ISSN 0012-2661, Differential Equations, 2016, Vol. 52, No. 5, pp. 644–666.

PARTIAL DIFFERENTIAL EQUATIONS

On the Solvability of a Boundary Value Problem for Nonlinear Wave Equations in Angular Domains

S. S. Kharibegashvili and O. M. Jokhadze
A. Razmadze Mathematical Institute,
Tbilisi Ivane Javakhishvili State University, Tbilisi, Georgia
Tbilisi State University, Tbilisi, Georgia
e-mail: kharibegashvili@yahoo.com, ojokhadze@yahoo.com
Received November 17, 2015

Abstract—For a one-dimensional wave equation with a weak nonlinearity, we study the Darboux boundary value problem in angular domains, for which we analyze the existence and uniqueness of a global solution and the existence of local solutions as well as the absence of global solutions.

DOI: 10.1134/S0012266116050104

1. STATEMENT OF THE PROBLEM

In the plane of independent variables x and t, consider the nonlinear wave equation

$$Lu := \Box u + f(x,t,u) = F(x,t), \tag{1.1}$$

where $f = f(x,t,s)$ is a given real function nonlinear with respect to the variable s, $F = F(x,t)$ is a given real function, and $u = u(x,t)$ is the unknown real function; moreover, we assume that f and F are continuous functions of their arguments and $\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$.

By $D:\ \gamma_2(t) < x < \gamma_1(t)$, $t > 0$, we denote the angular domain lying inside the characteristic angle $\Lambda_0:\ t > |x|$ and bounded by smooth noncharacteristic curves $\gamma_i:\ x = \gamma_i(t)$, $t \geq 0$, $i = 1,2$ (i.e., $|\gamma_i'(t)| \neq 1$, $t \geq 0$, $i = 1,2$), of the class C^2 issuing from the origin $O(0,0)$. Set $D_T := D \cap \{t < T\}$ and $\gamma_{i,T} := \gamma_i \cap \{t \leq T\}$, $T > 0$, $i = 1,2$.

For Eq. (1.1), we consider the Darboux boundary value problem, where the directional derivative of the solution of Eq. (1.1) is posed on $\gamma_{1,T}$, and a solution itself is posed on $\gamma_{2,T}$, in the following statement: find a solution $u = u(x,t)$ of that equation in the domain D_T with the boundary conditions

$$(l_1 u_x + l_2 u_t)|_{\gamma_{1,T}} = 0, \tag{1.2}$$

$$u|_{\gamma_{2,T}} = 0, \tag{1.3}$$

where l_1 and l_2 are given continuous functions; moreover, $(|l_1| + |l_2|)|_{\gamma_1} \neq 0$.

Note that, in the linear case in which the function f occurring in Eq. (1) is linear with respect to the variable s and the conditions

$$(\alpha_i u_x + \beta_i u_t)|_{\gamma_{i,T}} = 0, \qquad i = 1,2; \qquad u(0,0) = 0, \tag{1.4}$$

are considered instead of the boundary conditions (1.2) and (1.3), problem (1.1), (1.4) in the domain D_T was studied in [1–6]. Note also that problem (1.1)–(1.3) is equivalent to problem (1.1), (1.4) in which the direction (α_2, β_2) coincides with the direction of the tangent to the curve $\gamma_{2,T}$ at each of its points. In the case of Eq. (1.1) with a power-law nonlinearity in which the homogeneous Dirichlet conditions $u|_{\gamma_{i,T}} = 0$, $i = 1,2$, are posed on the curves γ_1 and γ_2, and moreover, one of these curves, either γ_1 or γ_2, is characteristic, this problem was considered in the papers [7–9], and the case in which both curves are noncharacteristic was studied in [10]. The special case of the boundary conditions (1.2) and (1.3) of the form $u_x|_{\gamma_{1,T}} = 0$ and $u|_{\gamma_{2,T}} = 0$, where $\gamma_{1,T}\colon\ x = 0,\ 0 \le t \le T$, and $\gamma_{2,T}$: $x = -t,\ 0 \le t \le T$, is a characteristic of Eq. (1.1) with a power-law nonlinearity, was considered in [11, 12]; moreover, the case in which $\gamma_{2,T}$ is a noncharacteristic curve was considered in [13, 14]. As was noted in [1, 6], such problems arise in the mathematical modeling of small harmonic oscillations of a wedge in a supersonic flow and oscillations of a string inside a cylinder filled with a viscous fluid.

In the present paper, we consider the more general case of a nonlinear function $f(x, t, s)$, smooth noncharacteristic curves γ_1 and γ_2, and the behavior of the vector field (l_1, l_2) in the boundary condition (1.2) as compared with the cases studied in the above-mentioned papers. Note that, in the case under consideration, the analysis of the solvability of problem (1.1)–(1.3) encounters additional difficulties of nontechnical character.

Set $\overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T) := \{v \in C^2(\overline{D}_T) :\ (l_1 v_x + l_2 v_t)|_{\gamma_{1,T}} = 0,\ v|_{\gamma_{2,T}} = 0\}$ and $\gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Definition 1.1. Let the conditions $f \in C(\overline{D}_T \times \mathbb{R})$, $F \in C(\overline{D}_T)$, and $l_1, l_2 \in C(\gamma_{1,T})$ be satisfied. A function u is called a *strong generalized solution* of problem (1.1)–(1.3) in the class C in the domain D_T if u belongs to $C(\overline{D}_T)$ and there exists a function sequence $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ such that $u_n \to u$ and $Lu_n \to F$ in the space $C(\overline{D}_T)$ as $n \to \infty$.

Remark 1.1. Obviously, a classical solution of problem (1.1)–(1.3) in the space $\overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ is a strong generalized solution of that problem in the class C in the domain D_T in the sense of Definition 1.1.

Definition 1.2. Let $f \in C(\overline{D}_\infty \times \mathbb{R})$, $F \in C(\overline{D}_\infty)$, and $l_1, l_2 \in C(\gamma_{1,\infty})$. We say that problem (1.1)–(1.3) is *globally solvable in the class* C if, for any finite $T > 0$, it has at least one strong generalized solution of the class C in the domain D_T in the sense of Definition 1.1.

Definition 1.3. Under the assumptions of Definition 1.2, a function $u \in C(\overline{D}_\infty)$ is called a *global strong generalized solution of problem* (1.1)–(1.3) *in the class* C *in the domain* D_∞ if, for any finite $T > 0$, the function $u|_{D_T}$ is a strong generalized solution of that problem in the class C in the domain D_T in the sense of Definition 1.1.

Definition 1.4. Under the assumptions of Definition 1.2, we say that problem (1.1)–(1.3) is *locally solvable in the class* C if there exists a positive number $T_0 = T_0(F)$ such that, for $T \le T_0$, it has at least one strong generalized solution of the class C in the domain D_T in the sense of Definition 1.1.

Remark 1.2. Under the above-mentioned assumptions, one can readily note that

$$-t < \gamma_2(t) < \gamma_1(t) < t, \quad t > 0; \qquad |\gamma_i'(t)| < 1, \quad t \ge 0, \quad \gamma_i(0) = 0, \quad i = 1, 2. \tag{1.5}$$

Remark 1.3. Below, without loss of generality, one can assume that $\gamma_1(t) \le 0$, $0 \le t \le T$, since otherwise, by virtue of condition (1.5), this could be achieved with the use of the Lorentz transformation

$$x' = \frac{x - k_0 t}{\sqrt{1 - k_0^2}}, \qquad t' = \frac{t - k_0 x}{\sqrt{1 - k_0^2}}, \qquad k_0 := \max_{0 \le t \le T} |\gamma_1'(t)| < 1,$$

which does not change the form of Eq. (1.1) and maps the characteristic angle $\Lambda_0\colon\ t > |x|$ into the characteristic angle $\Lambda_0'\colon\ t' > |x'|$.

Below, by virtue of Remark 1.3, in addition to condition (1.5), we require that

$$\gamma_2(t) < \gamma_1(t) \le 0, \qquad \gamma_1'(t) \le 0, \qquad \gamma_2'(t) < 0, \qquad t > 0. \tag{1.6}$$

In Section 2, we present conditions on the data of problem (1.1)–(1.3) under which we prove an a priori estimate for a strong generalized solution of that problem in the class C in the domain D_T. In Section 3, we study the global solvability of the problem in the class C in the domain D_T. We analyze the smoothness of the solution in Section 4 and study the uniqueness and existence of a global solution of problem (1.1)–(1.3) in the domain D_∞ in Section 5. Finally, in Section 6, we study the case of absence of a global solution as well as the local solvability of that problem.

2. A PRIORI ESTIMATE FOR THE SOLUTION OF PROBLEM (1.1)–(1.3)

Set

$$g(x,t,s) := \int_0^s f(x,t,s_1)\,ds_1, \qquad (x,t,s) \in \overline{D}_T \times \mathbb{R}. \tag{2.1}$$

In view of notation (2.1), consider the following conditions imposed on the nonlinear function f :

$$g(x,t,s) \ge -M_1 - M_2 s^2, \qquad (x,t,s) \in \overline{D}_T \times \mathbb{R}, \tag{2.2}$$

$$g_t(x,t,s) \le M_3 + M_4 s^2, \qquad (x,t,s) \in \overline{D}_T \times \mathbb{R}, \tag{2.3}$$

where $M_i := M_i(T) =$ const ≥ 0, $1 \le i \le 4$.

Remark 2.1. Let $f_0, f_{0t} \in C(\overline{D}_\infty)$, $f_0 \ge 0$, and $f_{0t} \le 0$. We present some classes of functions $f = f(x,t,s)$ that are often used in applications and satisfy conditions (2.2) and (2.3).

1. $f(x,t,s) = f_0(x,t)|s|^\alpha \operatorname{sgn} s$, where $\alpha > 0$, $\alpha \ne 1$. In this case, we have

$$g(x,t,s) = f_0(x,t)\frac{|s|^{\alpha+1}}{\alpha+1}.$$

2. $f(x,t,s) = f_0(x,t)\psi(s)$, where ψ belongs to $C(\mathbb{R})$, $\psi(s)\operatorname{sgn} s \ge 0$, and $s \in \mathbb{R}$. Here

$$g(x,t,s) = f_0(x,t)\int_0^s \psi(\tau)\,d\tau.$$

3. $f(x,t,s) = f_0(x,t)e^s$. In this case, we have $g(x,t,s) = f_0(x,t)(e^s - 1)$.

Now we subject the curve $\gamma_{1,T}$ to an additional constraint of the geometric nature, which depends on the direction of the vector (l_1, l_2) of the directional derivative occurring in the boundary condition (1.2),

$$[(l_1^2 + l_2^2)\nu_t + 2l_1 l_2 \nu_x](P) \ge 0, \qquad P \in \gamma_{1,T}, \tag{2.4}$$

where $\nu := (\nu_x, \nu_t)$ is the unit outward normal to ∂D_T at the point P.

Remark 2.2. By virtue of conditions (1.5) and (1.6), one can readily see that, on $\gamma_{1,T} \subset \partial D_T$, the unit vector $\nu := (\nu_x, \nu_t)$ of the outward normal to ∂D_T is defined by the relations

$$\nu_x = \frac{1}{\sqrt{1 + |\gamma_1'(t)|^2}} > 0, \qquad \nu_t = -\frac{\gamma_1'(t)}{\sqrt{1 + |\gamma_1'(t)|^2}} \ge 0, \qquad 0 \le t \le T. \tag{2.5}$$

It follows from relations (2.5) that condition (2.4) is satisfied for the case in which $l_1 l_2|_{\gamma_{1,T}} \ge 0$. In particular, if condition (1.2) is a homogeneous Neumann boundary condition, i.e., $u_\nu|_{\gamma_{1,T}} = (\nu_x u_x + \nu_t u_t)|_{\gamma_{1,T}} = 0$, then condition (2.4) is satisfied. One can also readily show that, for the case in which $l_1, l_2 =$ const , $l_2 = -k_0 l_1$, where $k_0 > 0$, $k_0 \ne 1$, and $\gamma_{1,T}\colon x = -kt$, $0 \le k =$ const < 1, condition (2.4) is equivalent to the condition $k \ge 2k_0/(1 + k_0^2)$.

