_{C(\overline{D}_{P_0})}+32t_0^2M_1+64t_0^2M_2\|\varphi_n\|^2_{C(\gamma_{P_0})}+8t_0^2\|\varphi_n'\|^2_{C(\gamma_{P_0})}\\
&\qquad+8t_0^2\|\psi_n\|^2_{C(\gamma_{P_0})}+16t_0^2\|G(\varphi_n)\|_{C(\gamma_{P_0})}\big).
\end{aligned}
$$

Hence, using the well-known inequality

$$\Big(\sum_{i=1}^n a_i^2\Big)^{\frac12}\le\sum_{i=1}^n|a_i|$$

we obtain

$$|u_n(x_0,t_0)|\le c_1\big(\|f_n\|_{C(\overline{D}_{P_0})}+\|\varphi_n\|_{C^1(\gamma_{P_0})}+\|G(\varphi_n)\|^{\frac12}_{C(\gamma_{P_0})}+\|\psi_n\|_{C(\gamma_{P_0})}\big)+c_2,$$

where

$$c_1^2:=\max\{4t_0^3\exp(t_0\alpha),2+64t_0^2M_2\exp(t_0\alpha),16t_0^2\exp(t_0\alpha)\},\quad c_2:=32t_0^2M_1\exp(t_0\alpha). \tag{2.16}$$

Passing in the last inequality to the limit for $n\to\infty$, in view of (2.4) and (2.8) we have

$$|u(x_0,t_0)|\le c_1\big(\|f\|_{C(\overline{D}_{P_0})}+\|\varphi\|_{C^1(\gamma_{P_0})}+\|G(\varphi)\|^{\frac12}_{C(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}\big)+c_2, \tag{2.17}$$

whence estimate (2.3) follows immediately. □

Remark 2.1. Let us consider some classes of the functions $g=g(s)$, occurring in applications and satisfying condition (2.2):
(1) $g(s)=|s|^\alpha\operatorname{sign}s$, where $\alpha>0$, $\alpha\ne1$. In this case $G(s)=|s|^{\alpha+1}/(\alpha+1)$, $s\in\mathbb{R}$, and condition (2.2) is fulfilled.
(2) $g\in C(\mathbb{R})$ and, if the inequality $g(s)\operatorname{sign}s\ge0$, $s\in\mathbb{R}$, is fulfilled, then condition (2.2) will also be fulfilled.
(3) $g(s)=e^s$, $s\in\mathbb{R}$. In this case $G(s)=e^s-1$, $s\in\mathbb{R}$, and therefore condition (2.2) is fulfilled.

3 Equivalent reduction of problem (1.1), (1.2) to a Volterra type nonlinear integral equation

Let $u\in C^2(\overline{\Omega})$ be a classical solution of problem (1.1), (1.2) and set

$$D_{x,t}:=\{(x_1,t_1):t_1+x-t<x_1<-t_1+x+t,\ t_1>0\},\quad (x,t)\in\Omega.$$

Note that $D_{P_0}=D_{x,t}$ for $x=x_0$, $t=t_0$. Using the initial conditions (1.2) and integrating equation (1.1) we get the equality (see e.g. [2])

$$u(x,t)+(\square^{-1}[g(u)])(x,t)=F(x,t),\quad (x,t)\in\overline{D}_{P_0}. \tag{3.1}$$

Here

$$\square:=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2},$$

while

$$F(x,t):=(l_1\varphi)(x,t)+(l_2\psi)(x,t)+(\square^{-1}f)(x,t). \tag{3.2}$$

The continuous operators

$$
\begin{aligned}
l_1&:C^k(\gamma_{P_0})\to C^k(\overline{D}_{P_0}), && k=0,1,2,\\
l_2&:C^k(\gamma_{P_0})\to C^{k+1}(\overline{D}_{P_0}), && k=0,1,\\
\square^{-1}&:C^k(\overline{D}_{P_0})\to C^{k+1}(\overline{D}_{P_0}), && k=0,1,
\end{aligned}
\tag{3.3}
$$

act by the formulas

$$
\begin{aligned}
(l_1\varphi)(x,t) &:= \frac{1}{2}[\varphi(x+t)+\varphi(x-t)],\\
(l_2\psi)(x,t) &:= \frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\xi)d\xi,\\
(\square^{-1}f)(x,t) &:= \frac{1}{2}\int\limits_{D_{x,t}} f(\xi,\tau)d\xi d\tau.
\end{aligned}
\tag{3.4}
$$

Remark 3.1. Equality (3.1) can be considered as a Volterra type nonlinear integral equation.

Lemma 3.1. *Let condition* (1.3) *be fulfilled. A function $u \in C(\overline{D}_{P_0})$ is a strong generalized solution of problem* (1.1), (1.2) *of class C in the domain D_{P_0} if and only if it is a continuous solution of the nonlinear integral equation* (3.1).

Proof. Indeed, let $u \in C(\overline{D}_{P_0})$ be a solution of equation (3.1). Since $u, f \in C(\overline{D}_{P_0})$ and the space $C^2(\overline{D}_{P_0})$ is dense in $C(\overline{D}_{P_0})$ (see e.g. [13, p. 37]), there exist sequences of functions $w_n, f_n \in C^2(\overline{D}_{P_0})$ such that $w_n \to u$ and $f_n \to f$ in the space $C(\overline{D}_{P_0})$ for $n \to \infty$.

Analogously, because $\varphi \in C^1(\gamma_{P_0})$ (resp. $\psi \in C(\gamma_{P_0})$), there exists a sequence of functions $\varphi_n \in C^2(\gamma_{P_0})$ (resp. $\psi_n \in C^1(\gamma_{P_0})$) such that $\varphi_n \to \varphi$ (resp. $\psi_n \to \psi$) in the space $C^1(\gamma_{P_0})$ (resp. $C(\gamma_{P_0})$) for $n \to \infty$.

Let

$$u_n := -\square^{-1}[g(w_n)] + l_1\varphi_n + l_2\psi_n + \square^{-1}f_n, \quad n = 1,2,\dots.$$

It is easy to verify that $u_n \in C^2(\overline{D}_{P_0})$. Since g is a continuous function and $l_i, \square^{-1}$, in view of (3.3), (3.4) are linear continuous operators in the corresponding spaces, and

$$\lim_{n\to\infty}\|w_n - u\|_{C(\overline{D}_{P_0})} = 0, \quad \lim_{n\to\infty}\|f_n - f\|_{C(\overline{D}_{P_0})} = 0, \quad \lim_{n\to\infty}\|\varphi_n - \varphi\|_{C^1(\gamma_{P_0})} = 0, \quad \lim_{n\to\infty}\|\psi_n - \psi\|_{C(\gamma_{P_0})} = 0,$$

we have

$$
\begin{aligned}
u_n &\to -\square^{-1}[g(u)] + l_1\varphi + l_2\psi + \square^{-1}f && \text{in } C(\overline{D}_{P_0}),\\
u_n(\cdot,0) &\to \varphi && \text{in } C^1(\gamma_{P_0}),\\
u_{nt}(\cdot,0) &\to \psi && \text{in } C(\gamma_{P_0}),
\end{aligned}
$$

for $n \to \infty$. But from equalities (3.1) and (3.2) it follows that $-\square^{-1}[g(u)] + l_1\varphi + l_2\psi + \square^{-1}f = u$. Thus we have

$$\lim_{n\to\infty}\|u_n - u\|_{C(\overline{D}_{P_0})} = 0.$$

On the other hand, we have $\square u_n = -g(w_n) + f_n$, whence due to

$$\lim_{n\to\infty}\|u_n - u\|_{C(\overline{D}_{P_0})} = 0, \quad \lim_{n\to\infty}\|w_n - u\|_{C(\overline{D}_{P_0})} = 0, \quad \lim_{n\to\infty}\|f_n - f\|_{C(\overline{D}_{P_0})} = 0$$

we obtain

$$Lu_n = \square u_n + g(u_n) = -g(w_n) + f_n + g(u_n) = -[g(w_n) - g(u)] + [g(u_n) - g(u)] + f_n \to f$$

taking into account (2.5) in the space $C(\overline{D}_{P_0})$ for $n \to \infty$. The converse is obvious. $\square$

4 Global solvability of problem (1.1), (1.2) in the class of continuous functions

As mentioned above, the operator $\square^{-1}$ from (3.4) is a linear continuous one, acting due to (3.3) from the space $C(\overline{D}_{P_0})$ into the space of continuously differentiable functions $C^1(\overline{D}_{P_0})$. Further, since the space $C^1(\overline{D}_{P_0})$ is compactly embedded into the space $C(\overline{D}_{P_0})$ (see e.g. [5, p. 135]), we easily obtain the validity of the following statement.

Lemma 4.1. *The operator $\square^{-1}: C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ from (3.4) is a linear compact operator.*

Rewrite equation (3.1) in the form

$$u = Au := -\square^{-1}[g(u)] + F, \tag{4.1}$$

where we observe that the operator $A: C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ is continuous and compact because the nonlinear operator $N: C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$, acting by the formula $Nu := g(u)$, is bounded and continuous and the linear operator $\square^{-1}: C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$, due to Lemma 4.1, is compact. At the same time, according to Lemma 2.1 and equalities (2.16), for any parameter $\tau \in [0, 1]$ and every solution $u \in C(\overline{D}_{P_0})$ of the equation $u = \tau Au$ the a priori estimate (2.3) is valid with the same positive constants c_1 and c_2 from (2.3), not dependent on u, φ, ψ, f and τ. Therefore, according to the Leray–Schauder Theorem (see e.g. [15, p. 375]), equation (4.1) in the conditions of Lemma 2.1 has at least one solution $u \in C(\overline{D}_{P_0})$. So, due to Lemma 3.1, we have proved the following theorem.

Theorem 4.1. *Let the conditions of Lemma 2.1 be fulfilled. Then problem* (1.1), (1.2) *is globally solvable in the class C in the sense of Definition 1.2, i.e. for any point $P_0 \in \Omega$ this problem has a strong generalized solution of class C in the domain D_{P_0}.*

5 Smoothness and uniqueness of a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0}. Existence of a global solution in Ω

The following lemma immediately follows from Lemma 3.1 and equalities (3.1)–(3.3).

Lemma 5.1. *Let $f \in C^1(\overline{\Omega})$, $g \in C^1(\mathbb{R})$, $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Then any strong generalized solution u of problem* (1.1), (1.2) *of class C in the domain D_{P_0} in the sense of Definition 1.1 is a classical one, i.e. belongs to the class $C^2(\overline{D}_{P_0})$.*

Consider the question of uniqueness of a strong generalized solution of problem (1.1), (1.2) of class C in the domain D_{P_0}.

