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Original article

The second Darboux problem for the wave equation with integral nonlinearity

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Abstract

For a one-dimensional wave equation with integral nonlinearity, the second Darboux problem is considered for which the questions on the existence and uniqueness of a global solution are investigated.

Keywords: Darboux problem; Integral nonlinearity; Wave equation; Global solvability

1. Statement of the problem

In a plane of independent variables x and t we consider the wave equation with integral nonlinearity of the type

$$L_\lambda u := u_{tt} - u_{xx} + \lambda g\left(x, t, u, \int_{\alpha(t)}^{\beta(t)} u(x,t)dx\right) = f(x,t), \tag{1.1}$$

where $\lambda \neq 0$ is the given real constant; g, α, β and f are the given and u is an unknown real functions of their arguments.

By $D_T := \{(x,t) \in \mathbb{R}^2 : -\widetilde{k}_2 t < x < \widetilde{k}_1 t, 0 < t < T; 0 < \widetilde{k}_i := \text{const} < 1, i = 1,2\}$ we denote a triangular domain lying inside of a characteristic angle $\Lambda := \{(x,t) \in \mathbb{R}^2 : t > |x|\}$ and bounded by the segments $\widetilde{\gamma}_{1,T} : x = \widetilde{k}_1 t, 0 \le t \le T$, $\widetilde{\gamma}_{2,T} : x = -\widetilde{k}_2 t, 0 \le t \le T$ and $\widetilde{\gamma}_{3,T} : t = T, -\widetilde{k}_2 T \le x \le \widetilde{k}_1 T$. For $T = +\infty$, $D_\infty := \{(x,t) \in \mathbb{R}^2 : -\widetilde{k}_2 t < x < \widetilde{k}_1 t, 0 < t < +\infty\}$ (Fig. 1.1).

For Eq. (1.1), let us consider the second Darboux problem on finding in the domain D_T a solution $u(x,t)$ of the above equation by the boundary conditions (see e.g., [1, p. 107]; [2, p. 228])

$$u|_{\widetilde{\gamma}_{i,T}} = 0, \quad i = 1,2. \tag{1.2}$$

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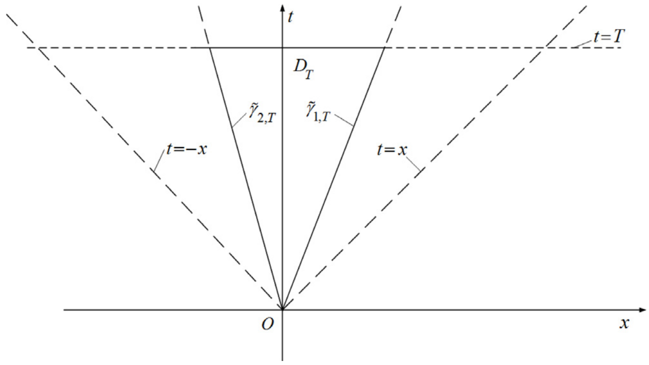

Fig. 1.1.

Below, when investigating problem (1.1), (1.2) it will be assumed that

$$-\widetilde{k}_2\, t \le \alpha(t) < \beta(t) \le \widetilde{k}_1\, t, \quad 0 < t < \infty. \tag{1.3}$$

For linear hyperbolic equations of second order with one spatial variable, a great number of works were devoted to the questions of the well-posedness of the Darboux problem (see, e.g., [2,3] and references therein). As it turned out, the presence of a weak nonlinearity in the equation affects the correctness of formulation even in the case of the first Darboux problem (see, e.g., [4–10]). Note that hyperbolic equations with nonlocal nonlinearities of type (1.1) have been considered in many works (see, e.g., [11–14] and references therein). In the present work it is shown that under definite conditions on the growth of nonlinear function $g = g(x, t, s_1, s_2)$ with respect to the variables s_1, s_2 the second Darboux problem (1.1), (1.2) is globally solvable.