Lemma 2.1. *Let $f \in C(\overline{D}_T \times \mathbb{R})$, $F \in C(\overline{D}_T)$, and $l_1, l_2 \in C(\gamma_{1,T})$, and let conditions* (1.5), (1.6), (2.2)–(2.4) *be satisfied. Then any strong generalized solution $u = u(x,t)$ of problem* (1.1)–(1.3) *of the class C in the domain D_T satisfies the a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|F\|_{C(\overline{D}_T)} + c_2 \tag{2.6}$$

with nonnegative constants $c_i := c_i(f, l_1, l_2, T)$, $i = 1, 2$, independent of u and F; moreover, $c_1 > 0$.

Proof. Let u be a strong generalized solution of problem (1.1)–(1.3) of the class C in the domain D_T. By Definition 1.1, there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|Lu_n - F\|_{C(\overline{D}_T)} = 0. \tag{2.7}$$

Consider the function $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ treated as a solution of the problem

$$Lu_n = F_n, \tag{2.8}$$

$$(l_1 u_{nx} + l_2 u_{nt})|_{\gamma_{1,T}} = 0, \tag{2.9}$$

$$u_n|_{\gamma_{2,T}} = 0, \tag{2.10}$$

where

$$F_n := Lu_n. \tag{2.11}$$

By multiplying both sides of Eq. (2.8) by u_{nt}, by integrating the resulting relation over the domain $D_\tau := \{(x,t) \in D_T : \ t < \tau\}$, $0 < \tau \le T$, and by using relation (2.1), we obtain

$$\frac{1}{2}\int_{D_\tau} (u_{nt}^2)_t \, dx\, dt - \int_{D_\tau} u_{nxx} u_{nt} \, dx\, dt + \int_{D_\tau} \frac{d}{dt}(g(x,t,u_n(x,t))\, dx\, dt - \int_{D_\tau} g_t(x,t,u_n(x,t))\, dx\, dt$$
$$= \int_{D_\tau} F_n u_{nt}\, dx\, dt. \tag{2.12}$$

Set $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. By integrating by parts on the left-hand side in relation (2.12) and by taking into account relations (2.1) and (2.10), we obtain

$$2\int_{D_\tau} F_n u_{nt}\, dx\, dt = \int_{\gamma_{1,\tau}} (u_{nt}^2 \nu_t - 2u_{nx}u_{nt}\nu_x + u_{nx}^2 \nu_t)\, ds$$
$$+ \int_{\gamma_{2,\tau}} \frac{1}{\nu_t}[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)]\, ds + \int_{\Omega_\tau} (u_{nt}^2 + u_{nx}^2)\, dx + 2\int_{\Omega_\tau} g(x,\tau,u_n(x,\tau))\, dx$$
$$+ 2\int_{\gamma_{1,\tau}} g(x,t,u_n(x,t))\nu_t\, ds - 2\int_{D_\tau} g_t(x,t,u_n(x,t))\, dx\, dt. \tag{2.13}$$

It follows from Eq. (2.9) that the relations

$$u_{nx} = -\lambda l_2, \qquad u_{nt} = \lambda l_1 \tag{2.14}$$

hold on $\gamma_{1,T}$, where λ is a proportionality coefficient.

By virtue of relations (2.4) and (2.14), we have

$$\int_{\gamma_{1,\tau}} (u_{nt}^2 \nu_t - 2u_{nx}u_{nt}\nu_x + u_{nx}^2\nu_t)\, ds = \int_{\gamma_{1,\tau}} \lambda^2[(l_1^2 + l_2^2)\nu_t + 2l_1 l_2 \nu_x]\, ds \ge 0. \tag{2.15}$$

Since $\nu_t \dfrac{\partial}{\partial x} - \nu_x \dfrac{\partial}{\partial t}$ is the operator of differentiation along the direction of the tangent to $\gamma_{2,T}$, i.e., is an internal differential operator on $\gamma_{2,T}$, it follows from conditions (2.10) that

$$(u_{nx}\nu_t - u_{nt}\nu_x)|_{\gamma_{2,\tau}} = 0. \tag{2.16}$$

By taking into account conditions (1.5) and (1.6), one can readily see that on $\gamma_{2,T} \subset \partial D_T$ the unit vector $\nu := (\nu_x, \nu_t)$ of the outward normal to ∂D_T is defined by the relations

$$\nu_x = -\frac{1}{\sqrt{1+|\gamma_2'(t)|^2}} < 0, \qquad \nu_t = \frac{\gamma_2'(t)}{\sqrt{1+|\gamma_2'(t)|^2}} < 0, \qquad 0 \le t \le T; \tag{2.17}$$

moreover,

$$(\nu_t^2 - \nu_x^2)|_{\gamma_{2,T}} < 0. \tag{2.18}$$

It follows from relations (2.16)–(2.18) that

$$\int\limits_{\gamma_{2,\tau}} \frac{1}{\nu_t}[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)]\,ds \ge 0. \tag{2.19}$$

By virtue of inequalities (2.2), (2.3), and (2.5),

$$\begin{aligned} \int\limits_{\Omega_\tau} g(x,\tau,u_n(x,\tau))\,dx + \int\limits_{\gamma_{1,\tau}} g(x,t,u_n(x,t))\nu_t\,ds - \int\limits_{D_\tau} g_t(x,t,u_n(x,t))\,dx\,dt \\ \ge -\int\limits_{\Omega_\tau}[M_1 + M_2|u_n(x,\tau)|^2]\,dx - \int\limits_{\gamma_{1,\tau}}[M_1 + M_2|u_n(x,t)|^2]\,ds \\ -\int\limits_{D_\tau}[M_3 + M_4|u_n(x,t)|^2]\,dx\,dt. \end{aligned} \tag{2.20}$$

Now, by taking into account inequalities (2.15), (2.19), and (2.20), from relation (2.13) we obtain

$$\begin{aligned} w_n(\tau) := \int\limits_{\Omega_\tau}(u_{nt}^2 + u_{nx}^2)\,dx \le 2M_1(\operatorname{mes}\Omega_T + \operatorname{mes}\gamma_{1,T}) + 2M_3\operatorname{mes}D_T \\ + 2M_2\int\limits_{\Omega_\tau}|u_n(x,\tau)|^2\,dx + 2M_2\int\limits_{\gamma_{1,\tau}}|u_n(x,t)|^2\,ds \\ + 2M_4\int\limits_{D_\tau}|u_n(x,t)|^2\,dx\,dt + 2\int\limits_{D_\tau}F_n u_{nt}\,dx\,dt. \end{aligned} \tag{2.21}$$

By virtue of conditions (1.5) and (1.6), one can readily see that

$$\operatorname{mes}\Omega_T \le T, \qquad \operatorname{mes}\gamma_{1,T} \le \sqrt{2}T, \qquad \operatorname{mes}D_T \le T^2. \tag{2.22}$$

Since $\Omega_\tau:\ \gamma_2(\tau) \le x \le \gamma_1(\tau),\ t = \tau$, and $\gamma_{2,T}$: $t = \gamma_2^{-1}(x),\ \gamma_2(T) \le x \le 0$, where γ_2^{-1} is the function inverse to γ_2, which is uniquely determined by virtue of condition (1.6), we can use relations (2.10) and the Newton–Leibniz formulas to obtain

$$u_n(x,\tau) = \int\limits_{\gamma_2^{-1}(x)}^{\tau} u_{nt}(x,t)\,dt, \qquad \gamma_2(\tau) \le x \le \gamma_1(\tau), \qquad (x,\tau) \in \Omega_\tau, \tag{2.23}$$

$$u_n(\gamma_1(t),t) = \int\limits_{\gamma_2(t)}^{\gamma_1(t)} u_{nx}(x,t)\,dx, \qquad 0 \le t \le \tau, \qquad (\gamma_1(t),t) \in \gamma_{1,\tau}. \tag{2.24}$$

This, together with the Schwarz inequality, implies that

$$|u_n(x,\tau)|^2 \le \int\limits_{\gamma_2^{-1}(x)}^{\tau} 1^2\,dt \int\limits_{\gamma_2^{-1}(x)}^{\tau} |u_{nt}(x,t)|^2\,dt \le T \int\limits_{\gamma_2^{-1}(x)}^{\tau} |u_{nt}(x,t)|^2\,dt, \qquad (x,\tau)\in\Omega_\tau, \quad (2.25)$$

$$|u_n(\gamma_1(t),t)|^2 \le \int\limits_{\gamma_2(t)}^{\gamma_1(t)} 1^2\,dx \int\limits_{\gamma_2(t)}^{\gamma_1(t)} |u_{nx}(x,t)|^2\,dx$$

$$\le T \int\limits_{\gamma_2(t)}^{\gamma_1(t)} |u_{nx}(x,t)|^2\,dx, \qquad 0\le t\le\tau, \qquad (\gamma_1(t),t)\in\gamma_{1,\tau}. \quad (2.26)$$

By integrating both sides of inequality (2.25) with respect to x on the closed interval $[\gamma_2(\tau),\gamma_1(\tau)]$, we obtain

$$\int\limits_{\Omega_\tau} |u_n(x,\tau)|^2\,dx \le T \int\limits_{\gamma_2(\tau)}^{\gamma_1(\tau)} \Bigg[\int\limits_{\gamma_2^{-1}(x)}^{\tau} |u_{nt}(x,t)|^2\,dt\Bigg]dx = T \int\limits_{D_\tau\cap\{x<\gamma_1(\tau)\}} |u_{nt}(x,t)|^2\,dx\,dt$$

$$\le T \int\limits_{D_\tau} |u_{nt}(x,t)|^2\,dx\,dt. \quad (2.27)$$

In a similar way, since $|\gamma_1'(t)|<1$, $t\ge 0$, it follows that, by integrating both sides of inequality (2.26) with respect to t over the interval $[0,\tau]$, we obtain

$$\int\limits_{\gamma_{1,\tau}} |u_n(x,t)|^2\,ds = \int\limits_0^{\tau} |u_n(\gamma_1(t),t)|^2 \sqrt{1+|\gamma_1'(t)|^2}\,dt \le \sqrt{2}\int\limits_0^{\tau} |u_n(\gamma_1(t),t)|^2\,dt$$

$$\le \sqrt{2}T \int\limits_0^{\tau} \Bigg[\int\limits_{\gamma_2(t)}^{\gamma_1(t)} |u_{nx}(x,t)|^2\,dx\Bigg]dt = \sqrt{2}T\int\limits_{D_\tau} |u_{nx}(x,t)|^2\,dx\,dt. \quad (2.28)$$

From inequality (2.27), we have

$$\int\limits_{D_\tau} u_n^2\,dx\,dt = \int\limits_0^{\tau}\Bigg[\int\limits_{\Omega_\sigma} u_n^2\,dx\Bigg]d\sigma \le T\int\limits_0^{\tau}\Bigg[\int\limits_{D_\sigma} u_{nt}^2\,dx\,dt\Bigg]d\sigma \le T^2\int\limits_{D_\tau} u_{nt}^2\,dx\,dt. \quad (2.29)$$

The relations $2F_nu_{nt}\le F_n^2+u_{nt}^2$, (2.22), and (2.27)–(2.29), together with inequality (2.21), imply the estimate

$$w_n(\tau) \le M_5 + M_6\int\limits_{D_\tau}(u_{nt}^2+u_{nx}^2)\,dx\,dt + \int\limits_{D_\tau} F_n^2\,dx\,dt. \quad (2.30)$$