Suppose that the function g satisfies the following conditions: for any $|s|, |s_1|, |s_2| \le r$

$$|g(s)| \le m(r), \quad |g(s_2) - g(s_1)| \le c(r)|s_2 - s_1|, \tag{5.1}$$

where $m(r)$ and $c(r)$ are some continuous non-negative functions of its argument $r \ge 0$.

Theorem 5.1. *Let conditions* (1.4) *and* (5.1) *be fulfilled. Then for any $P_0 \in \Omega$, problem* (1.1), (1.2) *cannot have more than one strong generalized solution of class C in the domain D_{P_0}.*

Proof. Indeed, suppose that problem (1.1), (1.2) has two possible different strong generalized solutions u^1, u^2 of class C in the domain D_{P_0}. According to Definition 1.1, there exists a sequence of functions $u_n^i \in C^2(\overline{D}_{P_0})$ such that

$$\begin{aligned} \lim_{n\to\infty} \|u_n^i - u^i\|_{C(\overline{D}_{P_0})} &= 0, \qquad \lim_{n\to\infty} \|Lu_n^i - f\|_{C(\overline{D}_{P_0})} = 0, \\ \lim_{n\to\infty} \|u_n^i(\,\cdot\,, 0) - \varphi\|_{C^1(\gamma_{P_0})} &= 0, \quad \lim_{n\to\infty} \|u_{nt}^i(\,\cdot\,, 0) - \psi\|_{C(\gamma_{P_0})} = 0 \end{aligned} \tag{5.2}$$

for any $P_0 := P_0(x_0, t_0) \in \Omega$.

Let $\omega_n := u_n^2 - u_n^1$. It is easy to see that the function $\omega_n \in C^2(\overline{D}_{P_0})$ satisfies the identities

$$\square \omega_n + g_n = f_n, \tag{5.3}$$

$$\omega_n|_{\gamma_{P_0}} = \tau_n, \quad \omega_{nt}|_{\gamma_{P_0}} = \nu_n, \tag{5.4}$$

where

$$g_n := g(u_n^2) - g(u_n^1), \quad f_n := Lu_n^2 - Lu_n^1, \quad \tau_n := (u_n^2 - u_n^1)|_{\gamma_{P_0}}, \quad \nu_n := (u_n^2 - u_n^1)_t|_{\gamma_{P_0}}. \tag{5.5}$$

In view of the first equality of (5.2) there exists a number $M := \text{const} > 0$, not dependent on the indices i and n, such that

$$\|u_n^i\|_{C(\overline{D}_{P_0})} \le M. \tag{5.6}$$

Due to equalities (5.2) and (5.5) we have

$$\lim_{n\to\infty} \|\tau_n\|_{C^1(\gamma_{P_0})} = 0, \quad \lim_{n\to\infty} \|\nu_n\|_{C(\gamma_{P_0})} = 0, \quad \lim_{n\to\infty} \|f_n\|_{C(\overline{D}_{P_0})} = 0.$$

According to the second inequality of (5.1), inequality (5.6) and the first equality of (5.5) it is easy to see that

$$|g_n| \le c(M)|\omega_n|. \tag{5.7}$$

Multiplying both sides of (5.3) by $2\omega_{nt}$ and integrating the resulting equality over the domain $D_{P_0,\tau}$, due to (5.4), in the same way as when obtaining inequality (2.9) from (2.6), (2.7), we get

$$\begin{aligned} v_n(\tau) &:= \int\limits_{\Omega_{P_0,\tau}} (\omega_{nx}^2 + \omega_{nt}^2)dx + \sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0,\tau}} \nu_t^{-1}(\omega_{nx}\nu_t - \omega_{nt}\nu_x)^2 ds \\ &= 2\int\limits_{D_{P_0,\tau}} (f_n - g_n)\omega_{nt}dxdt + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})}. \end{aligned} \tag{5.8}$$

In view of the Cauchy inequality and estimate (5.7) we have

$$\begin{aligned} 2\int\limits_{D_{P_0,\tau}} (f_n - g_n)\omega_{nt}dxdt &\le 2\int\limits_{D_{P_0,\tau}} \omega_{nt}^2 dxdt + \int\limits_{D_{P_0,\tau}} f_n^2 dxdt + \int\limits_{D_{P_0,\tau}} g_n^2 dxdt \\ &\le 2\int\limits_{D_{P_0,\tau}} \omega_{nt}^2 dxdt + \|f_n\|^2_{L_2(D_{P_0,\tau})} + c^2(M)\int\limits_{D_{P_0,\tau}} \omega_n^2 dxdt. \end{aligned} \tag{5.9}$$

Further, setting

$$v(x,t) = \begin{cases} \omega_{nt}(x,t), & (x,t) \in \overline{D}_{P_0,\tau}, \\ 0, & (x,t) \notin \overline{D}_{P_0,\tau}, \end{cases}$$

and taking into account that $t \le \tau$ for $(x,t) \in \overline{D}_{P_0,\tau}$, by a reasoning analogous to (2.12), we obtain

$$\begin{aligned} \int\limits_{D_{P_0,\tau}} \omega_n^2(x,t)dxdt &\le 2\tau\|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau \int\limits_{x_0-t_0}^{x_0+t_0} dx \int\limits_0^\tau \bigg(\int\limits_0^\tau v^2(x,\sigma)d\sigma\bigg)dt \\ &= 2\tau\|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau^2 \int\limits_{x_0-t_0}^{x_0+t_0} dx \int\limits_0^\tau v^2(x,t)dt \\ &= 2\tau\|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau^2 \int\limits_{D_{P_0,\tau}} \omega_{nt}^2(x,t)dxdt. \end{aligned} \tag{5.10}$$

From (5.8)–(5.10) it follows that

$$v_n(\tau) \le 2[1 + c^2(M)\tau^2]\int\limits_0^\tau v_n(\sigma)d\sigma + \|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2\tau c^2(M)\|\tau_n\|^2_{L_2(\gamma_{P_0})}.$$

Therefore, due to the Gronwall Lemma for $0 < \tau \le t_0$ we get

$$v_n(\tau) \le c_3\big(\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2\tau c^2(M)\|\tau_n\|^2_{L_2(\gamma_{P_0})}\big),$$

where $c_3 := \exp[2t_0(1 + c^2(M)t_0^2]$. Whence, taking into account (5.8), we have

$$\int\limits_{\gamma_{1,P_0}} (\omega_{nx}\nu_t - \omega_{nt}\nu_x)^2 ds \le \sqrt{2}c_2\big(\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2\tau c^2(M)\|\tau_n\|^2_{L_2(\gamma_{P_0})}\big), \quad 0 < \tau \le t_0.$$

Analogously to (2.17), and taking into account the explicit inequalities

$$\|f_n\|^2_{L_2(D_{P_0})} \le \|f_n\|^2_{C(\overline{D}_{P_0})} \operatorname{mes} D_{P_0} = t_0^2 \|f_n\|^2_{C(\overline{D}_{P_0})}, \quad \|\tau_n\|^2_{L_2(\gamma_{P_0})} \le 2t_0 \|\tau_n\|^2_{C(\gamma_{P_0})},$$

we obtain

$$\omega_n^2(x_0,t_0) \le 2\tau_n^2(x_0-t_0) + 4t_0^2 c_2 (t_0\|f_n\|^2_{C(\overline{D}_{P_0})} + 2\|\tau_n'\|^2_{C(\gamma_{P_0})} + 2\|\nu_n\|^2_{C(\gamma_{P_0})} + 4\tau c^2(M))\|\tau_n\|^2_{(C(\gamma_{P_0})}), \quad 0<\tau\le t_0.$$

Whence $\lim_{n\to\infty}\omega_n^2(x_0,t_0)=0$, i.e. $u^2(x_0,t_0)=u^1(x_0,t_0)$.

Analogously, we obtain $u^2(x,t)=u^1(x,t)$ for every point $(x,t)\in D_{P_0}$. □

Theorem 5.2. *Let the conditions of Lemma 5.1 and* (2.3) *be fulfilled. Then in the half-plane* Ω, *problem* (1.1), (1.2) *has a unique global classical solution* $u\in C^2(\overline{\Omega})$.

Proof. In the conditions of Theorem 5.2 according to Theorems 4.1, 5.1 and Lemma 5.1, in the domain D_{x_0,t_0} for $t_0=n$ there exists a unique classical solution $u_n\in C^2(\overline{D}_{x_0,n})$ of problem (1.1), (1.2). Since u_{n+1} is also a classical solution of problem (1.1), (1.2) in the domain $D_{x_0,n}$, due to Theorem 5.1 we have $u_{n+1}|_{D_{x_0,n}}=u_n$. Therefore the function u, constructed in the domain Ω by the rule $u(x,t)=u_n(x,t)$ for $n=[t]+1$, where $[t]$ is the entire part of the number t, and the point $(x,t)\in\Omega$, will be a unique classical solution of problem (1.1), (1.2) in the domain Ω of class $C^2(\overline{\Omega})$. □

6 The local solvability of problem (1.1), (1.2)

Theorem 6.1. *Let conditions* (1.3) *be fulfilled. Then for any fixed* $x_0\in\mathbb{R}$ *there exists a positive number* $T:=T(x_0;f,g,\varphi,\psi)$ *such that for* $t_0<T$ *problem* (1.1), (1.2) *will have at least one strong generalized solution* u *of class* C *in the domain* D_{P_0}.

Proof. In Sections 3 and 4, problem (1.1), (1.2) in the space $C(\overline{D}_{P_0})$ has been equivalently reduced to the functional equation (4.1), where the operator $A: C(\overline{D}_{P_0})\to C(\overline{D}_{P_0})$ is continuous and compact. Therefore for proving the solvability of equation (4.1), according to the Schauder Theorem it suffices to show that the operator A maps the ball

$$B_R := \{v\in C(\overline{D}_{P_0}) : \|v\|_{C(\overline{D}_{P_0})}\le R\}$$

with radius $R>0$, which is a closed and convex set in the Banach space $C(\overline{D}_{P_0})$, into itself. Let us show that this happens for sufficiently small t_0.