Definition 1.1. Let $\alpha, \beta \in C([0, T])$, $g \in C(\overline{D}_T \times \mathbb{R}^2)$, $f \in C(\overline{D}_T)$. The function u is said to be a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{D}_T)$, as $n \to \infty$, where $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{v \in C^2(\overline{D}_T) : v|_{\Gamma_T} = 0\}$, $\Gamma_T := \widetilde{\gamma}_{1,T} \cup \widetilde{\gamma}_{2,T}$.

Remark 1.1. Note that two different approximations with given properties define the same function in Definition 1.1. Obviously, the classical solution of problem (1.1), (1.2) from the space $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a strong generalized solution of that problem of the class C in the domain D_T. In its turn, if a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then it will be a classical solution of that problem, as well.

Definition 1.2. Let $\alpha, \beta \in C([0, \infty))$, $g \in C(\overline{D}_\infty \times \mathbb{R}^2)$, $f \in C(\overline{D}_\infty)$. We say that problem (1.1), (1.2) is globally solvable in the class C, if for any finite $T > 0$, this problem has a strong generalized solution of the class C in the domain D_T.

2. An a priori estimate of solution of problem (1.1), (1.2)

Let us consider the following condition imposed on the function g:

$$\left|g(x, t, s_1, s_2)\right| \le a + b|s_1| + c|s_2|, \quad (x, t, s_1, s_2) \in \overline{D}_T \times \mathbb{R}^2, \tag{2.1}$$

where $a, b, c = \text{const} \ge 0$.

Lemma 2.1. *Let the condition (2.1) be fulfilled. Then for a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T the following a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)} + c_2 \tag{2.2}$$

with nonnegative constants c_i, $i = 1, 2$, independent of u and f, is valid.

Proof. Let u be a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T. Then by virtue of Definition 1.1, there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\overline{D}_T)} = 0. \tag{2.3}$$

Denote

$$f_n := L_\lambda u_n. \tag{2.4}$$

Multiplying both parts of equality (2.4) by u_{nt} and integrating with respect to the domain $D_\tau := \{(x,t) \in D_T : t < \tau\}$, $0 < \tau \le T$, we obtain

$$\frac{1}{2}\int_{D_\tau} (u_{nt}^2)_t\,dx\,dt - \int_{D_\tau} u_{nxx}u_{nt}\,dx\,dt + \lambda\int_{D_\tau} g\Big(x,t,u_n,\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)u_{nt}\,dx\,dt = \int_{D_\tau} f_n u_{nt}\,dx\,dt.$$

Assume $\omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Then taking into account that $u_n|_{\Gamma_T} = 0$, the integration by parts of the left-hand side of the last equality yields

$$\begin{aligned}2\int_{D_\tau} f_n u_{nt}\,dx\,dt &= \int_{\Gamma_\tau} \frac{1}{\nu_t}\big[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)\big]ds \\ &\quad + \int_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)dx + 2\lambda\int_{D_\tau} g\Big(x,t,u_n,\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)u_{nt}\,dx\,dt,\end{aligned} \tag{2.5}$$

where $\nu := (\nu_x, \nu_t)$ is the unit vector of the outer normal to ∂D_τ, and $\Gamma_\tau := \Gamma_T \cap \{t \le \tau\}$.

Taking into account that $\nu_t \frac{\partial}{\partial x} - \nu_x \frac{\partial}{\partial t}$ is the inner differential operator on Γ_T and $u_n|_{\Gamma_T} = 0$, we have

$$(u_{nx}\nu_t - u_{nt}\nu_x)\big|_{\Gamma_\tau} = 0. \tag{2.6}$$

Since $D_\tau : -\widetilde{k}_2 t < x < \widetilde{k}_1 t$, $t < \tau$, it is easy to see that

$$(\nu_t^2 - \nu_x^2)\big|_{\Gamma_\tau} < 0, \quad \nu_t\big|_{\Gamma_\tau} < 0. \tag{2.7}$$

Bearing in mind (2.6) and (2.7), from (2.5) we obtain

$$w_n(\tau) := \int_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)dx \le 2\int_{D_\tau} f_n u_{nt}dx\,dt - 2\lambda\int_{D_\tau} g\Big(x,t,u_n,\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)u_{nt}\,dx\,dt. \tag{2.8}$$