Here

$$M_5 := 2(1+\sqrt{2})TM_1 + 2T^2M_3, \qquad M_6 := 2(1+\sqrt{2})M_2T + 2M_4T^2 + 1. \quad (2.31)$$

Since

$$\int\limits_{D_\tau}(u_{nt}^2+u_{nx}^2)\,dx\,dt = \int\limits_0^{\tau} w_n(\sigma)\,d\sigma,$$

it follows from inequality (2.30) that

$$w_n(\tau) \le M_6 \int\limits_0^\tau w_n(\sigma)\, d\sigma + M_5 + T^2 \|F_n\|^2_{C(\overline{D}_T)}, \qquad 0 < \tau \le T. \tag{2.32}$$

This, together with the Gronwall lemma, implies that

$$w_n(\tau) \le (M_5 + T^2 \|F_n\|^2_{C(\overline{D}_T)}) \exp(M_6 \tau), \qquad 0 < \tau \le T. \tag{2.33}$$

Next, by virtue of condition (2.10), for any $(x,t) \in \overline{D}_T \backslash O$ we have

$$u_n(x,t) = u_n(x,t) - u_n(\gamma_2(t), t) = \int\limits_{\gamma_2(t)}^x u_{nx}(\xi, t)\, d\xi,$$

whence, by analogy with the derivation of inequality (2.26), one can show that

$$|u_n(x,t)|^2 \le T \int\limits_{\gamma_2(t)}^x |u_{nx}(\xi,t)|^2\, d\xi, \qquad (x,t) \in \overline{D}_T \backslash O. \tag{2.34}$$

Inequality (2.34), together with the estimate (2.33) and the definition of the quantity w_n as the left-hand side in relation (2.21), implies that

$$|u_n(x,t)|^2 \le T \int\limits_{\Omega_t} u^2_{nx}\, dx \le T w_n(t) \le T(M_5 + T^2 \|F_n\|^2_{C(\overline{D}_T)}) \exp(M_6 t), \qquad (x,t) \in \overline{D}_T \backslash O. \tag{2.35}$$

By taking into account the estimate (2.35) and by using the obvious inequality

$$\left(\sum_{i=1}^m a_i^2 \right)^{1/2} \le \sum_{i=1}^m |a_i|,$$

we obtain

$$\|u_n\|_{C(\overline{D}_T)} \le c_1 \|F_n\|_{C(\overline{D}_T)} + c_2, \tag{2.36}$$

where

$$c_1 = T^{3/2} \exp(2^{-1} M_6 T), \qquad c_2 = (T M_5)^{1/2} \exp(2^{-1} M_6 T), \tag{2.37}$$

and M_5 and M_6 are the constants defined in (2.31). By taking into account relations (2.7) and (2.11) and by passing in inequality (2.36) to the limit as $n \to \infty$, we obtain the a priori estimate (2.6). The proof of Lemma 2.1 is complete.

Remark 2.3. In the linear case in which the function f occurring in Eq. (1.1) vanishes, one can introduce the notion of a strong generalized solution of problem (1.1)–(1.3) in a similar way. In this case, by virtue of relation (2.1), the function g vanishes and satisfies conditions (2.2) and (2.3) for $M_i = 0$, $1 \le i \le 4$; moreover, under conditions (1.5), (1.6), and (2.4), the a priori estimate (2.6) is valid and, by virtue of relations (2.31) and (2.37), acquires the form

$$\|u\|_{C(\overline{D}_T)} \le T^{3/2} \exp(2^{-1} T) \|F\|_{C(\overline{D}_T)}. \tag{2.38}$$

3. CASES OF GLOBAL SOLVABILITY OF PROBLEM (1.1)–(1.3) IN THE CLASS C

In the new independent variables $\xi = t + x$, $\eta = t - x$, the domain D_T becomes a curvilinear triangular domain G_T with vertices at the points $O(0,0)$, $Q_1(T + \gamma_1(T), T - \gamma_1(T))$, and $Q_2(T + \gamma_2(T), T - \gamma_2(T))$ of the plane of the variables ξ and η, and problem (1.1)–(1.3) becomes the problem

$$\widetilde{L}\widetilde{u} := \widetilde{u}_{\xi\eta} + \widetilde{f}(\xi, \eta, \widetilde{u}) = \widetilde{F}(\xi, \eta), \qquad (\xi, \eta) \in G_T, \tag{3.1}$$

$$(m_1\widetilde{u}_\xi + m_2\widetilde{u}_\eta)|_{\widetilde{\gamma}_{1,T}} = 0, \tag{3.2}$$

$$\widetilde{u}|_{\widetilde{\gamma}_{2,T}} = 0 \tag{3.3}$$

for the unknown function

$$\widetilde{u}(\xi, \eta) := u\left(\frac{\xi - \eta}{2}, \frac{\xi + \eta}{2}\right).$$

Here

$$\widetilde{f}(\xi, \eta, \widetilde{u}) := \frac{1}{4} f\left(\frac{\xi - \eta}{2}, \frac{\xi + \eta}{2}, \widetilde{u}\right), \qquad \widetilde{F}(\xi, \eta) := \frac{1}{4} F\left(\frac{\xi - \eta}{2}, \frac{\xi + \eta}{2}\right),$$

$$m_1 := l_2 + l_1, \quad m_2 := l_2 - l_1 \quad \text{on} \quad \widetilde{\gamma}_{1,T}, \tag{3.4}$$

and $\widetilde{\gamma}_{1,T}$ and $\widetilde{\gamma}_{2,T}$ are the images of the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ under that transformation issuing from the common point $O(0,0)$ with terminal points Q_1 and Q_2.

By analogy with Definition 1.1, one can introduce the notion of strong generalized solution $\widetilde{u}$ of problem (3.1)–(3.3) in the class C in the domain G_T.

By virtue of conditions (1.5) and (1.6), the smooth noncharacteristic curves $\widetilde{\gamma}_{1,T}$ and $\widetilde{\gamma}_{2,T}$ can be represented in the form

$$\widetilde{\gamma}_{1,T}: \ \eta = \lambda_1(\xi), \qquad 0 \le \xi \le \xi_0; \qquad \widetilde{\gamma}_{2,T}: \ \xi = \lambda_2(\eta), \qquad 0 \le \eta \le \eta_0, \tag{3.5}$$

where $\xi_0 := T + \gamma_1(T) < \eta_0 := T - \gamma_2(T)$ and

$$\lambda_1'(\xi) > 0, \qquad 0 \le \xi \le \xi_0; \qquad \lambda_2'(\eta) > 0, \qquad 0 \le \lambda_2(\eta) \le \eta, \qquad 0 \le \eta \le \eta_0; \tag{3.6}$$

$$\lambda_2(\lambda_1(\xi)) < \xi, \quad 0 < \xi \le \xi_0; \qquad \lambda_1(\lambda_2(\eta)) < \eta, \quad 0 < \eta \le \eta_0; \tag{3.7}$$

$$G_T := \{(\xi, \eta) \in (0, \xi_0) \times (0, \eta_0) : \ \lambda_1(\xi) < \eta, \ \lambda_2(\eta) < \xi, \ \xi + \eta < 2T\}. \tag{3.8}$$

Remark 3.1. Obviously, $u = u(x,t)$ is a strong generalized solution of problem (1.1)–(1.3) in the class C in the domain D_T if and only if $\widetilde{u}$ is a strong generalized solution of problem (3.1)–(3.3) in the class C in the domain G_T; moreover, under the assumptions of Lemma 2.1 this solution $\widetilde{u}$ satisfies an a priori estimate of the type (2.6),

$$\|\widetilde{u}\|_{C(\overline{G}_T)} = \|u\|_{C(\overline{D}_T)} \le c_1\|F\|_{C(\overline{D}_T)} + c_2 \le 4c_1\|\widetilde{F}\|_{C(\overline{G}_T)} + c_2 \tag{3.9}$$

with the same constants c_1 and c_2.

Further, we first consider the linear case of problem (3.1)–(3.3) for which the function $\widetilde{f}$ occurring in Eq. (3.1) vanishes,

$$\widetilde{\Box}\widetilde{w} := \widetilde{w}_{\xi\eta} = \widetilde{F}(\xi, \eta), \qquad (\xi, \eta) \in G_T, \tag{3.10}$$

$$(m_1\widetilde{w}_\xi + m_2\widetilde{w}_\eta)|_{\widetilde{\gamma}_{1,T}} = 0, \tag{3.11}$$

$$\widetilde{w}|_{\widetilde{\gamma}_{2,T}} = 0. \tag{3.12}$$

Remark 3.2. By Remarks 2.3 and 3.1, a strong generalized solution $\widetilde{w}$ of the linear problem (3.10)–(3.12) in the class C in the domain G_T satisfies the estimate

$$\|\widetilde{w}\|_{C(\overline{G}_T)} \le 4T^{3/2}\exp(2^{-1}T)\|\widetilde{F}\|_{C(\overline{G}_T)}. \tag{3.13}$$

In particular, the estimate (3.13) holds for a classical solution $\widetilde{w} \in C^2(\overline{G}_T)$ of that problem. The estimate (3.13) implies the uniqueness of both generalized and classical solutions of that problem.

Remark 3.3. It follows from the condition $(|l_1| + |l_2|)|_{\gamma_1} \neq 0$ that, by virtue of relations (3.4), at each point $P \in \widetilde{\gamma}_{1,T}$ at least one of the numbers $m_1(P)$ and $m_2(P)$ is nonzero. In what follows, we assume that $m_1|_{\gamma_1} \neq 0$; i.e.,

$$(l_2 + l_1)(P) \neq 0, \qquad P \in \gamma_{1,T}. \tag{3.14}$$

Condition (3.14) implies that the direction (l_1, l_2) is not a characteristic direction corresponding to the family of characteristics $x + t = \text{const}$ of Eq. (1.1).

Set

$$a(\xi) := \frac{m_2(\xi)}{m_1(\xi)} \lambda_2'(\lambda_1(\xi)), \qquad 0 \leq \xi \leq \xi_0, \tag{3.15}$$

and consider the equation

$$|a(0)| = \left| \frac{m_2(0)}{m_1(0)} \lambda_2'(0) \right| < 1. \tag{3.16}$$

Lemma 3.1. *Let conditions* (2.4) *and* (3.14) *be satisfied at the point* $P = O(0,0)$. *If either* $(l_1 l_2)(O) \neq 0$ *or* $(l_1 l_2)(O) = 0$ *but the curves* $\gamma_{1,T}$ *and* $\gamma_{2,T}$ *are not tangent to each other at the point* O *or are tangent but* $\gamma_2'(0) < 0$, *then condition* (3.16) *is also satisfied.*

Proof. By virtue of condition (3.6), we have

$$0 < \lambda_2'(0) \leq 1. \tag{3.17}$$

If $(l_1 l_2)(O) > 0$, then, obviously, $\left| \left(\dfrac{l_2 - l_1}{l_2 + l_1} \right)(O) \right| < 1$; therefore, by virtue of relations (3.4) and (3.17), inequality (3.16) is satisfied.

It follows from inequalities (2.4) and (2.5) at the point $P = O(0,0)$ that

$$\gamma_1'(0) \leq \frac{2 l_1 l_2}{l_1^2 + l_2^2}(O). \tag{3.18}$$

By virtue of the representations (3.5), one can readily show that

$$\lambda_1'(0) = \frac{1 - \gamma_1'(0)}{1 + \gamma_1'(0)}, \qquad \lambda_2'(0) = \frac{1 + \gamma_2'(0)}{1 - \gamma_2'(0)}. \tag{3.19}$$

Next, by virtue of conditions (3.6) and (3.7), we have $0 < \lambda_1'(0)\lambda_2'(0) \leq 1$, because $[\lambda_2(\lambda_1(\xi))]'(0) = [\lambda_1(\lambda_2(\eta))]'(0) = \lambda_1'(0)\lambda_2'(0)$. Therefore,

$$\lambda_2'(0) \leq \frac{1}{\lambda_1'(0)}. \tag{3.20}$$

For $(l_1 l_2)(O) < 0$, one can readily see that

$$\left| \left(\frac{l_2 - l_1}{l_2 + l_1} \right)(O) \right| > 1, \tag{3.21}$$

but nevertheless, as is shown below, inequality (3.16) remains valid.