Let us fix a number $T_0>0$ and set

$$M_0 := \sup_{\overline{D}_{(x_0,T_0)}} |f(x,t)|, \quad m_1 := \sup_{[x_0-T_0,x_0+T_0]} |\varphi(x)|, \quad m_2 := \sup_{[x_0-T_0,x_0+T_0]} |\psi(x)|, \quad g_0 := \sup_{|s|\le R}|g(s)|, \quad R>0.$$

Whence for $t_0\le T_0$, according to (3.2), (3.4), it follows that

$$\|\square^{-1}\|_{C(\overline{D}_{P_0})\to C(\overline{D}_{P_0})} \le 2^{-1}t_0^2, \quad \|F\|_{C(\overline{D}_{P_0})} \le m_1 + m_2 t_0 + 2^{-1}M_0 t_0^2$$

and therefore, due to equality (4.1), we have

$$\begin{aligned}\|Au\|_{C(\overline{D}_{P_0})} &\le \|\square^{-1}\|_{C(\overline{D}_{P_0})\to C(\overline{D}_{P_0})}\|g(u)\|_{C(\overline{D}_{P_0})} + \|F\|_{C(\overline{D}_{P_0})}\\ &\le m_1 + (2^{-1}g_0T_0 + m_2 + 2^{-1}M_0T_0)t_0. \end{aligned} \tag{6.1}$$

Now setting

$$R = 2m_1, \quad T=\min\{T_0, m_1 d^{-1}\},$$

where

$$d := 2^{-1}g_0T_0 + m_2 + 2^{-1}M_0T_0,$$

for $t_0<T$ from (6.1) we get

$$\|Au\|_{C(\overline{D}_{P_0})} \le m_1 + m_1 = 2m_1 = R.$$ □

7 Case of the nonexistence of a global solution of problem (1.1), (1.2)

Remark 7.1. Note that the violation of condition (2.2) may, generally speaking, cause the nonexistence of a global solution of problem (1.1), (1.2) in the sense of Definition 1.2, i.e. when for some point $P_0 \in \Omega$ the problem does not have a strong generalized solution of class C in the domain D_{P_0}. For certain conditions on the data of problem (1.1), (1.2) we will prove that for any fixed point $x_0 \in \mathbb{R}$, there exists a number $T^0 = T^0(x_0) > 0$ such that for $t_0 < T^0$ problem (1.1), (1.2) has a strong generalized solution of class C in the domain D_{P_0}, but for $t_0 > T^0$ it does not have such a solution in D_{P_0}.

Suppose that

$$g(s) = -|s|^\alpha s, \quad \alpha > 0, \quad s \in \mathbb{R}. \tag{7.1}$$

In this case, as it is easy to verify that condition (2.2) is violated.

Lemma 7.1. *Let conditions* (1.4), (7.1) *be fulfilled and let u be a strong generalized solution of problem* (1.1), (1.2) *of class C in the domain D_{P_0} in the sense of Definition 1.1. Then the integral equality*

$$\int\limits_{D_{P_0}} u\Box\chi dxdt = \int\limits_{D_{P_0}} |u|^\alpha u\chi dxdt + \int\limits_{D_{P_0}} f\chi dxdt + \int\limits_{x_0-t_0}^{x_0+t_0} [\psi(x)\chi(x,0) - \varphi(x)\chi_t(x,0)]dx \tag{7.2}$$

is valid for any function χ such that

$$\chi \in C^2(\overline{D}_{P_0}), \quad \chi|_{\gamma_{i,P_0}} = 0, \quad i = 1, 2. \tag{7.3}$$

Proof. By the definition of a strong generalized solution u of problem (1.1), (1.2) of class C in the domain D_{P_0}, the function $u \in C(\overline{D}_{P_0})$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$ such that (2.4) and (2.5) are valid for the functions g defined by (7.1).

Let $f_n := Lu_n$. Multiply both sides of $Lu_n = f_n$ by the function χ and integrate the obtained equality in the domain D_{P_0}. As a result of integration by parts of the left-hand side of this equality, taking into account (2.7) and (7.3), we obtain

$$\int\limits_{D_{P_0}} u_n\Box\chi dxdt = \int\limits_{D_{P_0}} |u_n|^\alpha u_n\chi dxdt + \int\limits_{D_{P_0}} f_n\chi dxdt + \int\limits_{x_0-t_0}^{x_0+t_0} [\psi_n(x)\chi(x,0) - \varphi_n(x)\chi_t(x,0)]dx.$$

Passing to the limit in this equality as $n \to \infty$, due to (2.4), (2.5), we get (7.2). □

Lemma 7.2. *Let conditions* (1.4) *be fulfilled, let $\alpha > 0$ and let the function $u \in C(\overline{D}_{P_0})$ be a strong generalized solution of problem* (1.1), (1.2) *of class C in the domain D_{P_0}. If $f \geq 0, \varphi \geq 0$ and $\psi \geq 0$, then $u \geq 0$ in the domain D_{P_0}.*

Proof. According to Lemma 3.1 and equality (4.1), the function u is a solution of the Volterra type integral equation

$$u(x,t) = \int\limits_{D_{x,t}} k(x,t;\xi,\tau)u(\xi,\tau)d\xi d\tau + F(x,t), \quad (x,t) \in \overline{D}_{P_0}, \tag{7.4}$$

where $k := \frac{1}{2}|u|^\alpha \geq 0$ and the function F is defined by (3.2) and (3.4). In the conditions of Lemma 7.2 we have

$$F \geq 0. \tag{7.5}$$

Assuming that the function k is given, consider the Volterra type linear integral equation

$$v(x,t) = \int\limits_{D_{x,t}} k(x,t;\xi,\tau)v(\xi,\tau)d\xi d\tau + F(x,t), \quad (x,t) \in \overline{D}_{P_0}, \tag{7.6}$$

in the class $C(\overline{D}_{P_0})$ with respect to an unknown function v. As is known (see e.g. [2]), equation (7.6) has a unique continuous solution $v(x,t)$ in the class $C(\overline{D}_{P_0})$, which for $(x,t) \in \overline{D}_{P_0}$ can be obtained by the method of approximation:

$$v_0 = 0, \quad v_{n+1} = \int_{D_{x,t}} k v_n d\xi d\tau + F, \quad n = 0, 1, \dots .$$

Whence in view of (7.5) we have $v_n(x,t) \geq 0$ in $\overline{D}_{P_0}$ for all $n = 0, 1, \dots$. On the other hand, we have $v_n \to v$ in the class $C(\overline{D}_{P_0})$ for $n \to \infty$. Therefore the limiting function $v \geq 0$ in the domain D_{P_0}. In view of equality (7.4), the function u is also a solution of equation (7.6), and therefore, by the uniqueness of the solution of this equation, we finally receive $u = v \geq 0$ in the domain D_{P_0}. □

In the conditions of Lemma 7.2, equality (7.2) can be rewritten in the form

$$\int_{D_{P_0}} |u| \Box \chi dxdt = \int_{D_{P_0}} |u|^p \chi dxdt + \int_{D_{P_0}} f\chi dxdt + \int_{x_0-t_0}^{x_0+t_0} [\psi(x)\chi(x,0) - \varphi(x)\chi_t(x,0)]dx, \quad p := \alpha + 1. \tag{7.7}$$

Let us use the method of test functions ([12, pp. 10–12]). Consider the function $\chi^0 := \chi^0(x,t)$ such that

$$\chi^0 \in C^2(\overline{D}_{(0,1)}), \quad \chi^0|_{D_{(0,1)}} > 0, \quad \chi^0|_{\gamma_{i,(0,1)}} = 0, \quad i = 1, 2, \tag{7.8}$$

and

$$\kappa_0 := \int_{D_{(0,1)}} \frac{|\Box \chi^0|^{p'}}{|\chi^0|^{p'-1}} dxdt < +\infty, \quad p' = 1 + \frac{1}{\alpha}. \tag{7.9}$$

It is easy to verify that the function χ^0, satisfying conditions (7.8) and (7.9), can be chosen by

$$\chi^0 = \chi^*(x,t) := [(1-t)^2 - x^2]^n, \quad (x,t) \in \overline{D}_{(0,1)}, \tag{7.10}$$

for a sufficiently large natural number n. Supposing that

$$\chi_{P_0}(x,t) = \chi^0\Big(\frac{x - x_0}{t_0}, \frac{t}{t_0}\Big),$$

in view of (7.8), it is easy to see that

$$\chi_{P_0} \in C^2(\overline{D}_{P_0}), \quad \chi_{P_0}|_{D_{P_0}} > 0, \quad \chi_{P_0}|_{\gamma_{i,P_0}} = 0, \quad i = 1, 2. \tag{7.11}$$

Assuming that the functions f, φ, ψ and the number x_0 are fixed, consider the function of one variable $t_0 > 0$

$$\zeta(t_0) := \int_{D_{P_0}} f\chi_{P_0} dxdt + \int_{x_0-t_0}^{x_0+t_0} \Big[\psi(x)\chi_{P_0}(x,0) - \varphi(x)\frac{\partial \chi_{P_0}(x,0)}{\partial t}\Big]dx. \tag{7.12}$$

The following theorem on the nonexistence of global solvability of problem (1.1), (1.2) is valid.

Theorem 7.1. *Let the conditions of Lemma 7.2 be fulfilled and let the function $u \in C(\overline{D}_{P_0})$ be a strong generalized solution of problem* (1.1), (1.2) *of class C in the domain D_{P_0}. In that case, if*

$$\liminf_{t_0 \to +\infty} \zeta(t_0) > 0, \tag{7.13}$$

then there exists a positive number $T_0 := T_0(x_0)$ such that for $t_0 > T_0$ problem (1.1), (1.2) *cannot have a strong generalized solution of class C in the domain D_{P_0}.*

Proof. Suppose that in the conditions of this theorem there exists a strong generalized solution u of problem (1.1), (1.2) of class C in the domain D_{P_0}. Then in view of Lemmas 7.1 and 7.2 we have equality (7.7), in which, due to (7.11), for the function χ the function $\chi = \chi_{P_0}$ can be chosen, i.e.

$$\int_{D_{P_0}} |u|^p \chi_{P_0} dxdt = \int_{D_{P_0}} |u| \Box \chi_{P_0} dxdt - \zeta(t_0). \tag{7.14}$$

If in the Young inequality with parameter $\varepsilon > 0$, i.e.