In view of (2.1), we have

$$\begin{aligned}\left|g\Big(x,t,u_n,\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)u_{nt}\right| &\le \Big(a + b|u_n| + c\Big|\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big|\Big)|u_{nt}| \\ &\le \frac{1}{2}\Big(a + b|u_n| + c\Big|\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big|\Big)^2 + \frac{1}{2}u_{nt}^2 \\ &\le \frac{3}{2}a^2 + \frac{3}{2}b^2u_n^2 + \frac{3}{2}c^2\Big(\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)^2 + \frac{1}{2}u_{nt}^2.\end{aligned} \tag{2.9}$$

If $(x,t) \in \overline{D}_T$, then owing to (1.3), $u_n|_{\Gamma_T} = 0$ and Schwartz inequality, we have

$$\begin{aligned}|u_n(x,t)| &= \left|u_n(-\widetilde{k}_2 t, t) + \int_{-\widetilde{k}_2 t}^{x} u_{nx}(s,t)ds\right| = \left|\int_{-\widetilde{k}_2 t}^{x} u_{nx}(s,t)ds\right| \\ &\le \Big(\int_{-\widetilde{k}_2 t}^{x} 1^2 ds\Big)^{\frac{1}{2}}\Big(\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\Big)^{\frac{1}{2}} \le \sqrt{2t}\Big(\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\Big)^{\frac{1}{2}},\end{aligned} \tag{2.10}$$

$$\Big(\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)^2 \le \int_{\alpha(t)}^{\beta(t)} 1^2dx\int_{\alpha(t)}^{\beta(t)} u_n^2(x,t)dx \le 2t\int_{\alpha(t)}^{\beta(t)} u_n^2(x,t)dx. \tag{2.11}$$

It follows from (2.8), (2.10) and (2.11) that

$$\left|\int_{\alpha(t)}^{\beta(t)} u_n^2(x,t)dx\right| \le 2t\int_{\alpha(t)}^{\beta(t)}\left[2t\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\right]dx$$
$$\le (2t)^2\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} dx\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds = 4t^3(\widetilde{k}_1+\widetilde{k}_2)\int_{\omega_t} u_{nx}^2 dx \le 8t^3\int_{\omega_t}(u_{nx}^2+u_{nt}^2)dx = 8t^3 w_n(t),$$

whence we get

$$\int_{D_\tau}\left|\int_{\alpha(t)}^{\beta(t)} u_n(\xi,t)d\xi\right|^2 dx\,dt = \int_0^\tau dt\int_{\omega_t}\left|\int_{\alpha(t)}^{\beta(t)} u_n(\xi,t)d\xi\right|^2 dx \le \int_0^\tau dt\int_{\omega_t} 8t^3 w_n(t)dx$$
$$= \int_0^\tau 8t^3 w_n(t)\,\mathrm{mes}\,\omega_t dt \le 16\tau^4\int_0^\tau w_n(t)dt. \tag{2.12}$$

From (2.9) and (2.12), we now obtain

$$\int_{D_\tau} g\left(x,t,u_n,\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\right)u_{nt}dx\,dt \le \frac{3}{2}a^2\,\mathrm{mes}\,D_\tau + \frac{3}{2}b^2\int_{D_\tau} u_n^2 dx\,dt$$
$$+\left(24c^2\tau^4+\frac{1}{2}\right)\int_0^\tau w_n(t)dt. \tag{2.13}$$

Further, in view of (2.10), we have

$$\int_{D_\tau} u_n^2 dx\,dt = \int_0^\tau dt\int_{\omega_t} u_n^2(x,t)dx \le \int_0^\tau dt\int_{\omega_t}\left(2t\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\right)dx$$
$$\le \int_0^\tau dt\int_{\omega_t}\left(2t\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds\right)dx \le \int_0^\tau \mathrm{mes}\,\omega_t\left(2t\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds\right)dt$$
$$\le 4\tau^2\int_0^\tau dt\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds = 4\tau^2\int_0^\tau dt\int_{\omega_t} u_{nx}^2 dx$$
$$\le 4\tau^2\int_0^\tau dt\int_{\omega_t}(u_{nx}^2+u_{nt}^2)dx = 4\tau^2\int_0^\tau w_n(t)dt. \tag{2.14}$$