Now, in view of the fact that $\mu(s) := (1+s)/(1-s)$, $s \in \mathbb{R}$, is an increasing function and by taking into account relations (3.4) and (3.18)–(3.21), we obtain the estimate

$$|a(0)| = \left|\left(\frac{l_2 - l_1}{l_2 + l_1}\right)(O)\right| \lambda_2'(0) \le \left|\left(\frac{l_2 - l_1}{l_2 + l_1}\right)(O)\right| \frac{1 + \gamma_1'(0)}{1 - \gamma_1'(0)}$$
$$\le \left|\left(\frac{l_2 - l_1}{l_2 + l_1}\right)(O)\right| \frac{1 + \frac{2l_1 l_2}{l_1^2 + l_2^2}(O)}{1 - \frac{2l_1 l_2}{l_1^2 + l_2^2}(O)} = \left|\left(\frac{l_2 + l_1}{l_2 - l_1}\right)(O)\right| < 1.$$

It remains to consider the case in which $(l_1 l_2)(O) = 0$. By virtue of inequalities (1.6), we have $\gamma_2'(0) \le \gamma_1'(0) \le 0$. Therefore, if the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ are not tangent at the point $O(0,0)$, then $\gamma_2'(0) \ne \gamma_1'(0) \le 0$ and hence $\gamma_2'(0) < 0$. This, together with relations (3.19), implies that

$$\lambda_2'(0) < 1. \tag{3.22}$$

Since the relation $(l_1 l_2)(O) = 0$ implies that $\left|\left(\frac{l_2 - l_1}{l_2 + l_1}\right)(O)\right| = 1$, it follows from the estimate (3.22) that

$$|a(0)| = \left|\left(\frac{l_2 - l_1}{l_2 + l_1}\right)(O)\right| \lambda_2'(0) = \lambda_2'(0) < 1.$$

In a similar way, one can consider the case in which $(l_1 l_2)(O) = 0$, the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ are tangent at the point O, but $\gamma_2'(0) < 0$. The proof of Lemma 3.1 is complete.

Remark 3.4. One can readily see that if $(l_1 l_2)(O) = 0$, then the relation $|a(0)| = 1$ holds if and only if $\gamma_2'(0) = 0$; in this case, by virtue of conditions (1.6), the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ are tangent at the common point O.

Let $G_{0,T} := \{(\xi, \eta) \in \mathbb{R}^2 : \; 0 < \xi < \xi_0, \; 0 < \eta < \eta_0\}$ be the characteristic rectangle in the plane of the variables ξ and η corresponding to Eq. (3.10). By virtue of (3.8), we have $G_T \subset G_{0,T}$. If $\widetilde{F}$ belongs to $C(\overline{G}_T)$, then we extend that function as a continuous function into the closed domain $\overline{G}_{0,T}$ and keep the previous notation for it by setting, for example, $\widetilde{F}(\xi, \eta) = \widetilde{F}(\xi, \lambda_1(\xi))$ for $0 \le \eta \le \lambda_1(\xi)$, $0 \le \xi \le \xi_0$, $\widetilde{F}(\xi, \eta) = \widetilde{F}(\lambda_2(\eta), \eta)$ for $0 \le \xi \le \lambda_2(\eta)$, $0 \le \eta \le \eta_0$, and $\widetilde{F}(\xi, \eta) = \widetilde{F}(2T - \eta, \eta)$ for $(\xi, \eta) \in G_{0,T} \cap \{\xi + \eta \ge 2T\}$. Since the space $C^1(\overline{G}_{0,T})$ is dense in the space $C(\overline{G}_{0,T})$ [15, p. 37 of the Russian translation], it follows that there exists a function sequence $\widetilde{F}_n$ such that

$$\widetilde{F}_n \in C^1(\overline{G}_{0,T}), \qquad \lim_{n \to \infty} \|\widetilde{F}_n - \widetilde{F}\|_{C(\overline{G}_{0,T})} = 0. \tag{3.23}$$

We introduce the function $\widetilde{u}_n \in C^2(\overline{G}_{0,T})$ that is the solution of the Goursat problem

$$\widetilde{\Box} \widetilde{u}_n = \widetilde{F}_n(\xi, \eta), \qquad (\xi, \eta) \in G_{0,T},$$
$$\widetilde{u}_n(\xi, 0) = \varphi_n(\xi), \qquad 0 \le \xi \le \xi_0; \qquad \widetilde{u}_n(0, \eta) = \psi_n(\eta), \qquad 0 \le \eta \le \eta_0,$$

where $\varphi_n \in C^2([0, \xi_0])$ and $\psi_n \in C^2([0, \eta_0])$ are some functions satisfying the matching condition

$$\varphi_n(0) = \psi_n(0) = 0. \tag{3.24}$$

It is well known that the unique solution of this problem can be represented in the form [16, p. 246]

$$\widetilde{u}_n(\xi, \eta) = \varphi_n(\xi) + \psi_n(\eta) + \int\limits_0^{\xi} d\xi' \int\limits_0^{\eta} \widetilde{F}_n(\xi', \eta')\, d\eta', \qquad (\xi, \eta) \in \overline{G}_{0,T}. \tag{3.25}$$

By assuming that

$$\gamma_i \in C^i([0, T]), \qquad l_i \in C^1(\gamma_{1,T}), \qquad i = 1, 2, \tag{3.26}$$

we readily obtain

$$\lambda_1 \in C^1([0,\xi_0]), \qquad \lambda_2 \in C^2([0,\eta_0]), \qquad m_i \in C^1(\widetilde{\gamma}_{1,T}), \qquad i=1,2. \tag{3.27}$$

Now we construct functions $\varphi_n \in C^2([0,\xi_0])$ and $\psi_n \in C^2([0,\eta_0])$ such that the function $\widetilde{w}=\widetilde{u}_n$ defined by relation (3.25) satisfies the boundary conditions (3.11) and (3.12). By differentiating relation (3.12) in the direction of the tangent to $\widetilde{\gamma}_{2,T}$ with regard of (3.5), we obtain

$$\lambda_2'(\eta)\widetilde{u}_{n\xi}(\lambda_2(\eta),\eta)+\widetilde{u}_{n\eta}(\lambda_2(\eta),\eta)=0, \qquad 0\le\eta\le\eta_0. \tag{3.28}$$

Obviously, relation (3.28), together with the condition $\widetilde{u}_n(0,0)=0$, is equivalent to condition (3.12). By substituting the expression for $\widetilde{u}_n$ in (3.25) into relations (3.11) and (3.28) and by using the representations (3.5), for the functions φ_n' and ψ_n' we obtain the system of functional equations

$$m_1(\xi)\varphi_n'(\xi)+m_2(\xi)\psi_n'(\lambda_1(\xi))=\omega_{1n}(\xi), \qquad 0\le\xi\le\xi_0, \tag{3.29}$$

$$\lambda_2'(\eta)\varphi_n'(\lambda_2(\eta))+\psi_n'(\eta)=\omega_{2n}(\eta), \qquad 0\le\eta\le\eta_0. \tag{3.30}$$

Here

$$\omega_{1n}(\xi):=-m_1(\xi)\int\limits_0^{\lambda_1(\xi)}\widetilde{F}_n(\xi,\eta')\,d\eta'-m_2(\xi)\int\limits_0^{\xi}\widetilde{F}_n(\xi',\lambda_1(\xi))\,d\xi', \qquad 0\le\xi\le\xi_0, \tag{3.31}$$

$$\omega_{2n}(\eta):=-\lambda_2'(\eta)\int\limits_0^{\eta}\widetilde{F}_n(\lambda_2(\eta),\eta')\,d\eta'-\int\limits_0^{\lambda_2(\eta)}\widetilde{F}_n(\xi',\eta)\,d\xi', \qquad 0\le\eta\le\eta_0. \tag{3.32}$$

If condition (3.14) is satisfied, which is equivalent to the condition $m_1|_{\widetilde{\gamma}_{1,T}}\ne 0$, then, by eliminating the function ψ_n' from the system of equations (3.29) and (3.30), for $\varphi_{0n}:=\varphi_n'$, we obtain the functional equation

$$\varphi_{0n}(\xi)-a(\xi)\varphi_{0n}(\lambda_2(\lambda_1(\xi)))=\omega_n(\xi), \qquad 0\le\xi\le\xi_0. \tag{3.33}$$

Here $a(\xi)$, $0\le\xi\le\xi_0$, is the function defined by relation (3.15), and

$$\omega_n(\xi):=\frac{1}{m_1(\xi)}[\omega_{1n}(\xi)-m_2(\xi)\omega_{2n}(\lambda_1(\xi))], \qquad 0\le\xi\le\xi_0. \tag{3.34}$$

By setting

$$\tau(\xi):=\lambda_2(\lambda_1(\xi)), \qquad 0\le\xi\le\xi_0, \tag{3.35}$$

and by taking into account relations (3.7) and (3.27), we obtain

$$\tau\in C^1([0,\xi_0]), \quad \tau(0)=0, \quad \tau(\xi)<\xi \quad \text{if} \quad 0<\xi\le\xi_0. \tag{3.36}$$

Since $a\in C([0,\xi_0])$, it follows that, under condition (3.16), there exists a positive number ε such that

$$|a(\xi)|\le q:=\text{ const }<1 \quad \text{if} \quad 0\le\xi\le\varepsilon. \tag{3.37}$$

From relations (3.36), we find that if $\tau_k(\xi):=\tau(\tau_{k-1}(\xi))$ and $\tau_0(\xi):=\xi$, $0\le\xi\le\xi_0$, then the function sequence $\{\tau_k(\xi)\}_{k=1}^{\infty}$ converges uniformly to zero on the interval $[0,\xi_0]$; i.e., there exists a positive integer $n_0=n_0(\varepsilon)$ such that

$$\tau_k(\xi)\le\varepsilon, \qquad 0\le\xi\le\xi_0, \qquad k\ge n_0. \tag{3.38}$$

By $\Lambda:\ C([0,\xi_0])\to C([0,\xi_0])$ we denote the linear continuous operator acting by the rule

$$(\Lambda\omega_n)(\xi):=a(\xi)\omega_n(\tau(\xi)), \qquad 0\le\xi\le\xi_0. \tag{3.39}$$

Obviously,

$$(\Lambda^k\omega_n)(\xi) = a(\xi)a(\tau(\xi))\cdots a(\tau_{k-1}(\xi))\omega_n(\tau_k(\xi)), \qquad k \ge 2, \tag{3.40}$$

and for $k = 1$ and $k = 0$, we set

$$\Lambda^1 = \Lambda \quad \text{and} \quad \Lambda^0 = I, \tag{3.41}$$

where I is the identity operator.