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{p' \varepsilon^{p'-1}} b^{p'}, \quad a, b \ge 0, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p := \alpha + 1 > 1,$$

we take

$$a = |u| \chi_{P_0}^{\frac{1}{p}} \quad \text{and} \quad b = \frac{|\Box \chi_{P_0}|}{\chi_{P_0}^{\frac{1}{p}}},$$

then we obtain

$$|u \Box \chi_{P_0}| = |u| \chi_{P_0}^{\frac{1}{p}} \frac{|\Box \chi_{P_0}|}{\chi_{P_0}^{\frac{1}{p}}} \le \frac{\varepsilon}{p} |u|^p \chi_{P_0} + \frac{1}{p' \varepsilon^{p'-1}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}}.$$

By (7.14) and the last inequality, we have

$$\left(1 - \frac{\varepsilon}{p}\right) \int_{D_{P_0}} |u|^p \chi_{P_0} \, dx dt \le \frac{1}{p' \varepsilon^{p'-1}} \int_{D_{P_0}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}} dx dt - \zeta(t_0),$$

whence for $\varepsilon < p$ we get

$$\int_{D_{P_0}} |u|^p \chi_{P_0} \, dx dt \le \frac{p}{(p - \varepsilon) p' \varepsilon^{p'-1}} \int_{D_{P_0}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}} dx dt - \frac{p}{p - \varepsilon} \zeta(t_0). \tag{7.15}$$

Since

$$\min_{0 < \varepsilon < p} \frac{p}{(p - \varepsilon) p' \varepsilon^{p'-1}} = 1,$$

which is reached for $\varepsilon = 1$, from (7.15) it follows that

$$\int_{D_{P_0}} |u|^p \chi_{P_0} \, dx dt \le \int_{D_{P_0}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}} dx dt - p' \zeta(t_0). \tag{7.16}$$

Due to (7.9), after the substitution of variables $x = t_0 x' + x_0$, $t = t_0 t'$ it is easy to verify that

$$\int_{D_{P_0}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}} dx dt = t_0^{-2(p'-1)} \int_{D_{(0,1)}} \frac{|\Box \chi^0|^{p'}}{|\chi^0|^{p'-1}} dx' dt' = t_0^{-2(p'-1)} \kappa_0 < +\infty.$$

Whence in view of (7.11), from inequality (7.16) we obtain

$$0 \le \int_{D_{P_0}} |u|^p \chi_{P_0} \, dx dt \le t_0^{-2(p'-1)} \kappa_0 - p' \zeta(t_0). \tag{7.17}$$

Because $p' > 1$ and due to (7.9), we have

$$\lim_{t_0 \to +\infty} t_0^{-2(p'-1)} \kappa_0 = 0.$$

Therefore, in view of (7.13), there exists a positive number $T_0 := T_0(x_0)$ such that for $t_0 > T_0$ the right-hand side of inequality (7.17) will be negative, while the left-hand side of this inequality is non-negative. This means that if there exists a strong generalized solution u of problem (1.1), (1.2) of class C in the domain D_{P_0}, then $t_0 \le T_0$ necessarily, which proves Theorem 7.1. □

Remark 7.2. In accordance with Remark 7.1, let us denote by $T^0 := T^0(x_0)$ the upper bound of those $t_0 > 0$, for which problem (1.1), (1.2) is solvable in the domain D_{P_0}. According to Theorems 6.1 and 7.1, we have $0 < T^0 \le T_0$; moreover, problem (1.1), (1.2) is solvable in the domain D_{P_0} for $t_0 < T^0$ and does not have a solution for $t_0 > T^0$.

Remark 7.3. It is easy to verify that if $f, \psi \in C(\overline{\Omega})$, $f, \psi \geq 0$, $\varphi = 0$ and additionally one of the conditions

$$\begin{aligned} &(1) \quad f(x,t) \geq ct^{-m}, \qquad x \in \mathbb{R}, \quad t \geq 1, \quad 0 \leq m := \text{const} \leq 2, \quad \psi = 0,\\ &(2) \quad \psi(x) \geq c, \qquad x \in \mathbb{R},\\ &(3) \quad f(x,t) \geq c, \qquad (x,t) \in \overline{\Omega}, \end{aligned} \tag{7.18}$$

is fulfilled, where $c := \text{const} > 0$, then condition (7.13) will be fulfilled, and therefore in this case problem (1.1), (1.2) for sufficiently large t_0 will not have a strong generalized solution u of class C in the domain D_{P_0}.

Indeed, in case (1) of (7.18) applying in (7.12) the transformation of independent variables x and t by the formulas $x = t_0\xi + x_0$, $t = t_0\tau$, after some simple transformations we have

$$\begin{aligned} \zeta(t_0) &= t_0^2 \int\limits_{D_{(0,1)}} f(t_0\xi + x_0, t_0\tau)\chi^0(\xi,\tau)\,d\xi d\tau \\ &\geq ct_0^{2-m} \int\limits_{D_{(0,1)}\cap\{\tau \geq t_0^{-1}\}} \tau^{-m}\chi^0(\xi,\tau)\,d\xi d\tau + t_0^2 \int\limits_{D_{(0,1)}\cap\{\tau < t_0^{-1}\}} f(t_0\xi + x_0, \tau)\chi^0(\xi,\tau)\,d\xi d\tau \end{aligned}$$

in supposition that $t_0 > 1$.

Let now $T_1 > 1$ be an arbitrary fixed number. Then from the last inequality for the function ζ we have

$$\zeta(t_0) \geq ct_0^{2-m} \int\limits_{D_{(0,1)}\cap\{\tau \geq t_0^{-1}\}} \tau^{-m}\chi^0(\xi,\tau)\,d\xi d\tau \geq cT_1^{2-m} \int\limits_{D_{(0,1)}\cap\{\tau \geq T_1^{-1}\}} \tau^{-m}\chi^0(\xi,\tau)\,d\xi d\tau \tag{7.19}$$

if $t_0 \geq T_1 > 1$ and $m \leq 2$. From (7.19) in view of (7.8) the validity of inequality (7.13) immediately follows.

Further, in case (2) of (7.18) considering in the first integral of (7.12) the transformation of the independent variable x according to the formula $x = x_0 + t_0\tau$, after some transformations we have (χ^* is defined by (7.10))

$$\begin{aligned} \zeta(t_0) &\geq \int\limits_{x_0-t_0}^{x_0+t_0} \psi(x)\chi_{P_0}(x,0)\,dx = t_0 \int\limits_{-1}^{1} \psi(x_0 + t_0\tau)\chi^*(\tau,0)\,d\tau \\ &\geq ct_0 \int\limits_{-1}^{1} (1-\tau^2)^n d\tau = 2ct_0 \int\limits_{0}^{1} (1-\tau^2)^n d\tau = ct_0 B(2^{-1}, n+1) > 0, \end{aligned} \tag{7.20}$$

where $B(a,b)$ is the well-known Euler integral of the first kind. From (7.20) the validity of inequality (7.13) immediately follows.

Case (3) of (7.18) can be considered analogously.

8 A priori estimate of a strong generalized solution of the Cauchy–Darboux problem (1.1), (1.7) of class C in the domain Λ_T

Lemma 8.1. *Let $g \in C(\mathbb{R})$, $f \in C(\overline{\Lambda}_T)$ and let conditions* (1.5), (1.6), (2.2) *be fulfilled. Then for any strong generalized solution $u = u(x,t)$ of problem* (1.1), (1.7) *of class C in the domain Λ_T the a priori estimate*

$$\|u\|_{C(\overline{\Lambda}_T)} \leq c_1 \|f\|_{C(\overline{\Lambda}_T)} + c_2 \tag{8.1}$$

is valid with non-negative constants $c_i := c_i(g,T)$, $i = 1,2$, not dependent on u and f, and $c_1 > 0$.

Proof. Let u be a strong generalized solution of problem (1.1), (1.7) of class C in the domain Λ_T. We know that, by Definition 1.4, there is a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{\Lambda}_T, \gamma_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{\Lambda}_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{\Lambda}_T)} = 0. \tag{8.2}$$

Consider the function $u_n \in \overset{\circ}{C}^2(\overline{\Lambda}_T, \gamma_T)$, as a solution of the problem

$$Lu_n = f_n, \tag{8.3}$$

$$u_{nx}|_{\gamma_{1,T}} = 0, \quad u_n|_{\gamma_{2,T}} = 0. \tag{8.4}$$

Here

$$f_n := Lu_n. \tag{8.5}$$

Multiplying both sides of equality (8.3) by u_{nt} and integrating over the domain

$$\Lambda_\tau := \{(x,t) \in \Lambda_T : t < \tau\}, \quad 0 < \tau \le T,$$

by (2.1) we get

$$\frac{1}{2}\int\limits_{\Lambda_\tau}(u_{nt}^2)_t dxdt - \int\limits_{\Lambda_\tau} u_{nxx}u_{nt} + \int\limits_{\Lambda_\tau}[G(u_n)]_t dxdt = \int\limits_{\Lambda_\tau} f_n u_{nt} dxdt.$$

Let $\Omega_\tau := \overline{\Lambda}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Integrating the left-hand side of the last equality by parts and taking into account (8.4), we have

$$2\int\limits_{\Lambda_\tau} f_n u_{nt} dxdt = \int\limits_{\gamma_{2,\tau}} \frac{1}{\nu_t}[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)]ds + \int\limits_{\Omega_\tau}(u_{nt}^2 + u_{nx}^2)dx + 2\int\limits_{\Omega_\tau} G(u_n)dx. \tag{8.6}$$

Since $\nu_t\frac{\partial}{\partial x} - \nu_x\frac{\partial}{\partial t}$ is the differentiation operator along the direction tangent to $\gamma_{2,T}$, i.e. the inner differential operator on $\gamma_{2,T}$, by view of the second equality from (8.4) we have

$$(u_{nx}\nu_t - u_{nt}\nu_x)|_{\gamma_{2,\tau}} = 0. \tag{8.7}$$

Due to (1.5) and (1.6), it is easy to verify that on $\gamma_{2,T} \subset \partial\Lambda_T$ the unit vector $\nu := (\nu_x, \nu_t)$ of the outer normal to $\partial\Lambda_T$ satisfies the conditions

$$\gamma_{2,T} : \nu_x = -\frac{1}{\sqrt{1 + [\gamma_2'(t)]^2}} < 0, \quad \nu_t = \frac{\gamma_2'(t)}{\sqrt{1 + [\gamma_2'(t)]^2}} < 0, \quad 0 \le t \le T, \quad (\nu_t^2 - \nu_x^2)|_{\gamma_{2,T}} < 0. \tag{8.8}$$

From (8.7) and (8.8) it follows that

$$\int\limits_{\gamma_{2,\tau}} \frac{1}{\nu_t}[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)]ds \ge 0.$$

Using the last inequality, (2.2) and the inequality $\operatorname{mes}\Omega_T \le T$, from (8.6) we have

$$w_n(\tau) := \int\limits_{\Omega_\tau}(u_{nt}^2 + u_{nx}^2)dx \le 2M_1T + 2M_2\int\limits_{\Omega_\tau} u_n^2 dx + 2\int\limits_{\Lambda_\tau} f_n u_{nt} dxdt. \tag{8.9}$$