Taking into account (2.13), (2.14) and the fact that $\mathrm{mes}\,D_\tau \le \tau^2 \le T^2$, $2f_n u_{nt} \le u_{nt}^2 + f_n^2$, as well as

$$\int_{D_\tau} u_{nt}^2 dx\,dt \le \int_0^\tau w_n(t)dt,$$

from (2.8) we get

$$w_n(\tau) \le |\lambda|\left(3a^2T^2 + 12b^2T^2\int_0^\tau w_n(t)dt + 48c^2T^4\int_0^\tau w_n(t)dt + \int_0^\tau w_n(t)dt\right) + \int_0^\tau w_n(t)dt$$
$$+\|f_n\|_{L_2(D_T)}^2 \le \left[|\lambda|\left(12b^2T^2+48c^2T^4+1\right)+1\right]\int_0^\tau w_n(t)dt + 3|\lambda|a^2T^2$$
$$+\|f_n\|_{L_2(D_T)}^2,\quad 0<\tau\le T.$$

Hence according to the Gronwall's lemma, it follows that

$$w_n(\tau) \le \left(3|\lambda|a^2T^2 + \|f_n\|_{L_2(D_T)}^2\right)\exp\left(T\left[|\lambda|\left(12b^2T^2+48c^2T^4+1\right)+1\right]\right),\quad 0<\tau\le T. \tag{2.15}$$

If $(x,t)\in\overline{D}_T$, then owing to (2.8), (2.10) and (2.15), we have

$$\left|u_n(x,t)\right|^2 \le 2t\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds \le 2T\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t}(u_{nx}^2+u_{nt}^2)dx = 2T\,w_n(t)$$
$$\le 2T\left(3|\lambda|a^2T^2+\|f_n\|_{L_2(D_T)}^2\right)\exp\left(T\left[|\lambda|\left(12b^2T^2+48c^2T^4+1\right)+1\right]\right).$$

This implies that

$$\|u_n\|_{C(\overline{D}_T)} \le c_1 \|f_n\|_{C(\overline{D}_T)} + c_2, \tag{2.16}$$

where

$$\begin{aligned} c_1 &= \sqrt{2T}\,\exp\Big(\frac{T}{2}\Big[|\lambda|\big(12\,b^2T^2 + 48\,c^2T^4 + 1\big) + 1\Big]\Big),\\ c_2 &= a\,T\sqrt{6T|\lambda|}\,\exp\Big(\frac{T}{2}\Big[|\lambda|\big(12\,b^2T^2 + 48\,c^2T^4 + 1\big) + 1\Big]\Big). \end{aligned} \tag{2.17}$$

By virtue of (2.3), passing in inequality (2.16) to the limit, as $n \to \infty$, we obtain the estimate (2.2) which proves Lemma 2.1. □

Remark 2.1. If in inequality (2.1) the number $a = 0$, then in the a priori estimate (2.2) the value $c_2 = 0$. In this case estimate (2.2) takes the form

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)},$$

hence from $f = 0$ it follows that $u = 0$, which in a linear case implies the uniqueness of a solution of problem (1.1), (1.2).

3. Equivalent reduction of problem (1.1), (1.2) to a nonlinear integral equation of Volterra type

In new independent variables $\xi = \frac{1}{2}(t + x)$, $\eta = \frac{1}{2}(t - x)$ the domain D_T will go over to a triangular domain G_T with vertices at the points $O(0, 0)$, $Q_1\big(\frac{1+\widetilde{k}_1}{2}\,T, \frac{1-\widetilde{k}_1}{2}\,T\big)$, $Q_2\big(\frac{1-\widetilde{k}_2}{2}\,T, \frac{1+\widetilde{k}_2}{2}\,T\big)$ of the plane of variables ξ, η, and problem (1.1), (1.2) will go over to the problem

$$\widetilde{L}_\lambda v := v_{\xi\eta} + \lambda\, K\, v = \widetilde{f}(\xi, \eta), \quad (\xi, \eta) \in G_T, \tag{3.1_λ}$$

$$v\big|_{\gamma_{i,T}} = 0, \quad \gamma_{i,T} := O\,Q_i, \quad i = 1, 2, \tag{3.2_λ}$$

with respect to a new unknown function $v(\xi, \eta) := u(\xi - \eta, \xi + \eta)$; $\widetilde{f}(\xi, \eta) := f(\xi - \eta, \xi + \eta)$.