By virtue of relations (3.36)–(3.41), we have the estimate

$$\begin{aligned}|(\Lambda^k\omega_n)(\xi)| &\le [a(\xi)a(\tau(\xi))\cdots a(\tau_{n_0-1}(\xi))][a(\tau_{n_0}(\xi))\cdots a(\tau_{k-1}(\xi))]\omega_n(\tau_k(\xi)) \\ &\le \|a\|^{n_0}_{C([0,\xi])} q^{k-n_0}\|\omega_n\|_{C([0,\xi])}, \qquad 0 \le \xi \le \xi_0, \qquad k > n_0,\end{aligned}$$

whence we obtain

$$\|\Lambda^k\|_{C([0,\xi_0])\to C([0,\xi_0])} \le M_0 q^k, \qquad k > n_0, \tag{3.42}$$

where

$$M_0 := (q^{-1}\|a\|_{C([0,\xi_0])})^{n_0}.$$

It follows from inequality (3.42), where $q < 1$, that if condition (3.16) is satisfied, then the Neumann series

$$(I - \Lambda)^{-1} = \sum_{k=0}^{\infty} \Lambda^k$$

of the operator Λ is convergent in the space $C([0, \xi_0])$, and by (3.35), the unique solution $\varphi_{0n} \in C([0, \xi_0])$ of Eq. (33) can be represented in the form

$$\varphi_{0n}(\xi) = \left[\sum_{k=0}^{\infty} \Lambda^k\omega_n\right](\xi), \qquad 0 \le \xi \le \xi_0. \tag{3.43}$$

Remark 3.5. Note that, by virtue of Remark 3.4, if, in the case $(l_1 l_2)(O) = 0$, we have $\gamma_2'(0) = 0$, which is equivalent to the condition $\lambda_2'(0) = 1$, then the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ are tangent at the point O; moreover, $|a(0)| = 1$, $\tau'(0) = \lambda_2'(0)\lambda_1'(0) = 1$, and Eq. (3.33) is not solvable in the class $C([0, \xi_0])$ for any right-hand side $\omega_n \in C([0, \xi_0])$. In this case, a necessary and sufficient condition for the solvability of Eq. (3.33) in the class $C([0, \xi_0])$ is given by the uniform convergence of the series on the right-hand side in relation (3.43) on the interval $[0, \xi_0]$, which is not necessarily true for any function $\omega_n \in C([0, \xi_0])$.

Remark 3.6. One can readily see that if we additionally require that the functions a, τ, and ω_n belong to $C^1([0, \xi_0])$, then the solution φ_{0n} of Eq. (3.33), which can be represented as the convergent series (3.43) in $C([0, \xi_0])$, belongs to the space $C^1([0, \xi_0])$ as well; moreover, its derivative $\chi_n := \varphi_{0n}'$ can be found from the functional equation

$$\chi_n(\xi) - a_1(\xi)\chi_n(\tau(\xi)) = \widetilde{\omega}_{1n}(\xi), \qquad 0 \le \xi \le \xi_0, \tag{3.44}$$

where $a_1(\xi) := a(\xi)\tau'(\xi)$ and $\widetilde{\omega}_{1n}(\xi) := \omega_n'(\xi) + a'(\xi)\varphi_{0n}(\tau(\xi))$, $0 \le \xi \le \xi_0$, and since $|\tau'(0)| \le 1$ by virtue of relations (3.36), we have $|a_1(0)| < 1$ under condition (3.16); consequently, by analogy with (3.43), the solution χ_n of Eq. (3.44) can be represented in the form

$$\chi_n = \sum_{k=0}^{\infty} \Lambda_1^k \widetilde{\omega}_{1n}, \tag{3.45}$$

where $(\Lambda_1\widetilde{\omega}_{1n})(\xi) := a_1(\xi)\widetilde{\omega}_{1n}(\tau(\xi))$, $0 \le \xi \le \xi_0$. By setting

$$\widetilde{\varphi}_{0n}(\xi) := \int_0^{\xi} \chi_n(\xi')\,d\xi' + \varphi_{0n}(0), \qquad 0 \le \xi \le \xi_0, \tag{3.46}$$

and by integrating Eq. (3.44), we obtain

$$\widetilde{\varphi}_{0n}(\xi) - \varphi_{0n}(0) - \int\limits_0^\xi a(\xi')\,d\widetilde{\varphi}_{0n}(\tau(\xi')) = \int\limits_0^\xi a'(\xi')\varphi_{0n}(\tau(\xi'))\,d\xi' + \omega_n(\xi) - \omega_n(0), \qquad 0 \le \xi \le \xi_0.$$

By integrating the third term on the left-hand side in the last relation, we obtain

$$\widetilde{\varphi}_{0n}(\xi) - \varphi_{0n}(0) - a(\xi)\widetilde{\varphi}_{0n}(\tau(\xi)) + a(0)\widetilde{\varphi}_{0n}(\tau(0)) + \int\limits_0^\xi a'(\xi')\widetilde{\varphi}_{0n}(\tau(\xi'))\,d\xi'$$
$$= \int\limits_0^\xi a'(\xi')\varphi_{0n}(\tau(\xi'))\,d\xi' + \omega_n(\xi) - \omega_n(0), \qquad 0 \le \xi \le \xi_0.$$

By using relation (3.35), by subtracting relation (3.33) from the last relation, and by taking into account the equalities $\tau(0) = 0$ and $\widetilde{\varphi}_{0n}(0) = \varphi_{0n}(0)$ in view of relations (3.36) and (3.46), for $\psi_{0n} := \widetilde{\varphi}_{0n} - \varphi_{0n}$, we obtain the Volterra integro-functional equation

$$\psi_{0n}(\xi) - a(\xi)\psi_{0n}(\tau(\xi)) + \int\limits_0^\xi a'(\xi')\psi_{0n}(\tau(\xi'))\,d\xi' = 0, \qquad 0 \le \xi \le \xi_0.$$

By applying the standard successive approximation method [4] to that equation, we obtain $\psi_{0n} = 0$; i.e., $\widetilde{\varphi}_{0n} = \varphi_{0n}$, and therefore,

$$\varphi_{0n}(\xi) = \int\limits_0^\xi \chi_n(\xi')\,d\xi' + \varphi_{0n}(0), \qquad 0 \le \xi \le \xi_0,$$

taking into account the representation (3.46). Hence it follows that φ_{0n} belongs to $C^1([0,\xi_0])$. Since $\varphi_{0n} := \varphi_n'$, we have

$$\psi_n'(\eta) = \omega_{2n}(\eta) - \lambda_2'(\eta)\varphi_{0n}(\lambda_2(\eta)), \qquad 0 \le \eta \le \eta_0, \tag{3.47}$$

by virtue of relation (3.30); by relations (3.24), (3.27), and (3.32), we have

$$\varphi_n(\xi) = \int\limits_0^\xi \varphi_{0n}(\xi')\,d\xi' \in C^2([0,\xi_0]), \qquad \psi_n(\eta) = \int\limits_0^\eta \psi_n'(\eta')\,d\eta' \in C^2([0,\eta_0]). \tag{3.48}$$

Remark 3.7. By keeping the same notation for the restrictions of the functions $\widetilde{u}_n$ and $\widetilde{F}_n$ to the subdomain G_T of the domain $G_{0,T}$ and by taking into account their definition, we find that the function $\widetilde{u}_n \in C^2(\overline{G}_T)$ is a classical solution of the linear problem (3.10)–(3.12) for $\widetilde{F} = \widetilde{F}_n$; by Remark 3.2 and the estimate (3.13), the following inequality holds:

$$\|\widetilde{u}_n - \widetilde{u}_k\|_{C(\overline{G}_T)} \le 4T^{3/2}\exp(2^{-1}T)\|\widetilde{F}_n - \widetilde{F}_k\|_{C(\overline{G}_T)}.$$

This, together with relations (3.23), implies that the function sequence $\widetilde{u}_n \in C^2(\overline{G}_T)$ is a Cauchy sequence in the complete space $C(\overline{G}_T)$; therefore, the exists a function $\widetilde{w} \in C(\overline{G}_T)$ such that

$$\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{w}\|_{C(\overline{G}_T)} = 0. \tag{3.49}$$

By virtue of relations (3.23) and (3.49), the function $\widetilde{w}$ thus defined is a strong generalized solution of the linear problem (3.10)–(30.12) in the class C in the domain G_T, whose uniqueness follows from the estimate (3.13). We denote this solution $\widetilde{w}$ by $\widetilde{\square}^{-1}\widetilde{F}$; i.e.,

$$\widetilde{w} = \widetilde{\square}^{-1}\widetilde{F}, \tag{3.50}$$

where the linear operator $\widetilde{\square}^{-1}:\ C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous, and by (3.13), its norm satisfies the estimate

$$\|\widetilde{\square}^{-1}\|_{C(\overline{G}_T)\to C(\overline{G}_T)} \le 4T^{3/2}\exp(2^{-1}T). \tag{3.51}$$

Moreover, it follows from relations (3.31), (3.32), (3.34), (3.39)–(3.41), and (3.43)–(3.45) that the operator $\widetilde{\square}^{-1}$ occurring in relation (3.55) indeed maps any continuous function $\widetilde{F} \in C(\overline{G}_T)$ to a function $\widetilde{w} \in C^1(\overline{G}_T)$ and the linear operator

$$\widetilde{\square}^{-1}:\ C(\overline{G}_T) \to C^1(\overline{G}_T)$$

is also continuous. [For details on the smoothness of $\widetilde{w}$ in (3.50), see Section 4, the representation (4.10).] The above-performed argument implies that, for the validity of the representation (3.50), i.e., for the unique solvability of the linear problem (3.10)–(3.12) in the class C, it suffices to require that $f \in C(\overline{D}_T \times \mathbb{R})$, $F \in C(\overline{D}_T)$, conditions (1.5), (1.6), and (2.4) are satisfied at the point O, and relations (3.14) and (3.26) and assumptions of Lemma 3.1 are valid.

Remark 3.8. Since the space $C^1(\overline{G}_T)$ is compactly embedded in $C(\overline{G}_T)$ [17, p. 135 of the Russian translation], it follows from Remark 3.7 that the linear operator $\widetilde{\square}^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is compact, and its norm can be estimated as (3.51).

Remark 3.9. By virtue of Remarks 3.1 and 3.7 and relation (3.50), the function $u = u(x,t)$ is a strong generalized solution of problem (1.1)–(1.3) of the class C in the domain D_T if and only if $\widetilde{u}(\xi,\eta) := u\left(\dfrac{\xi-\eta}{2}, \dfrac{\xi+\eta}{2}\right)$ is a continuous solution of the functional equation

$$\widetilde{u} = K_0\widetilde{u} := \widetilde{\square}^{-1}(-\widetilde{f}(\xi,\eta,\widetilde{u}) + \widetilde{F}) \tag{3.52}$$

in the class $C(\overline{G}_T)$, where $K_0:\ C(\overline{G}_T) \to C(\overline{G}_T)$ is a continuous compact operator, because the nonlinear operator $N:\ C(\overline{G}_T) \to C(\overline{G}_T)$ acting by the rule $N\widetilde{u} = -\widetilde{f}(\xi,\eta,\widetilde{u}) + \widetilde{F}$, where $\widetilde{f} \in C(\overline{G}_T \times \mathbb{R})$ and $\widetilde{F} \in C(\overline{G}_T)$, is bounded and continuous, and the linear operator $\widetilde{\square}^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is compact by virtue of Remark 3.8. At the same time, by virtue of the estimate (3.9) and relations (2.37), the same a priori estimate (3.9) with the same constants c_1 and c_2 is valid for any parameter $\tau \in [0,1]$ and for any solution $\widetilde{u} \in C(\overline{G}_T)$ of the equation $\widetilde{u} = \tau K_0\widetilde{u}$. Therefore, by the Leray–Schauder theorem [18, p. 375], Eq. (3.52) has at least one solution $\widetilde{u} \in C(\overline{G}_T)$. Therefore, in view of Remarks 3.1 and 3.9, we have thereby proved the following assertion.

Theorem 3.1. *Let the conditions $f \in C(\overline{D}_T \times \mathbb{R})$ and $F \in C(\overline{D}_T)$ as well as conditions* (1.5), (1.6), (2.2)–(2.4), (3.14), *and* (3.26) *be satisfied*; *moreover*, *in the case of* $(l_1 l_2)(O) = 0$, *assume that the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ either are not tangent at the point O or are tangent but $\gamma_2'(0) < 0$. Then problem* (1.1)–(1.3) *has at least one strong generalized solution u of the class C in the domain D_T in the sense of Definition* 1.1.