Since $\Omega_\tau : \gamma_2(\tau) \le x \le 0,\ t = \tau$, and $\gamma_{2,T} : t = \gamma_2^{-1}(x),\ \gamma_2(T) \le x \le 0$, where γ_2^{-1} is a function and the inverse of γ_2 is uniquely defined by (1.6), in view of (8.4) and the Newton–Leibniz formula we have

$$u_n(x,\tau) = \int\limits_{\gamma_2^{-1}(x)}^{\tau} u_{nt}(x,t)dt, \quad \gamma_2(\tau) \le x \le 0, \quad (x,\tau) \in \Omega_\tau.$$

Using the Schwarz inequality, for $(x,\tau) \in \Omega_\tau$ we get

$$|u_n(x,\tau)|^2 \le \int\limits_{\gamma_2^{-1}(x)}^{\tau} 1^2 dt \int\limits_{\gamma_2^{-1}(x)}^{\tau} |u_{nt}(x,t)|^2 dt \le T\int\limits_{\gamma_2^{-1}(x)}^{\tau} |u_{nt}(x,t)|^2 dt.$$

Integrating both parts of the last inequality with respect to x on the segment $[\gamma_2(\tau), 0]$ we have

$$\int\limits_{\Omega_\tau} u_n^2 dx \le T\int\limits_{\gamma_2(\tau)}^{0}\Bigg[\int\limits_{\gamma_2^{-1}(x)}^{\tau} |u_{nt}(x,t)|^2\Bigg]dx = T\int\limits_{\Lambda_\tau\cap\{x<0\}} |u_{nt}(x,t)|^2 dxdt \le T\int\limits_{\Lambda_\tau} |u_{nt}(x,t)|^2 dxdt, \tag{8.10}$$

whence it follows that

$$\int\limits_{\Lambda_\tau} u_n^2 dxdt = \int\limits_0^\tau d\sigma\int\limits_{\Omega_\sigma} u_n^2 dx \le T\int\limits_0^\tau d\sigma\Bigg[\int\limits_{\Lambda_\sigma} u_{nt}^2 dxdt\Bigg] \le T^2\int\limits_{\Lambda_\tau} u_{nt}^2 dxdt. \tag{8.11}$$

Using the inequality $2f_n u_{nt} \le f_n^2 + u_{nt}^2$ and (8.9)–(8.11) we get

$$w_n(\tau) \le M_3 + M_4 \int\limits_{\Lambda_\tau} (u_{nt}^2 + u_{nx}^2)dxdt + \int\limits_{\Lambda_\tau} f_n^2 dxdt. \tag{8.12}$$

Here

$$M_3 := 2M_1 T, \quad M_4 := 2M_2 T + 1. \tag{8.13}$$

Due to

$$\int\limits_{\Lambda_\tau} (u_{nt}^2 + u_{nx}^2)dxdt = \int\limits_0^\tau w_n(\sigma)d\sigma$$

and that $\operatorname{mes} \Lambda_T \le T^2$, from (8.12) we have

$$w_n(\tau) \le M_4 \int\limits_0^\tau w_n(\sigma)d\sigma + M_3 + T^2 \|f_n\|^2_{C(\overline{\Lambda}_T)}, \quad 0 < \tau \le T.$$

Whence, according to the Gronwall Lemma, it follows that

$$w_n(\tau) \le (M_3 + T^2 \|f_n\|^2_{C(\overline{\Lambda}_T)}) \exp(M_4 \tau), \quad 0 < \tau \le T. \tag{8.14}$$

Further, in view of the second equality from (8.4), for any $(x,t) \in \overline{\Lambda}_T \setminus O$ we have

$$u_n(x,t) = u_n(x,t) - u_n(\gamma_2(t),t) = \int\limits_{\gamma_2(t)}^x u_{nx}(\xi,t)d\xi,$$

whence it follows that

$$|u_n(x,t)|^2 \le T \int\limits_{\gamma_2(t)}^x |u_{nx}(\xi,t)|^2 d\xi, \quad (x,t) \in \overline{D}_T \setminus O.$$

Whence, due to (8.14), for $(x,t) \in \overline{D}_T \setminus O$ we have

$$|u_n(x,t)|^2 \le T \int\limits_{\Omega_t} u_{nx}^2 dx \le T w_n(t) \le T(M_3 + T^2 \|f_n\|^2_{C(\overline{\Lambda}_T)}) \exp(M_4 t).$$

Taking into account the obvious inequality

$$\left(\sum_{i=1}^m a_i^2\right)^{\frac{1}{2}} \le \sum_{i=1}^m |a_i|,$$

we now get

$$\|u_n\|_{C(\overline{\Lambda}_T)} \le c_1 \|f_n\|_{C(\overline{\Lambda}_T)} + c_2, \tag{8.15}$$

where

$$c_1 := T^{\frac{3}{2}} \exp\Big(\frac{1}{2} M_4 T\Big), \quad c_2 := (TM_3)^{\frac{1}{2}} \exp\Big(\frac{1}{2} M_4 T\Big), \tag{8.16}$$

and the constants M_3 and M_4 are defined by (8.13). By (8.2) and (8.5), passing in (8.15) to the limit as $n \to \infty$, we get the a priori estimate (8.1). □

Remark 8.1. In the linear case, i.e. when $g = 0$ in equation (1.1), in an analogous manner we can introduce the notion of a strong generalized solution of problem (1.1), (1.7). Then, due to (2.1) the function $G = 0$ and for $M_i = 0$, $i = 1, 2$, condition (2.2) is valid for it. Moreover, when conditions (1.5) and (1.6) are fulfilled, then the a priori estimate (8.1) is also fulfilled and due to (8.13), estimate (8.15) takes the form

$$\|u\|_{C(\overline{\Lambda}_T)} \le T^{\frac{3}{2}} \exp\Big(\frac{T}{2}\Big) \|f\|_{C(\overline{\Lambda}_T)}.$$

9 Cases of the global solvability of problem (1.1), (1.7) in the class C

Introducing new independent variables $\xi = t + x$, $\eta = t - x$ the domain Λ_T is transformed into the curved triangular domain G_T with the vertices at the points $O(0,0)$, $Q_1(T,T)$, $Q_2(T+\gamma_2(T), T-\gamma_2(T))$ of the plane of variables ξ, η, while problem (1.1), (1.7) is transformed into the problem

$$\widetilde{L}\widetilde{u} := \widetilde{u}_{\xi\eta} + \widetilde{g}(\widetilde{u}) = \widetilde{f}(\xi,\eta), \quad (\xi,\eta) \in G_T, \tag{9.1}$$

$$(\widetilde{u}_\xi - \widetilde{u}_\eta)|_{\widetilde{\gamma}_{1,T}} = 0, \quad \widetilde{u}|_{\widetilde{\gamma}_{2,T}} = 0 \tag{9.2}$$

with respect to the new unknown function

$$\widetilde{u}(\xi,\eta) := u\Big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\Big), \quad (\xi,\eta) \in G_T.$$

Here

$$\widetilde{g} := \frac{1}{4}g, \quad \widetilde{f}(\xi,\eta) := \frac{1}{4}f\Big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\Big), \quad (\xi,\eta) \in G_T,$$

and $\widetilde{\gamma}_{1,T}$ and $\widetilde{\gamma}_{2,T}$ are the images of the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ during this transformation, outgoing from the common point $O(0,0)$ with end points O_1 and O_2.

Analogously to Definition 1.4, we introduce the notion of a strong generalized solution $\widetilde{u}$ of problem (9.1), (9.2) of class C in the domain G_T.

Due to (1.5) and (1.6), the smooth curves $\widetilde{\gamma}_{1,T}$ and $\widetilde{\gamma}_{2,T}$ are representable in the form

$$\begin{aligned} \widetilde{\gamma}_{1,T}: \quad & \eta = \xi, && 0 \le \xi \le \xi_0, \\ \widetilde{\gamma}_{2,T}: \quad & \xi = \tau(\eta), && 0 \le \eta \le \eta_0, \end{aligned} \tag{9.3}$$

where $\xi_0 := T < \eta_0 := T - \gamma_2(T)$ and

$$\tau'(\eta) > 0, \quad 0 \le \eta \le \eta_0, \quad \tau(0) = 0, \quad \tau(\eta) < \eta, \quad 0 < \eta \le \eta_0, \tag{9.4}$$

$$G_T := \{(\xi,\eta) \in (0,\xi_0) \times (0,\eta_0) : \xi < \eta,\ \tau(\eta) < \xi,\ \xi + \eta < T\}. \tag{9.5}$$

Remark 9.1. It is obvious that $u = u(x,t)$ is a strong generalized solution of problem (1.1), (1.7) of class C in the domain Λ_T if and only if $\widetilde{u}$ is a strong generalized solution of problem (9.1), (9.2) of class C in the domain G_T, i.e. when there exists a sequence of functions $\widetilde{u}_n \in \{v \in C^2(\overline{G}_T) : (v_\xi - v_\eta)|_{\widetilde{\gamma}_{1,T}} = 0,\ v|_{\widetilde{\gamma}_{2,T}} = 0\}$ such that

$$\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{C(\overline{G}_T)} = 0, \quad \lim_{n\to\infty} \|\widetilde{L}\widetilde{u}_n - \widetilde{f}\|_{C(\overline{G}_T)} = 0;$$

moreover, if the conditions of Lemma 8.1 are fulfilled, for $\widetilde{u}$ the a priori estimate of type (8.1)

$$\|\widetilde{u}\|_{C(\overline{G}_T)} \le c_1 \|\widetilde{f}\|_{C(\overline{G}_T)} + c_2 \tag{9.6}$$

holds with the same constants c_1 and c_2.

Below we will consider the linear case of problem (9.1), (9.2) when in equation (9.1) the function $\widetilde{g} = 0$:

$$\widetilde{\Box}\widetilde{u} := \widetilde{u}_{\xi\eta} = \widetilde{f}(\xi,\eta), \quad (\xi,\eta) \in G_T, \tag{9.7}$$

$$(\widetilde{u}_\xi - \widetilde{u}_\eta)|_{\widetilde{\gamma}_{1,T}} = 0, \quad \widetilde{u}|_{\widetilde{\gamma}_{2,T}} = 0. \tag{9.8}$$

Remark 9.2. By Remarks 8.1 and 9.1, for a strong generalized solution $\widetilde{u}$ of the linear problem (9.7), (9.8) of class C in the domain G_T the estimate

$$\|\widetilde{u}\|_{C(\overline{G}_T)} \le T^{\frac{3}{2}} \exp\Big(\frac{T}{2}\Big) \|\widetilde{f}\|_{C(\overline{G}_T)} \tag{9.9}$$

is valid.