Here, the operator K acts by the formula

$$(K\,v)(\xi, \eta) = g\bigg(\xi - \eta, \xi + \eta, v, \int_{\alpha(\xi+\eta)}^{\beta(\xi+\eta)} v(\xi, \eta)d\xi - v(\xi, \eta)d\eta\bigg), \tag{3.3}$$

$$\begin{aligned} \gamma_{1,T} &: \eta = k_1\xi, \quad 0 \le \xi \le \xi_0 := 2^{-1}(1 + \widetilde{k}_1)T,\\ \gamma_{2,T} &: \xi = k_2\eta, \quad 0 \le \eta \le \eta_0 := 2^{-1}(1 + \widetilde{k}_2)T, \end{aligned} \tag{3.4}$$

$$0 < k_i := \frac{1 - \widetilde{k}_i}{1 + \widetilde{k}_i} < 1, \quad i = 1, 2. \tag{3.5}$$

Analogously to Definition 1.1, we introduce the notion of a strong generalized solution v of problem (3.1_λ), (3.2_λ) of the class C in the domain G_T.

If $P_0(\xi, \eta) \in G_T$, we denote by $P_1M_0P_0N_0$ a rectangle, characteristic with respect to Eq. (3.1_λ) whose vertices N_0 and M_0 lie, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{2,T}$, that is, by virtue of (3.4): $N_0 := (\xi, k_1\xi)$, $M_0 := (k_2\eta, \eta)$, $P_1 := (k_2\eta, k_1\xi)$. Since $P_1 \in G_T$, we construct analogously the characteristic rectangle $P_2M_1P_1N_1$ whose vertices N_1 and M_1 lie, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{2,T}$. Continuing this process, we obtain the characteristic rectangle $P_{i+1}M_iP_iN_i$ for which $N_i \in \gamma_{1,T}$, $M_i \in \gamma_{2,T}$, and $N_i := (\xi_i, k_1\xi_i)$, $M_i := (k_2\eta_i, \eta_i)$, $P_{i+1} := (k_2\eta_i, k_1\xi_i)$ if $P_i := (\xi_i, \eta_i)$, $i > 0$ (Fig. 3.1).

It is not difficult to see that

$$\begin{aligned} P_{2n} &= \big((k_1k_2)^n\xi, (k_1k_2)^n\eta\big), & P_{2n+1} &= \big((k_1k_2)^nk_2\eta, (k_1k_2)^nk_1\xi\big), & n &= 0, 1, 2, \ldots,\\ M_{2n} &= \big((k_1k_2)^nk_2\eta, (k_1k_2)^n\eta\big), & M_{2n+1} &= \big((k_1k_2)^{n+1}\xi, (k_1k_2)^nk_1\xi\big), & n &= 0, 1, 2, \ldots,\\ N_{2n} &= \big((k_1k_2)^n\xi, (k_1k_2)^nk_1\xi\big), & N_{2n+1} &= \big((k_1k_2)^nk_2\eta, (k_1k_2)^{n+1}\eta\big), & n &= 0, 1, 2, \ldots. \end{aligned} \tag{3.6}$$