Remark 3.10. One can readily see that if the assumptions of Theorem 3.1 are true for $T = \infty$, then problem (1.1)–(1.3) is globally solvable in the class C in the sense of Definition 1.2.

4. SMOOTHNESS OF SOLUTION OF PROBLEM (1.1)–(1.3)

Now let us study the smoothness of the strong generalized solution of the nonlinear problem (1.1)–(1.3) depending on the smoothness of the data of that problem. To this end, under the assumptions of Theorem 3.1 with regard of Remark 3.1, we trace the scheme of the construction of a strong generalized solution $\widetilde{w}$ of the linear problem (3.10)–(3.12) in the class C in the domain G_T and show that such a solution actually belongs to the class $C^1(\overline{G}_T)$, and the boundary conditions (3.11) and (3.12) are satisfied pointwise. Indeed, by virtue of relations (3.31), (3.32), and (3.34), the right-hand side ω_n of Eq. (33) can be represented in the form

$$\omega_n(\xi) = -\frac{1}{m_1(\xi)}\Bigg[m_1(\xi)\int\limits_0^{\lambda_1(\xi)} \widetilde{F}_n(\xi,\eta')\,d\eta' + m_2(\xi)\int\limits_0^{\xi} \widetilde{F}_n(\xi',\lambda_1(\xi))\,d\xi'$$
$$- m_2(\xi)\lambda_2'(\lambda_1(\xi))\int\limits_0^{\lambda_1(\xi)} \widetilde{F}_n(\tau(\xi),\eta')\,d\eta' - m_2(\xi)\int\limits_0^{\tau(\xi)} \widetilde{F}_n(\xi',\lambda_1(\xi))\,d\xi'\Bigg], \quad 0 \le \xi \le \xi_0. \quad (4.1)$$

This, together with conditions (3.23), implies that

$$\lim_{n\to\infty} \|\omega_n - \omega\|_{C(\overline{G}_T)} = 0, \quad (4.2)$$

where

$$\omega(\xi) := -\frac{1}{m_1(\xi)}\Bigg[m_1(\xi)\int\limits_0^{\lambda_1(\xi)} \widetilde{F}(\xi,\eta')\,d\eta' + m_2(\xi)\int\limits_0^{\xi} \widetilde{F}(\xi',\lambda_1(\xi))\,d\xi'$$
$$- m_2(\xi)\lambda_2'(\lambda_1(\xi))\int\limits_0^{\lambda_1(\xi)} \widetilde{F}(\tau(\xi),\eta')\,d\eta' - m_2(\xi)\int\limits_0^{\tau(\xi)} \widetilde{F}(\xi',\lambda_1(\xi))\,d\xi'\Bigg], \quad 0 \le \xi \le \xi_0. \quad (4.3)$$

In turn, it follows from relations (3.39)–(3.43), (4.1)–(4.3) that

$$\lim_{n\to\infty} \|\varphi_{0n} - \varphi_0\|_{C([0,\xi_0])} = 0, \quad (4.4)$$

where $\varphi_{0n} := \varphi_n'$ and

$$\varphi_0 := \left[\sum_{k=0}^{\infty} \Lambda^k \omega\right] \in C([0,\xi_0]). \quad (4.5)$$

Since the derivative ψ_n' of the function ψ_n occurring in the representation (3.25) is defined by relation (3.47), it follows from (3.23), (3.32), and (4.4) that

$$\lim_{n\to\infty} \|\psi_n' - \psi_0\|_{C([0,\eta_0])} = 0, \quad (4.6)$$

where

$$\psi_0 \in C([0,\eta_0]), \qquad \psi_0(\eta) := \omega_2(\eta) - \lambda_2'(\eta)\varphi_0(\lambda_2(\eta)), \qquad 0 \le \eta \le \eta_0, \quad (4.7)$$

$$\omega_2(\eta) := -\lambda_2'(\eta)\int\limits_0^{\eta} \widetilde{F}(\lambda_2(\eta),\eta')\,d\eta' - \int\limits_0^{\lambda_2(\eta)} \widetilde{F}(\xi',\eta)\,d\xi', \qquad 0 \le \eta \le \eta_0. \quad (4.8)$$

Finally, by using Remark 3.7 and the limit relations (3.23), (3.49), (4.4), (4.6), and (3.48) in the notation

$$\varphi(\xi) := \int\limits_0^{\xi} \varphi_0(\xi')\,d\xi', \quad 0 \le \xi \le \xi_0, \quad \psi(\eta) := \int\limits_0^{\eta} \psi_0(\eta')\,d\eta', \quad 0 \le \eta \le \eta_0, \quad (4.9)$$

and by passing to the limit in relation (3.25), for the strong generalized solution $\widetilde{w}$ of the linear problem (3.10)–(3.12) in the class C in the domain G_T we obtain the representation

$$\widetilde{w}(\xi,\eta)=\varphi(\xi)+\psi(\eta)+\int\limits_0^{\xi} d\xi'\int\limits_0^{\eta}\widetilde{F}(\xi',\eta')\,d\eta',\qquad (\xi,\eta)\in\overline{G}_T. \tag{4.10}$$

If $\widetilde{F}$ belongs to $C(\overline{G}_T)$, then, by virtue of relations (4.5) and (4.7), it follows from the representation (4.10) that

$$w\in C^1(\overline{G}_T).$$

Next, by virtue of relations (4.2), (4.4) and (3.33), (3.35), the function φ_0 satisfies the functional equation

$$\varphi_0(\xi)-a(\xi)\varphi_0(\tau(\xi))=\omega(\xi),\qquad 0\le\xi\le\xi_0. \tag{4.11}$$

Remark 4.1. If the function $\widetilde{F}$ belongs to $C^1(\overline{G}_T)$ and the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ are not tangent at the point O, then, by [19, p. 595], one can extend that function in the rectangle $\overline{G}_{0,T}$ (keeping the same notation for it) so as to ensure that the function $\widetilde{F}$ belongs to $C^1(\overline{G}_{0,T})$. In the case of tangency of the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ at the point O, throughout the following we assume that such an extension is possible.

It follows from relation (4.2) that if condition (3.26) is satisfied and one additionally requires that the function $\widetilde{F}$ belongs to $C^1(\overline{G}_T)$, then the right-hand side ω of Eq. (4.11) belongs to the class $C^1([0,\xi_0])$. This, together with the argument carried out in Remark 3.6, implies that the solution of Eq. (4.11) belongs to the space $C^1([0,\xi_0])$; consequently, by (4.7) and (4.8), the function ψ_0 belongs to the space $C^1([0,\eta_0])$ as well. Therefore, under the above-stipulated assumptions with regard of notation (4.9), we find that the function $\widetilde{w}$ occurring in (4.10) belongs to the space $C^2(\overline{G}_T)$. Thus, in view of Remark 3.7, we have proved the following assertion.

Theorem 4.1. *If conditions* (1.5), (1.6), (2.4), (3.14), *and* (3.26) *are satisfied*, $\widetilde{F}\in C(\overline{G}_T)$, *and moreover*, *for* $(l_1l_2)(O)=0$ *the curves* $\gamma_{1,T}$ *and* $\gamma_{2,T}$ *either are not tangent at the point* O *or are tangent but* $\gamma_2'(0)<0$, *then the strong generalized solution* $\widetilde{w}$ *of the linear problem* (3.10)–(3.12) *in the class* C *in the domain* G_T *belongs to the space* $C^1(\overline{G}_T)$; *i.e.*, *by relation* (3.50), $\widetilde{w}=\widetilde{\Box}^{-1}\widetilde{F}$ *in the class* $C^1(\overline{G}_T)$; *and if it is additionally required that the function* $\widetilde{F}$ *belongs to* $C^1(\overline{G}_T)$, *then* $\widetilde{w}$ *belongs to* $C^2(\overline{G}_T)$; *in addition*, *the boundary conditions* (3.11) *and* (3.12) *are valid pointwise in both cases.*

The following assertion is a consequence of Remarks 3.1 and 3.9, relation (3.52), and Theorem 4.1.

Theorem 4.2. *If the assumptions of Theorem* 3.1 *are satisfied*, *then a strong generalized solution* u *of problem* (1.1)–(1.3) *in the class* C *in the domain* D_T *belongs to the space* $C^1(\overline{D}_T)$; *under the additional requirements* $f\in C^1(\overline{D}_T\times\mathbb{R})$ *and* $F\in C^1(\overline{D}_T)$, *this solution belongs to the space* $C^2(\overline{D}_T)$, *i.e.*, *is classical*; *moreover*, *in both cases the boundary conditions* (1.1) *and* (1.3) *are satisfied pointwise.*

5. UNIQUENESS THEOREM. EXISTENCE OF GLOBAL SOLUTION OF PROBLEM (1.1)–(1.3) IN THE DOMAIN D_∞

By definition, a function $f=f(x,t,s)$ satisfies the local Lipschitz condition with respect to the variable s on the set $\overline{D}_T\times\mathbb{R}$ if

$$|f(x,t,s_2)-f(x,t,s_1)|\le M(T,r)|s_2-s_1|,\quad (x,t)\in\overline{D}_T,\quad |s_i|\le r,\quad i=1,2, \tag{5.1}$$

where $M(T,r):=\text{const}\ge 0$.

Theorem 5.1. *Let condition* (2.4) *be satisfied, let the function* $f \in C(\overline{D}_T \times \mathbb{R})$ *satisfy condition* (5.1), *let* F *belong to* $C(\overline{D}_T)$, *and let* l_1 *and* l_2 *belong to the class* $C(\gamma_{1,T})$. *Then problem* (1.1)–(1.3) *has at most one strong generalized solution in the class* C *in the domain* D_T *in the sense of Definition* 1.1.

Proof. Indeed, assume that problem (1.1)–(1.3) has two possible distinct strong generalized solutions u^1 and u^2 in the class C in the domain D_T. Then, by Definition 1.1, there exist sequences of functions $u_n^i \in \mathring{C}^2(\overline{D}_T, \gamma_T)$, $i = 1, 2$, such that

$$\lim_{n\to\infty} \|u_n^i - u^i\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|Lu_n^i - F\|_{C(\overline{D}_T)} = 0, \qquad i = 1, 2. \tag{5.2}$$

Set $\omega_n := u_n^2 - u_n^1$. One can readily see that the function $\omega_n \in \mathring{C}^2(\overline{D}_T, \gamma_T)$ is a classical solution of the problem

$$\Box\omega_n + g_n = F_n, \qquad (l_1\omega_{nx} + l_2\omega_{nt})|_{\gamma_{1,T}} = 0, \qquad \omega_n|_{\gamma_{2,T}} = 0. \tag{5.3}$$

Here

$$g_n := f(x, t, u_n^2) - f(x, t, u_n^1), \qquad F_n := Lu_n^2 - Lu_n^1. \tag{5.4}$$