In particular, the classical solution $\widetilde{u} \in C^2(\overline{G}_T)$ of this problem satisfies estimate (9.9). This estimate implies the uniqueness of both a generalized and a classical solution of problem (9.7), (9.8).

Let

$$a(\xi) := -\tau'(\xi), \quad 0 \le \xi \le \xi_0. \tag{9.10}$$

In view of (1.6) and that

$$\tau'(0) = \frac{1 + \gamma_2'(0)}{1 - \gamma_2'(0)},$$

and also using (9.10), we have

$$|a(0)| = |\tau'(0)| < 1. \tag{9.11}$$

Let $G_{0,T} := \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \xi_0,\ 0 < \eta < \eta_0\}$ be the characteristic rectangle in the plane of variables ξ and η, corresponding to equation (9.7). Due to (9.5) we have $G_T \subset G_{0,T}$. For $\widetilde{f} \in C(\overline{G}_T)$ we extend this function continuously to the closed domain $\overline{G}_{0,T}$, retaining the same notations, for example, letting $\widetilde{f}(\xi, \eta) = \widetilde{f}(\xi, \xi)$ for $0 \le \eta \le \xi$, $0 \le \xi \le \xi_0$ and $\widetilde{f}(\xi, \eta) = \widetilde{f}(\tau(\eta), \eta)$ for $0 \le \xi \le \tau(\eta)$, $0 \le \eta \le \eta_0$. Since the space $C^1(\overline{G}_{0,T})$ is dense in $C(\overline{G}_{0,T})$ (see [13, p. 37]), there exists a sequence of functions $\widetilde{f}_n$ such that

$$\widetilde{f}_n \in C^1(\overline{G}_{0,T}), \quad \lim_{n\to\infty} \|\widetilde{f}_n - \widetilde{f}\|_{C(\overline{G}_{0,T})} = 0. \tag{9.12}$$

Consider the function $\widetilde{u}_n \in C^2(\overline{G}_{0,T})$, which is a solution of the Goursat problem

$$\widetilde{\Box}\widetilde{u}_n = \widetilde{f}_n(\xi, \eta), \quad (\xi, \eta) \in G_{0,T}, \tag{9.13}$$

$$\widetilde{u}_n(\xi, 0) = \varphi_n(\xi), \quad 0 \le \xi \le \xi_0, \quad \widetilde{u}_n(0, \eta) = \psi_n(\eta), \quad 0 \le \eta \le \eta_0, \tag{9.14}$$

where $\varphi_n \in C^2([0, \xi_0])$ and $\psi_n \in C^2([0, \eta_0])$ are the functions satisfying the compatibility conditions

$$\varphi_n(0) = \psi_n(0) = 0. \tag{9.15}$$

As is known, the unique solution of problem (9.13), (9.14) is representable in the form (see [2, p. 246])

$$\widetilde{u}_n(\xi, \eta) = \varphi_n(\xi) + \psi_n(\eta) + \int_0^\xi d\xi' \int_0^\eta \widetilde{f}_n(\xi', \eta')\,d\eta', \quad (\xi, \eta) \in \overline{G}_{0,T}. \tag{9.16}$$

Let us find the functions $\varphi_n \in C^2([0, \xi_0])$ and $\psi_n \in C^2([0, \eta_0])$ such that the function $\widetilde{u} = \widetilde{u}_n$ defined by equality (9.16) would satisfy the boundary conditions (9.2). Differentiating the second equality from (9.2) along the direction tangent to $\widetilde{\gamma}_{2,T}$, in view of (9.3) we get

$$\tau'(\eta)\widetilde{u}_\xi(\tau(\eta), \eta) + \widetilde{u}_\eta(\tau(\eta), \eta) = 0, \quad 0 \le \eta \le \eta_0. \tag{9.17}$$

It is obvious that equality (9.17) together with the condition $\widetilde{u}(0, 0) = 0$ is equivalent to the second condition in (9.2). Substituting the expression for $\widetilde{u} = \widetilde{u}_n$ from (9.16) into (9.17) and the first condition in (9.2), and using (9.3), for the functions φ_n and ψ_n we obtain the following system of functional equations:

$$\varphi_n'(\xi) - \psi_n'(\xi) = \omega_{1n}(\xi), \quad 0 \le \xi \le \xi_0, \tag{9.18}$$

$$\tau'(\eta)\varphi_n'(\tau(\eta)) + \psi_n'(\eta) = \omega_{2n}(\eta), \quad 0 \le \eta \le \eta_0. \tag{9.19}$$

Here

$$\omega_{1n}(\xi) := \int_0^\xi \widetilde{f}_n(\xi', \xi)\,d\xi' - \int_0^\xi \widetilde{f}_n(\xi, \eta')\,d\eta', \qquad 0 \le \xi \le \xi_0, \tag{9.20}$$

$$\omega_{2n}(\eta) := -\tau'(\eta)\int_0^\eta \widetilde{f}_n(\tau(\eta), \eta')\,d\eta' - \int_0^{\tau(\eta)} \widetilde{f}_n(\xi', \eta)\,d\xi', \quad 0 \le \eta \le \eta_0. \tag{9.21}$$

Eliminating the function ψ_n' in system (9.18), (9.19), for $\varphi_{0n} := \varphi_n'$ we obtain the functional equation

$$\varphi_{0n}(\xi) - a(\xi)\varphi_{0n}(\tau(\xi)) = \omega_n(\xi), \quad 0 \le \xi \le \xi_0. \tag{9.22}$$

Here the function $a(\xi)$, $0 \le \xi \le \xi_0$, is defined by expression (9.10) as

$$\omega_n(\xi) := \omega_{1n}(\xi) + \omega_{2n}(\xi), \quad 0 \le \xi \le \xi_0. \tag{9.23}$$

Since $a \in C([0,\xi_0])$ when condition (9.11) is fulfilled, there exists a positive number ε such that

$$|a(\xi)| \le q := \text{const} < 1 \quad \text{for } 0 \le \xi \le \varepsilon. \tag{9.24}$$

Due to (9.4), if $\tau_k(\xi) := \tau(\tau_{k-1}(\xi))$, $\tau_1(\xi) := \tau(\xi)$, $0 \le \xi \le \xi_0$, then the sequence of functions $\{\tau_k(\xi)\}_{k=1}^{\infty}$ on the segment $[0,\xi_0]$ uniformly converges to zero, i.e. for any $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$ such that

$$\tau_k(\xi) \le \varepsilon, \quad 0 \le \xi \le \xi_0, \; k \ge n_0. \tag{9.25}$$

Denote by $\Lambda : C([0,\xi_0]) \to C([0,\xi_0])$ the linear continuous operator, acting by the formula

$$(\Lambda\omega_n)(\xi) := a(\xi)\omega_n(\tau(\xi)), \quad 0 \le \xi \le \xi_0. \tag{9.26}$$

It is obvious that

$$(\Lambda_k\omega_n)(\xi) = a(\xi)a(\tau(\xi)) \cdots a(\tau_{k-1}(\xi))\omega_n(\tau_k(\xi)), \quad k \ge 2, \tag{9.27}$$

and for $k = 1$ and $k = 0$ we assume that

$$\Lambda^1 = \Lambda \quad \text{and} \quad \Lambda^0 := I, \tag{9.28}$$

where I is the unit operator.

Due to (9.4), (9.24)–(9.28) the estimate

$$|(\Lambda^k\omega_n)(\xi)| \le [a(\xi)a(\tau(\xi)) \cdots a(\tau_{k-1}(\xi))][a(\tau_{n_0}(\xi)) \cdots a(\tau_{k-1}(\xi))]\omega_n(\tau_k(\xi)) \le \|a\|^{n_0}_{C([0,\xi])} q^{k-n_0} \|\omega_n\|_{C([0,\xi])}$$

for $0 \le \xi \le \xi_0$ and $k > n_0$ is valid, whence we get

$$\|\Lambda^k\|_{C([0,\xi_0]) \to C([0,\xi_0])} \le M_0 q^k, \quad k > n_0, \tag{9.29}$$

where

$$M_0 := (q^{-1}\|a\|_{C([0,\xi_0])})^{n_0}.$$

From (9.29), where $q < 1$, it follows that when condition (9.11) is fulfilled, the Neumann series

$$(I - \Lambda)^{-1} = \sum_{k=0}^{\infty} \Lambda^k$$

of the operator Λ converges in the space $C([0,\xi_0])$ and the unique solution $\varphi_{0n} \in C([0,\xi_0])$ of equation (9.22), in view of (9.24), is representable in the form

$$\varphi_{0n}(\xi) = \left[\sum_{k=0}^{\infty} \Lambda^k\omega_n\right](\xi), \quad 0 \le \xi \le \xi_0. \tag{9.30}$$

Remark 9.3. It is easy to verify that if we additionally assume that the curve γ_2 belongs to the class C^2, i.e.

$$\gamma_2 \in C^2([0,\eta_0]), \tag{9.31}$$

then $\tau \in C^2([0,\eta_0])$. Therefore, due to (9.10), (9.12), (9.20), (9.21) and (9.23) the functions a and ω_n belong to $C^1([0,\xi_0])$, then the solution φ_{0n} of equation (9.22), representable in the form of the convergent series (9.30) in $C([0,\xi_0])$, will also belong to the space $C^1([0,\xi_0])$. Indeed, differentiating formally equation (9.22) with respect to $\chi_n := \varphi'_{0n}$, we get the functional equation

$$\chi_n(\xi) - a_1(\xi)\chi_n(\tau(\xi)) = \widetilde{\omega}_{1n}(\xi), \quad 0 \le \xi \le \xi_0, \tag{9.32}$$

where $a_1(\xi) := a(\xi)\tau'(\xi)$ and $\widetilde{\omega}_{1n}(\xi) := \omega'_n(\xi) + a'(\xi)\varphi_{0n}(\tau(\xi))$ for $0 \le \xi \le \xi_0$. In view of (9.11) we have $|a_1(0)| < 1$ and the solution χ_n of equation (9.32), analogously to (9.30), is representable in the form

$$\chi_n = \sum_{k=0}^{\infty} \Lambda_1^k \widetilde{\omega}_{1n}, \tag{9.33}$$

where $(\Lambda_1\widetilde{\omega}_{1n})(\xi) := a_1(\xi)\widetilde{\omega}_{1n}(\tau(\xi))$, $0 \le \xi \le \xi_0$. Assuming that

$$\widetilde{\varphi}_{0n}(\xi) := \int_0^\xi \chi_n(\xi')d\xi' + \varphi_{0n}(0), \quad 0 \le \xi \le \xi_0,$$

and integrating equation (9.32), we find that

$$\widetilde{\varphi}_{0n}(\xi) - \varphi_{0n}(0) - \int_0^\xi a(\xi')d\widetilde{\varphi}_{0n}(\tau(\xi')) = \int_0^\xi a'(\xi')\varphi_{0n}(\tau(\xi'))d\xi' + \omega_n(\xi) - \omega_n(0), \quad 0 \le \xi \le \xi_0.$$