By virtue of relations (5.2), there exists a number $m :=$ const > 0 independent of the indices i and n such that $\|u_n^i\|_{C(\overline{D}_T)} \le m$, which, together with relations (5.1) and (5.2), implies that

$$|g_n| \le M(T, m)|\omega_n|. \tag{5.5}$$

By virtue of relations (5.2) and the second relation in (5.4), we have

$$\lim_{n\to\infty} \|F_n\|_{C(\overline{D}_T)} = 0. \tag{5.6}$$

By multiplying both sides of the first relation in (5.3) by ω_{nt}, by integrating the resulting relation over the domain

$$D_\tau := \{(x, t) \in D_T : \ t < \tau\}, \qquad 0 < \tau \le T,$$

and by following the derivation of relation (2.13) in (2.8)–(2.10), we obtain

$$\begin{aligned} w_n(\tau) := \int\limits_{\Omega_\tau} (\omega_{nt}^2 + \omega_{nx}^2)\, dx = -\int\limits_{\gamma_{1,\tau}} (\omega_{nt}^2\nu_t - 2\omega_{nx}\omega_{nt}\nu_x + \omega_{nx}^2\nu_t) ds \\ - \int\limits_{\gamma_{2,\tau}} \frac{1}{\nu_t}[(\omega_{nx}\nu_t - \omega_{nt}\nu_x)^2 + \omega_{nt}^2(\nu_t^2 - \nu_x^2)]\, ds + 2\int\limits_{D_\tau} (F_n - g_n)\omega_{nt}\, dx\, dt. \end{aligned} \tag{5.7}$$

By virtue of inequality (5.5) and the Cauchy inequality, we have the estimate

$$\begin{aligned} \left| 2\int\limits_{D_\tau} (F_n - g_n)\omega_{nt}\, dx\, dt \right| &\le \int\limits_{D_\tau} (F_n - g_n)^2\, dx\, dt + \int\limits_{D_\tau} \omega_{nt}^2\, dx\, dt \\ &\le 2\int\limits_{D_\tau} F_n^2\, dx\, dt + 2M^2(T, m)\int\limits_{D_\tau} \omega_n^2\, dx\, dt + \int\limits_{D_\tau} \omega_{nt}^2\, dx\, dt. \end{aligned} \tag{5.8}$$

Since inequalities (2.15) and (2.19), true for u_n, also hold for ω_n, it follows from relations (5.7) and (5.8) that

$$w_n(\tau) \le 2M^2(T, m)\int\limits_{D_\tau} \omega_n^2\, dx\, dt + \int\limits_{D_\tau} \omega_{nt}^2\, dx\, dt + 2\int\limits_{D_\tau} F_n^2\, dx\, dt. \tag{5.9}$$

Since inequality (2.29), true for u_n, also holds for ω_n, from the estimate (5.9), we have

$$w_n(\tau) \le (2M^2(T,m)T^2+1)\int\limits_{D_\tau}\omega_{nt}^2\,dx\,dt + 2\int\limits_{D_T}F_n^2\,dx\,dt$$
$$\le M_0\int\limits_{D_\tau}(\omega_{nt}^2+\omega_{nx}^2)\,dx\,dt + 2\int\limits_{D_T}F_n^2\,dx\,dt, \tag{5.10}$$

where $M_0 := 2M^2(T,m)T^2+1$.

By taking into account the relation

$$\int\limits_{D_\tau}(\omega_{nt}^2+\omega_{nx}^2)\,dx\,dt = \int\limits_0^\tau w_n(\sigma)\,d\sigma,$$

from inequality (5.10) we obtain

$$w_n(\tau) \le M_0\int\limits_0^\tau w_n(\sigma)\,d\sigma + 2\|F_n\|^2_{C(\overline{D}_T)}\operatorname{mes} D_T, \qquad 0<\tau\le T.$$

This, together with the Gronwall lemma, implies that

$$w_n(\tau) \le 2\|F_n\|^2_{C(\overline{D}_T)}(\operatorname{mes} D_T)\exp(M_0T), \qquad 0<\tau\le T. \tag{5.11}$$

Since inequality (2.34), true for u_n, also holds for ω_n, it follows from the estimate (5.11) and inequality (2.35) that

$$|\omega_n(x,t)|^2 \le Tw_n(t) \le 2T\|F_n\|^2_{C(\overline{D}_T)}(\operatorname{mes} D_T)\exp(M_0T), \qquad (x,t)\in\overline{D}_T\backslash O. \tag{5.12}$$

By using relations (5.2) and (5.6) and the relation $\omega_n := u_n^2 - u_n^1$ and by passing in inequality (5.12) to the limit as $n\to\infty$, we obtain $|(u^2-u^1)(x,t)|^2 \le 0$, $(x,t)\in\overline{D}_T\backslash O$; i.e., $u^2=u^1$, which contradicts the above assumption. The proof of Theorem 5.1 is complete.

Remark 5.1. Obviously, condition (5.1) is satisfied if $f\in C^1(\overline{D}_T\times\mathbb{R})$.

Theorems 3.1, 4.2, and 5.1 and Remark 5.1 imply the following assertion.

Theorem 5.2. *Let $f\in C^1(\overline{D}_T\times\mathbb{R})$ and $F\in C^1(\overline{D}_T)$, and let conditions* (1.5), (1.6), (2.2)–(2.4), (3.14), *and* (3.26) *be satisfied. Moreover, assume that in the case of $(l_1l_2)(O)=0$ either the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ are not tangent at the point O or $\gamma_2'(0)<0$. Then problem* (1.1)–(1.3) *has a unique classical solution $u\in C^2(\overline{D}_T)$ in the domain D_T.*

Corollary 5.1. *If the assumptions of Theorem* 5.2 *hold for $T=\infty$, then problem* (1.1)–(1.3) *has a unique global classical solution $u\in C^2(\overline{D}_\infty)$.*

Indeed, by Theorem 5.2, problem (1.1)–(1.3) for $T=n$ has a unique classical solution u_n in the domain D_n. Since u_{n+1} is a classical solution of that problem in the domain D_n as well, we have $u_{n+1}|_{D_n}=u_n$ by virtue of the uniqueness Theorem 5.1. Therefore, the function u constructed in the domain D_∞ by the rule $u(x,t)=u_n(x,t)$ for $n=[t]+1$, where $[t]$ is the integer part of the number t and $(x,t)\in\overline{D}_\infty$, is the unique global classical solution of problem (1.1)–(1.3) in the domain D_∞.

6. CASES OF THE ABSENCE OF GLOBAL SOLVABILITY OF PROBLEM (1.1)–(1.3) AND ITS LOCAL SOLVABILITY

In what follows, we show that if condition (2.2) fails, then problem (1.1)–(1.3) is not necessarily globally solvable in the class C in the sense of Definition 1.2. To this end, we use the method of test functions described in [20, pp. 10–14].

Lemma 6.1. *Let u be a strong generalized solution of problem* (1.1)–(1.3) *in the class C in the domain D_T in the sense of Definition* 1.1. *Then the integral relation*

$$\int\limits_{D_T} u\Box\varphi\,dx\,dt + \int\limits_{D_T} f(x,t,u)\varphi\,dx\,dt = \int\limits_{D_T} F\varphi\,dx\,dt \tag{6.1}$$

holds for any test function φ such that

$$\varphi\in C^2(\overline{D}_T),\qquad \varphi|_{\partial D_T}=0,\qquad \nabla\varphi|_{\partial D_T}=0, \tag{6.2}$$

where $\nabla := \left(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\right)$.

Proof. By the definition of a strong generalized solution u of problem (1.1)–(1.3) in the class C, in the domain D_T, we have $u\in C(\overline{D}_T)$, and there exists a function sequence $u_n\in \overset{\circ}{C}{}^2(\overline{D}_T,\gamma_T)$ such that the limit relations (2.7) hold.

Set $F_n := Lu_n$. We multiply both sides of the relation $Lu_n = F_n$ by the function φ and integrate the resulting relation over the domain D_T. By virtue of condition (6.2), after the integration by parts in the resulting integral relation, we obtain

$$\int\limits_{D_T} u_n\Box\varphi\,dx\,dt + \int\limits_{D_T} f(x,t,u_n)\varphi\,dx\,dt = \int\limits_{D_T} F_n\varphi\,dx\,dt. \tag{6.3}$$

By taking into account the limit relations (2.7) and by passing in relation (6.3) to the limit as $n\to\infty$, we obtain the desired relation (6.1). The proof of Lemma 6.1 is complete.

Consider the following condition imposed on the function f:

$$f(x,t,s)\le -\lambda|s|^{\alpha+1},\qquad (x,t,s)\in\overline{D}_\infty\times\mathbb{R};\qquad \lambda,\alpha:=\text{ const }>0. \tag{6.4}$$

One can readily see that condition (2.2) fails in case (6.4).

We introduce the function $\varphi^0 := \varphi^0(x,t)$ satisfying the conditions

$$\varphi^0\in C^2(\overline{D}_\infty),\quad \varphi^0|_{D_{T=1}}>0,\quad \varphi^0|_{\partial D_{T=1}}=0,\quad \nabla\varphi^0|_{\partial D_{T=1}}=0,\quad \varphi^0|_{t\ge 1}=0 \tag{6.5}$$

and

$$\kappa_0 := \int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx\,dt<\infty,\qquad p'=1+\frac{1}{\alpha}. \tag{6.6}$$

To simplify the exposition, we consider the case in which the curves γ_1 and γ_2 are rays; i.e.,

$$\gamma_i:\ x=-k_it,\qquad k_i:=\text{ const },\qquad i=1,2;\qquad 0<k_1<k_2<1. \tag{6.7}$$

One can readily see that, in the case of (6.7), for the function φ^0 satisfying conditions (6.5) and (6.6) one can take the function

$$\varphi^0(x,t)=\begin{cases}[(x+k_1t)(x+k_2t)(1-t)]^m & \text{for}\quad (x,t)\in D_{T=1},\\ 0 & \text{for}\quad t\ge 1\end{cases}$$

for a sufficiently large $m:=\text{ const }>0$.

By setting $\varphi_T(x,t):=\varphi^0\left(\dfrac{x}{T},\dfrac{t}{T}\right)$, $T>0$, and by using conditions (6.5), one can readily see that

$$\varphi_T\in C^2(\overline{D}_\infty),\quad \varphi_T|_{D_T}>0,\quad \varphi_T|_{\partial D_T}=0,\quad \nabla\varphi_T|_{\partial D_T}=0,\quad \varphi_T|_{t\ge T}=0. \tag{6.8}$$

By assuming that $F\in C(\overline{D}_\infty)$ is a fixed function, we introduce the following function of one variable T:

$$\zeta(T):=\int\limits_{D_T} F\varphi_T\,dx\,dt,\qquad T>0. \tag{6.9}$$

We have the following assertion on the absence of the global solvability of problem (1.1)–(1.3).

Theorem 6.1. *Let the function $f \in C(\overline{D}_\infty \times \mathbb{R})$ satisfy condition* (6.4), *let F belong to $C(\overline{D}_\infty)$, $F \geq 0$, in the domain D_∞, and let*

$$\liminf_{T\to\infty} \zeta(T) > 0. \tag{6.10}$$

Then there exists a positive number $T_0 = T_0(F)$ such that problem (1.1)–(1.3) *with $T > T_0$ cannot have a strong generalized solution in the class C in the domain D_T in the sense of Definition* 1.1.