Integrating the third summand on the left-hand side of the last equality we have

$$\widetilde{\varphi}_{0n}(\xi) - \varphi_{0n}(0) - a(\xi)\widetilde{\varphi}_{0n}(\tau(\xi)) + a(0)\widetilde{\varphi}_{0n}(\tau(0)) + \int_0^\xi a'(\xi')\widetilde{\varphi}_{0n}(\tau(\xi'))d\xi'$$
$$= \int_0^\xi a'(\xi')\varphi_{0n}(\tau(\xi'))d\xi' + \omega_n(\xi) - \omega_n(0), \quad 0 \le \xi \le \xi_0.$$

Subtracting equality (9.22) from the last equality, for $\psi_{0n} := \widetilde{\varphi}_{0n} - \varphi_{0n}$ we get the following Volterra type homogeneous integro-functional equation:

$$\psi_{0n}(\xi) - a(\xi)\psi_{0n}(\tau(\xi)) + \int_0^\xi a'(\xi')\psi_{0n}(\tau(\xi'))d\xi' = 0, \quad 0 \le \xi \le \xi_0.$$

Applying the standard approximation method [10] to this equation we get $\psi_{0n} = 0$, i.e. $\widetilde{\varphi}_{0n} = \varphi_{0n}$ and, therefore,

$$\varphi_{0n}(\xi) = \int_0^\xi \chi_n(\xi')d\xi' + \varphi_{0n}(0), \quad 0 \le \xi \le \xi_0,$$

whence it follows that $\varphi_{0n} \in C^1([0, \xi_0])$. Therefore, taking into account that due to (9.19)

$$\psi_n'(\eta) = \omega_{2n}(\eta) - \tau'(\eta)\varphi_{0n}(\tau(\eta)), \quad 0 \le \eta \le \eta_0, \tag{9.34}$$

where $\omega_{2n} = \omega_{2n}(\eta)$ is defined from (9.21), and by (9.15)

$$\varphi_n(\xi) = \int_0^\xi \varphi_{0n}(\xi')d\xi' \in C^2([0, \xi_0]), \quad \psi_n(\eta) = \int_0^\eta \psi_n'(\eta')d\eta' \in C^2([0, \eta_0]), \tag{9.35}$$

in view of (9.20), (9.21), (9.23), (9.26)–(9.28) and (9.30), (9.32), (9.33), equality (9.16) can be rewritten as

$$\widetilde{u}_n(\xi, \eta) = (K\widetilde{f}_n)(\xi, \eta), \quad (\xi, \eta) \in \overline{G}_{0,T},$$

where the linear operator $K : C^1(\overline{G}_{0,T}) \to C^2(\overline{G}_{0,T})$ is continuous.

Remark 9.4. Retaining the same notations for the restrictions of the functions $\widetilde{u}_n$, $\widetilde{f}_n$ on the sub-domain G_T of the domain $G_{0,T}$, due to the construction, the function $\widetilde{u}_n \in C^2(\overline{G}_T)$ will be a classical solution of the linear problem (9.7), (9.8) for $\widetilde{f} = \widetilde{f}_n$ and, by Remark 9.2 and estimate (9.9), we have

$$\|\widetilde{u}_n - \widetilde{u}_k\|_{C(\overline{G}_T)} \le T^{\frac{3}{2}} \exp\Big(\frac{T}{2}\Big)\|\widetilde{f}_n - \widetilde{f}_k\|_{C(\overline{G}_T)}. \tag{9.36}$$

From (9.36) and (9.12) it follows that the sequence of functions $\widetilde{u}_n \in C^2(\overline{G}_T)$ is fundamental in the complete space $C(\overline{G}_T)$ and therefore there exists a function $\widetilde{u} \in C(\overline{G}_T)$ such that

$$\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{C(\overline{G}_T)} = 0. \tag{9.37}$$

Due to (9.12) and (9.37), the function $\tilde{u}$ constructed in such a way will be a strong generalized solution of the linear problem (9.7), (9.8) of class C in the domain G_T, the uniqueness of which follows from estimate (9.9). Denoting this solution $\tilde{u}$ by $\tilde{L}_0^{-1}\tilde{f}$, i.e.

$$\tilde{u} = \tilde{L}_0^{-1}\tilde{f}, \tag{9.38}$$

the linear operator $\tilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous and for its norm due to (9.9) we have the estimate

$$\|\tilde{L}_0^{-1}\|_{C(\overline{G}_T)\to C(\overline{G}_T)} \le T^{\frac{3}{2}} \exp\Big(\frac{T}{2}\Big). \tag{9.39}$$

Moreover, from (9.20), (9.21), (9.23), (9.26)–(9.28) and (9.30), (9.32), (9.33) it follows that the operator $\tilde{L}_0^{-1}$ from (9.38) in fact transforms the function $\tilde{f} \in C(\overline{G}_T)$ to the function $\tilde{u} \in C^1(\overline{G}_T)$ and the linear operator $\tilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is also continuous.

Remark 9.5. Since the space $C^1(\overline{G}_T)$ is compactly embedded into $C(\overline{G}_T)$ (see [5, p. 135]), due to Remark 9.4 the linear operator $\tilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is also compact and for its norm the estimate (9.39) is valid.

Remark 9.6. In view of Remarks 9.1, 9.4 and equality (9.38), the function $u = u(x,t)$ is a strong generalized solution of problem (1.1), (1.7) of class C in the domain Λ_T if and only if $\tilde{u}$ is a solution of the functional equation

$$\tilde{u} = K_0\tilde{u} := \tilde{L}_0^{-1}(-\tilde{g}(\tilde{u}) + \tilde{f}) \tag{9.40}$$

in the class $C(\overline{G}_T)$, where the operator $K_0 : C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous and compact since the nonlinear operator $N : C(\overline{G}_T) \to C(\overline{G}_T)$, acting by the formula $N\tilde{u} = -\tilde{g}(\tilde{u}) + \tilde{f}$ for $g \in C(\mathbb{R})$ and $\tilde{f} \in C(\overline{G}_T)$ is bounded and continuous, and the linear operator $\tilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ due to Remark 9.5 is compact. At the same time, due to estimate (9.6) and equalities (8.16), for any parameter $\tau \in [0,1]$ and any solution $\tilde{u} \in C(\overline{G}_T)$ of the equation $\tilde{u} = \tau K_0\tilde{u}$ the same a priori estimate (9.6) is valid with the same constants c_1 and c_2. Therefore, according to the Leray–Schauder Theorem [15, p. 375], equation (9.40) has at least one solution $\tilde{u} \in C(\overline{G}_T)$.

In view of Remarks 9.1 and 9.6 we have proved the following theorem.

Theorem 9.1. *Let $g \in C(\mathbb{R})$, $f \in C(\overline{\Lambda}_T)$ and let conditions* (1.5), (1.6), (2.2), (9.31) *be fulfilled. Then problem* (1.1), (1.7) *has at least one strong generalized solution u of class C in the domain Λ_T in the sense of Definition 1.4.*

Remark 9.7. It is easy to see that if the conditions of Theorem 9.1 are fulfilled for $T = +\infty$, then problem (1.1), (1.7) is globally solvable in the class C in the sense of Definition 1.5.

10 Smoothness of the solution of problem (1.1), (1.7)

First, let us consider the question of the smoothness of a strong generalized solution of the linear problem (9.7), (9.8) depending on the smoothness of the problem data. Aiming at this, when the conditions of Theorem 9.1 are fulfilled, taking into account Remark 9.1, let us follow the scheme of construction of a strong generalized solution $\tilde{u}$ of the linear problem (9.7), (9.8) of class C in the domain G_T and show that in fact this solution belongs to the class $C^1(\overline{G}_T)$ and the boundary conditions (9.8) are fulfilled pointwise. Indeed, due to (9.20), (9.21) and (9.23), the right-hand side ω_n of equation (9.22) is representable in the form

$$\omega_n(\xi) = -\int_0^{\xi} \tilde{f}_n(\xi,\eta')d\eta' + \int_0^{\xi} \tilde{f}_n(\xi',\xi)d\xi' + \tau'(\xi)\int_0^{\xi} \tilde{f}_n(\tau(\xi),\eta')d\eta' + \int_0^{\tau(\xi)} \tilde{f}_n(\xi',\xi)d\xi', \quad 0 \le \xi \le \xi_0. \tag{10.1}$$

Due to (9.12), from (10.1) we have

$$\lim_{n\to\infty} \|\omega_n - \omega\|_{C(\overline{G}_T)} = 0, \tag{10.2}$$

where

$$\omega(\xi) := -\int_0^{\xi} \tilde{f}(\xi,\eta')d\eta' + \int_0^{\xi} \tilde{f}(\xi',\xi)d\xi' + \tau'(\xi)\int_0^{\xi} \tilde{f}(\tau(\xi),\eta')d\eta' + \int_0^{\tau(\xi)} \tilde{f}(\xi',\xi)d\xi', \quad 0 \le \xi \le \xi_0. \tag{10.3}$$

In turn, from (9.26)–(9.30), (10.1)–(10.3) it follows that

$$\lim_{n\to\infty} \|\varphi_{0n} - \varphi_0\|_{C([0,\xi_0])} = 0, \tag{10.4}$$

where

$$\varphi_0 := \Big[\sum_{k=0}^{\infty} \Lambda^k \omega\Big] \in C([0,\xi_0]). \tag{10.5}$$

Since the derivative ψ_n' of the function ψ_n from representation (9.16) is given by equality (9.34), due to (9.12), (9.21) and (10.4) we have

$$\lim_{n\to\infty} \|\psi_n' - \psi_0\|_{C([0,\eta_0])} = 0, \tag{10.6}$$

where

$$\psi_0 \in C([0,\eta_0]), \quad \psi_0(\eta) := \omega_2(\eta) - \lambda_2'(\eta)\varphi_0(\lambda_2(\eta)), \quad 0 \le \eta \le \eta_0, \tag{10.7}$$

with

$$\omega_2(\eta) := -\tau'(\eta)\int_0^{\eta} \widetilde{f}(\tau(\eta),\eta')d\eta' - \int_0^{\tau(\eta)} \widetilde{f}(\xi',\eta)d\xi', \quad 0 \le \eta \le \eta_0. \tag{10.8}$$

Finally, using Remark 9.4, the limit equalities (9.12), (9.37), (10.4), (10.6), and also (9.35), using the notation

$$\varphi(\xi) := \int_0^{\xi} \varphi_0(\xi')d\xi', \quad 0 \le \xi \le \xi_0, \quad \psi(\eta) := \int_0^{\eta} \psi_0(\eta')d\eta', \quad 0 \le \eta \le \eta_0, \tag{10.9}$$

and passing in equality (9.16) to the limit, for a strong generalized solution $\widetilde{u}$ of the linear problem (9.7), (9.8) of class C in the domain G_T we get the representation

$$v(\xi,\eta) = \varphi(\xi) + \psi(\eta) + \int_0^{\xi} d\xi' \int_0^{\eta} \widetilde{f}(\xi',\eta')d\eta', \quad (\xi,\eta) \in \overline{G}_T. \tag{10.10}$$

If $\widetilde{f} \in C(\overline{G}_T)$, then, due to (10.5) and (10.7), from representation (10.10) it follows that

$$v \in C^1(\overline{G}_T).$$

Further, in view of (10.2), (10.4) and (9.22) the function φ_0 satisfies the functional equation

$$\varphi_0(\xi) - a(\xi)\varphi_0(\tau(\xi)) = \omega(\xi), \quad 0 \le \xi \le \xi_0. \tag{10.11}$$

Remark 10.1. If $\widetilde{f} \in C^1(\overline{G}_T)$, then since due to (9.11) the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ do not have a common tangent line at the point O, as is known [4, p. 595], this function, retaining the same notations, can be extended to the rectangle $\overline{G}_{0,T}$ in such a way that $\widetilde{f} \in C^1(\overline{G}_{0,T})$.

From (10.3) it follows that when condition (9.31) is fulfilled, if we additionally require that $\widetilde{f} \in C^1(\overline{G}_T)$, then the right-hand side ω of equation (10.11) will belong to the class $C^1([0,\xi_0])$. Whence, by Remark 9.4 it follows that the solution φ_0 of equation (10.11) belongs to the space $C^1([0,\xi_0])$ and, due to (10.7) and (10.8), the function $\psi_0 \in C^1([0,\eta_0])$. Therefore, due to the assumptions made above and taking into account (10.9) we get that the function $\widetilde{u}$ from (10.10) will belong to the space $C^2(\overline{G}_T)$. Thus we have proved the following theorem.

Theorem 10.1. *If conditions* (1.5), (1.6) *and* (9.31) *are fulfilled, then the strong generalized solution $\widetilde{u}$ of the linear problem* (9.7), (9.8) *of class C in the domain G_T belongs to the space $C^1(\overline{G}_T)$, i.e. in accordance with* (9.38)

$$\widetilde{u} = \widetilde{L}_0^{-1}\widetilde{f} \in C^1(\overline{G}_T),$$

and if we additionally require that $\widetilde{f} \in C^1(\overline{G}_T)$, then

$$\widetilde{u} = \widetilde{L}_0^{-1}\widetilde{f} \in C^2(\overline{G}_T).$$

Moreover, in both cases the boundary conditions (9.8) *are fulfilled pointwise.*

The corollary of Remarks 9.1 and 9.6, equality (9.38) and Theorem 10.1 is the following theorem.

Theorem 10.2. *If the conditions of Theorem 9.1 are fulfilled, then the strong generalized solution u of problem* (1.1), (1.7) *of class C in the domain Λ_T belongs to the space $C^1(\overline{\Lambda}_T)$, and for the additional requirement that $g \in C^1(\mathbb{R})$ and $f \in C^1(\overline{\Lambda}_T)$ this solution also belongs to the space $C^2(\overline{\Lambda}_T)$, i.e. is classical. Moreover, in both cases the boundary conditions* (1.7) *are fulfilled pointwise.*

11 Uniqueness, existence and nonexistence of a global solution. Local solvability

Reasonings analogous to that used in proving the theorems on the uniqueness, existence and nonexistence of a global solution, and also on the local solvability of the Cauchy problem (1.1), (1.2) enable us to prove the following propositions.

Theorem 11.1. *Let the function g satisfy condition* (5.1) *and let $f \in C(\overline{\Lambda}_T)$. Then problem* (1.1), (1.7) *can have at most one strong generalized solution of class C in the domain Λ_T in the sense of Definition 1.4.*

Theorem 11.2. *Let $g \in C^1(\mathbb{R})$, $f \in C^1(\overline{\Lambda}_T)$ and let conditions* (1.5), (1.6), (2.2), (9.31) *be fulfilled. Then problem* (1.1), (1.7) *has in the domain Λ_T, $0 < T \leq \infty$, a unique classical solution $u \in C^2(\overline{\Lambda}_T)$.*

Before formulating the theorem on the nonexistence of a global solution of problem (1.1), (1.7), we impose the condition

$$g(s) \leq -\lambda |s|^{\alpha+1}, \quad s \in \mathbb{R}, \quad \lambda, \alpha = \text{const} > 0, \tag{11.1}$$

on the nonlinear function g, and for $\gamma_2 : x = -kt$, $0 < k = \text{const} < 1$, consider the function (see [12])

$$\varphi^0(x,t) = \begin{cases} [x(x+kt)(1-t)]^m, & (x,t) \in \Lambda_{T=1}, \\ 0, & t \geq 1, \end{cases}$$

where m is a sufficiently large positive number.

Let

$$\varphi_T(x,t) := \varphi^0\Big(\frac{x}{T}, \frac{t}{T}\Big), \quad \zeta(T) := \int\limits_{\Lambda_T} f\varphi_T \, dx\, dt, \quad T > 0.$$

Theorem 11.3. *Let the function $g \in C(\mathbb{R})$ satisfy condition* (11.1), *$f \in C(\overline{\Lambda}_\infty)$, $f \geq 0$ in the domain Λ_∞ and*

$$\liminf_{T\to\infty} \zeta(T) > 0. \tag{11.2}$$

Then there exists a positive number $T_0 := T_0(f)$ such that for $T > T_0$ problem (1.1), (1.7) *cannot have a strong generalized solution of class C in the domain Λ_T in the sense of Definition 1.4.*

Remark 11.1. It is easy to verify that if $f \in C(\overline{\Lambda}_\infty)$, $f \geq 0$ and $f(x,t) \geq ct^{-m}$ for $t \geq 1$, where $c = \text{const} > 0$ and $0 \leq m = \text{const} \leq 2$, then condition (11.2) will be fulfilled and, according to Theorem 11.3, problem (1.1), (1.7) cannot have a strong generalized solution of class C in the domain Λ_T for sufficiently large T.

Corollary 11.1. *When the conditions of Theorem 11.3 are fulfilled, problem* (1.1), (1.7) *is not globally solvable in the class C in the sense of Definition 1.5, and it does not have a global solution of class C in the domain Λ_∞ in the sense of Definition 1.6.*

Note that when condition (2.2) (which guarantees the global solvability of problem (1.1), (1.7)) is violated, the local solvability of this problem remains in force. Indeed, the following theorem on the local solvability of problem (1.1), (1.7) is valid.

Theorem 11.4. *Let $g \in C(\mathbb{R})$, $f \in C(\overline{\Lambda}_\infty)$ and let conditions* (1.5), (1.6) *be fulfilled. Then problem* (1.1), (1.7) *is locally solvable in the class C in the sense of Definition 1.7, i.e. there exists a positive number $T_0 := T_0(f)$ such that for $T \leq T_0$ this problem has at least one strong generalized solution u of class C in the domain Λ_T.*

Funding: This research was supported by the Shota Rustaveli National Science Foundation (Grant No. 31/32).

References

[1] G. K. Berikelashvili, O. M. Jokhadze, B. G. Midodashvili and S. S. Kharibegashvili, On the existence and nonexistence of global solutions of the first Darboux problem for nonlinear wave equations (in Russian), *Differ. Uravn.* **44** (2008), no. 3, 359–372, 430; translation in *Differ. Equ.* **44** (2008), no. 3, 374–389.

[2] A. V. Bitsadze, *Some Classes of Partial Differential Equations* (in Russian), Nauka, Moscow, 1981.

[3] Y. Ebihara, M. Nakao and T. Nanbu, On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 = f$, *Pacific J. Math.* **60** (1975), no. 2, 63–70.

[4] G. M. Fichtenholz, *The Course of Differential and Integral Calculus. Vol. II* (in Russian), Nauka, Moscow, 1966.

[5] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin, 1983.

[6] O. G. Goman, Equation of reflected wave (in Russian), *Vestn. Mosk. Univ. Ser. I* **23** (1968), no. 2, 84–87.

[7] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Math. Appl. (Berlin) 26, Springer-Verlag, Berlin, 1997.

[8] O. Jokhadze, Cauchy–Goursat problem for one-dimensional semilinear wave equations, *Comm. Partial Differential Equations* **34** (2009), no. 4–6, 367–382.

[9] O. M. Jokhadze and S. S. Kharibegashvili, On the first Darboux problem for second-order nonlinear hyperbolic equations (in Russian), *Mat. Zametki* **84** (2008), no. 5, 693–712; translation in *Math. Notes* **84** (2008), no. 5–6, 646–663.

[10] S. Kharibegashvili, Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems, *Mem. Differential Equations Math. Phys.* **4** (1995), 1–127.

[11] Z. O. Mellnik, Example of a nonclassical boundary value problem for the vibrating string equation (in Russian), *Ukrain. Mat. Zh.* **32** (1980), no. 5, 671–674.

[12] È. Mitidieri and S. I. Pokhozhaev, A priori estimates and the absence of solutions to nonlinear partial differential equations and inequalities, *Tr. Mat. Inst. Steklova* **234** (2001), 1–384; translation in *Proc. Steklov Inst. Math.* **234** (2001), 1–362.

[13] R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland Math. Libr. 35, North-Holland, Amsterdam, 1985.

[14] K. Ono, On the global existence and decay of solutions for semilinear telegraph equations, *Int. J. Appl. Math.* **2** (2000), no. 9, 1121–1136.

[15] V. A. Trenogin, *Functional Analysis* (in Russian), 2nd ed., Nauka, Moscow, 1993.

[16] S. D. Troitskaya, On a boundary value problem for hyperbolic equations (in Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* **62** (1998), no. 2, 193–224; translation in *Izv. Math.* **62** (1998), no. 2, 399–428.

Received July 29, 2013; accepted January 23, 2014.