Proof. Suppose that, under assumptions of the theorem, there exists a strong generalized solution u of problem (1.1)–(1.3) in the class C in the domain D_T. Then, by Lemma 6.1, relation (6.1) holds, where, by virtue of condition (6.8), for the test function φ one can take the function φ_T; i.e.,

$$-\int_{D_T} f(x,t,u)\varphi_T\,dx\,dt + \int_{D_T} F\varphi_T\,dx\,dt = \int_{D_T} u\Box\varphi_T\,dx\,dt. \tag{6.11}$$

Since the function φ_T is positive in the domain D_T, it follows from condition (6.4), notation (6.9), and relation (6.11) that

$$\lambda\int_{D_T} |u|^p\varphi_T\,dx\,dt \leq \int_{D_T} |u|\,|\Box\varphi_T|\,dx\,dt - \zeta(T), \qquad p := \alpha+1. \tag{6.12}$$

If in the Young inequality with parameter $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon}{p}\,a^p + \frac{1}{p'\varepsilon^{p'-1}}\,b^{p'}; \qquad a,b \geq 0, \quad \frac{1}{p}+\frac{1}{p'} = 1, \quad p > 1,$$

we take $a = |u|\varphi_T^{1/p}$ and $b = |\Box\varphi_T|/\varphi_T^{1/p}$, then, by virtue of the relation $p'/p = p'-1$, we obtain

$$|u\Box\varphi_T| = |u|\varphi_T^{1/p}\frac{|\Box\varphi_T|}{\varphi_T^{1/p}} \leq \frac{\varepsilon}{p}|u|^p\varphi_T + \frac{1}{p'\varepsilon^{p'-1}}\,\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}. \tag{6.13}$$

It follows from inequalities (6.12) and (6.13) that

$$\left(\lambda - \frac{\varepsilon}{p}\right)\int_{D_T} |u|^p\varphi_T\,dx\,dt \leq \frac{1}{p'\varepsilon^{p'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \zeta(T),$$

whence for $\varepsilon < \lambda p$ we obtain

$$\int_{D_T} |u|^p\varphi_T\,dx\,dt \leq \frac{p}{(\lambda p-\varepsilon)p'\varepsilon^{p'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p}{\lambda p-\varepsilon}\zeta(T). \tag{6.14}$$

By taking into account the relations $p' = \dfrac{p}{p-1}$, $p = \dfrac{p'}{p'-1}$, and the minimum

$$\min_{0<\varepsilon<\lambda p}\frac{p}{(\lambda p-\varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{\lambda^{p'}},$$

which is attained for $\varepsilon = \lambda$, we rewrite inequality (6.14) in the form

$$\int_{D_T} |u|^p\varphi_T\,dx\,dt \leq \frac{1}{\lambda^{p'}}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p'}{\lambda}\zeta(T). \tag{6.15}$$

Since $\varphi_T(x,t) := \varphi^0\left(\frac{x}{T}, \frac{t}{T}\right)$, it follows that, by taking into account relation (6.6) and by performing the change of variables $x = Tx_1$ and $t = Tt_1$, one can represent the integral on the right-hand side in inequality (6.15) in the form

$$\int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt = T^{-2(p'-1)} \int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx_1\,dt_1 = T^{-2(p'-1)}\kappa_0 < \infty. \tag{6.16}$$

By virtue of relations (6.8) and (6.16), it follows from inequality (6.15) that

$$0 \le \int\limits_{D_T} |u|^p \varphi_T\,dx\,dt \le \frac{1}{\lambda^{p'}}\,T^{-2(p'-1)}\kappa_0 - \frac{p'}{\lambda}\,\zeta(T). \tag{6.17}$$

Since $p' > 1$, we have $-2(p'-1) < 0$, and by virtue of (6.6),

$$\lim_{T\to\infty} \frac{1}{\lambda^{p'}}\,T^{-2(p'-1)}\kappa_0 = 0. \tag{6.18}$$

By virtue of relations (6.10) and (6.18), there exists a positive number $T_0 = T_0(F)$ such that for $T > T_0$ the right-hand side of inequality (6.17) is negative, while the left-hand side of this inequality is nonnegative. Hence it follows that if there exists a strong generalized solution of problem (1.1)–(1.3) in the class C in the domain D_T, then the inequality $T \le T_0$ is necessarily true. The proof of Theorem 6.1 is complete.

Remark 6.1. One can readily see that if the conditions $F \in C(\overline{D}_\infty)$, $F \ge 0$, and $F(x,t) \ge ct^{-m}$ are satisfied for $t \ge 1$, where $c :=$ const > 0 and $0 \le m :=$ const ≤ 2, then inequality (6.10) holds, and by Theorem 6.1 problem (1.1)–(1.3) does not have a strong generalized solution in the class C in the domain D_T for sufficiently large T in this case.

Corollary 5.2. *Under the assumptions of Theorem* 6.1, *problem* (1.1)–(1.3) *is not globally solvable in the class C in the sense of Definition* 1.2; *i.e., it cannot have a global strong generalized solution in the class C in the domain D_∞ in the sense of Definition* 1.3.

In what follows, we show that, although the global solvability of problem (1.1)–(1.3) has been proved under condition (2.2), the local solvability of this problem remains valid if that condition fails.

Theorem 6.2. *Let $f \in C(\overline{D}_\infty \times \mathbb{R})$ and $F \in C(\overline{D}_\infty)$, and let conditions* (1.5), (1.6), (3.14), *and* (3.26) *be satisfied; moreover, suppose that in the case $(l_1 l_2)(O) = 0$ the curves γ_1 and γ_2 either are not tangent at the point O or are tangent but $\gamma_2'(0) < 0$. Then problem* (1.1)–(1.3) *is locally solvable in the class C in the sense of Definition* 1.4; *i.e., there exists a positive number $T_0 = T_0(F)$ such that this problem with $T \le T_0$ has at least one strong generalized solution u in the class C in the domain D_T.*

Proof. By Remarks 3.7 and 3.9, a function $u \in C(\overline{D}_T)$ is a strong generalized solution of problem (1.1)–(1.3) in the class C in the domain D_T if and only if

$$\widetilde{u}(\xi,\eta) := u\left(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\right)$$

is a solution of the functional equation (3.52) in the class $C(\overline{G}_T)$, where $K_0 :\ C(\overline{G}_T) \to C(\overline{G}_T)$ is a continuous compact operator. Therefore, by virtue of the Schauder theorem [18, p. 370], for the solvability of Eq. (3.52) in the space $C(\overline{G}_T)$, it suffices to show that the operator K_0 maps some ball

$$B(O,r) := \{w \in C(\overline{G}_T) :\ \|w\|_{C(\overline{G}_T)} \le r\}$$

of radius $r > 0$ (which is a closed convex set in the Banach space $C(\overline{G}_T)$) into itself for sufficiently small T.

Take an arbitrary positive number T_* and assume that $T \le T_*$. By virtue of relations (3.15) and (3.52) for

$$\|w\|_{C(\overline{G}_T)} \le r, \qquad f^* := \sup_{\substack{(\xi,\eta)\in\overline{G}_{T_*}\\ |s|\le r}} |\widetilde{f}(\xi,\eta,s)|, \qquad F^* := \|\widetilde{F}\|_{C(\overline{G}_{T_*})} \tag{6.19}$$

with regard of the embedding $D_T \subset D_{T_*}$, we obtain

$$\begin{aligned}\|K_0 w\|_{C(\overline{G}_T)} &\le \|\widetilde{\Box}^{-1}\|_{C(\overline{G}_T)\to C(\overline{G}_T)} \sup_{\substack{(\xi,\eta)\in\overline{G}_{T_*}\\ |s|\le r}} |\widetilde{f}(\xi,\eta,s)| + \|\widetilde{\Box}^{-1}\|_{C(\overline{G}_T)\to C(\overline{G}_T)} \|\widetilde{F}\|_{C(\overline{G}_T)} \\ &\le 4T^{3/2}\exp(2^{-1}T)(f^* + F^*). \end{aligned} \tag{6.20}$$

It follows from relations (6.19) and (6.20) that if

$$T \le T_0 := \min\{T_*, h^{-1}[4^{-1} r (f^* + F^*)^{-1}]\},$$

where h^{-1} is the function inverse to $h(s) := s^{3/2}\exp(2^{-1}s)$, $s > 0$, then $\|K_0 w\|_{C(\overline{G}_T)} \le r$ for $\|w\|_{C(\overline{G}_T)} \le r$. The proof of Theorem 6.2 is complete.

ACKNOWLEDGMENTS

The research was supported by the National Scientific Foundation of Georgia (project no. 31/32).

REFERENCES

1. Goman, O.G., Equations of Reflected Wave, *Vestn. Moskov. Univ. Mat. Mekh.*, 1968, no. 2, pp. 84–87.
2. Mel'nik, Z.O., Example of a Nonclassical Boundary Value Problem for the Vibrating String Equation, *Ukrain. Mat. Zh.*, 1980, vol. 32, no. 5, pp. 671–674.
3. Kharibegashvili, S.S., A Boundary Value Problem for a Second-Order Hyperbolic Equation, *Dokl. Akad. Nauk SSSR*, 1985, vol. 280, no. 6, pp. 1313–1316.
4. Kharibegashvili, S., Goursat and Darboux Type Problems for Linear Hyperbolic Partial Differential Equations and Systems, *Mem. Differential Equations Math. Phys.*, 1995, vol. 4, pp. 1–127.
5. Troitskaya, S.D., On a Well-Posed Boundary Value Problem for Hyperbolic Equations with Two Independent Variables, *Uspekhi Mat. Nauk*, 1995, vol. 50, no. 4, pp. 124–125.
6. Troitskaya, S.D., On a Boundary Value Problem for Hyperbolic Equations, *Izv. Ross. Akad. Nauk Ser. Mat.*, 1998, vol. 62, no. 2, pp. 193–224.
7. Berikelashvili, G.K., Dzhokhadze, O.M., Midodashvili, B.G., and Kharibegashvili, S.S., On the Existence and Nonexistence of Global Solutions of the First Darboux Problem for Nonlinear Wave Equations, *Differ. Uravn.*, 2008, vol. 44, no. 3, pp. 359–372.
8. Jokhadze, O. and Midodashvili, B., The First Darboux Problem for Nonlinear Wave Equations with a Nonlinear Positive Source Term, *Nonlinear Anal.*, 2008, vol. 69, pp. 3005–3015.
9. Dzhokhadze, O.M. and Kharibegashvili, S.S., On the First Darboux Problem for Second-Order Nonlinear Hyperbolic Equations, *Mat. Zametki*, 2008, vol. 81, no. 5, pp. 693–712.
10. Kharibegashvili, S.S. and Dzhokhadze, O.M., Second Darboux Problem for the Wave Equation with a Power-Law Nonlinearity, *Differ. Uravn.*, 2013, vol. 49, no. 12, pp. 1623–1640.
11. Jokhadze, O., On Existence and Nonexistence of Global Solutions of Cauchy–Goursat Problem for Nonlinear Wave Equations, *J. Math. Anal. Appl.*, 2008, vol. 340, pp. 1033–1045.
12. Jokhadze, O., The Cauchy–Goursat Problem for One-Dimensional Semilinear Wave Equations, *Commun. Partial Differential Equations*, 2009, vol. 34, no. 4, pp. 367–382.
13. Kharibegashvili, S.S. and Dzhokhadze, O.M., The Cauchy–Darboux Problem for a One-Dimensional Wave Equation with Power Nonlinearity, *Sibirsk. Mat. Zh.*, 2013, vol. 54, no. 6, pp. 1407–1426.
14. Jokhadze, O. and Kharibegashvili, S., On the Cauchy and Cauchy–Darboux Problems for Semilinear Wave Equations, *Georgian Math. J.*, 2015, vol. 22, no. 1, pp. 81–104.

15. Narasimhan, R., *Analysis on Real and Complex Manifolds*, Amsterdam: North-Holland Publ., 1968. Translated under the title *Analiz na deistvitel'nykh i kompleksnykh mnogoobraziyakh*, Moscow, 1971.
16. Vladimirov, V.S., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow: Nauka, 1971.
17. Gilbarg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Berlin: Springer-Verlag, 1983. Translated under the title *Ellipticheskie differentsial'nye uravneniya s chastnymi proizvodnymi vtorogo poryadka*, Moscow: Nauka, 1989.
18. Trenogin, V.A., *Funktsional'nyi analiz* (Functional Analysis), Moscow: Nauka, 1993.
19. Fikhtengol'ts, G.M., *Kurs differentsial'nogo i integral'nogo ischisleniya* (A Course of Differential and Integral Calculus), Moscow, 1969.
20. Mitidieri, E. and Pokhozhaev, S.I., A Priori Estimates and the Absence of Solutions of Nonlinear Partial Differential Equations and Inequalities, *Tr. Mat. Inst. Steklova*, 2001, vol. 234, pp. 1–384.