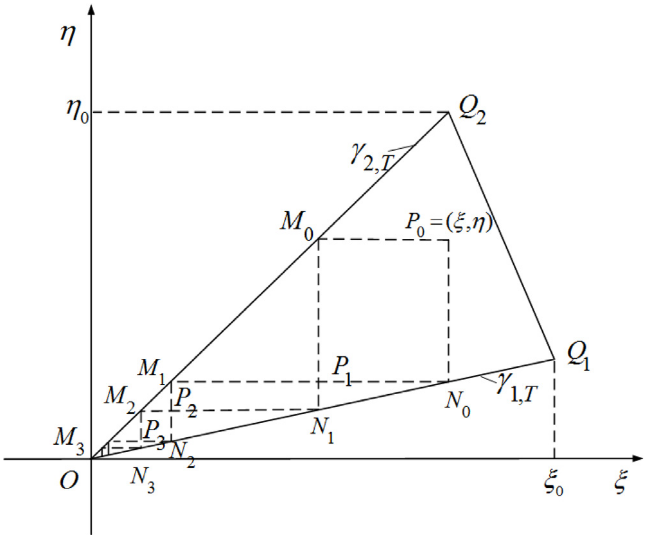

Fig. 3.1.

Consider first a linear case, i.e., when in problem (3.1$_\lambda$), (3.2$_\lambda$) the parameter $\lambda = 0$. If v is a strong generalized solution of problem (3.1$_0$), (3.2$_0$) of the class C in the domain G_T, then considering the function v as a solution of the Goursat problem for equation (3.1$_0$), in the rectangle $P_{i+1}M_iP_iN_i$ with data on characteristic segments $P_{i+1}N_i$ and $P_{i+1}M_i$, we have (see, e.g., [15, p. 173]),

$$v(P_i) = v(M_i) + v(N_i) - v(P_{i+1}) + \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad i = 0, 1, \ldots .$$

Thus, by virtue of equality (3.2$_0$), it follows that

$$\begin{aligned} v(\xi,\eta) &= v(P_0) = v(M_0) + v(N_0) - v(P_1) + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &= -v(P_1) + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &= -v(M_1) - v(N_1) + v(P_2) - \int_{P_2M_1P_1N_1} \widetilde{f}\, d\xi_1\, d\eta_1 + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &= v(P_2) - \int_{P_2M_1P_1N_1} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &\quad + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 = \cdots = (-1)^n v(P_n) \\ &\quad + \sum_{i=0}^{n-1} (-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad (\xi,\eta) \in G_T. \end{aligned} \tag{3.7}$$

Since the point P_n from (3.7) tends to the point $O(0,0)$, as $n \to \infty$, by (3.2$_0$), we have $\lim_{n\to\infty} v(P_n) = 0$. Hence, passing in equality (3.7) to the limit, as $n \to \infty$, for a strong generalized solution v of problem (3.1$_0$), (3.2$_0$) of the class C in the domain G_T, we obtain the following integral representation:

$$v(\xi,\eta) = \sum_{i=0}^{\infty} (-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad (\xi,\eta) \in G_T. \tag{3.8}$$

Remark 3.1. Since $\widetilde{f} \in C(\overline{G}_T)$ and there take place inequalities (3.5), and moreover, owing to (3.6),

$$\operatorname{mes} P_{i+1}M_iP_iN_i = (k_1k_2)^i(\xi - k_2\eta)(\eta - k_1\xi), \tag{3.9}$$

the series in the right-hand side of equality (3.8) is uniformly and absolutely convergent.

Remark 3.2. From the above reasoning it follows that for any $\widetilde{f} \in C(\overline{G}_T)$, linear problem (3.1_0), (3.2_0) has a unique strong generalized solution v of the class C in the domain G_T which is representable in the form of uniformly and absolutely converging series (3.8).

Introduce into consideration the operator $\widetilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ acting by the formula

$$(\widetilde{L}_0^{-1}\widetilde{f})(\xi,\eta) := \sum_{i=0}^{\infty}(-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad (\xi,\eta) \in G_T. \tag{3.10}$$

Remark 3.3. According to (3.10) and Remark 3.2, a unique strong generalized solution v of problem (3.1_0), (3.2_0) of the class C in the domain G_T is representable in the form $v = \widetilde{L}_0^{-1}\widetilde{f}$, and owing to (3.5), (3.9), we have the estimate

$$\begin{aligned} \big|v(\xi,\eta)\big| &\le \sum_{i=0}^{\infty}\int_{P_{i+1}M_iP_iN_i} |\widetilde{f}|\, d\xi_1\, d\eta_1 \le (\xi+\eta)^2\|\widetilde{f}\|_{C(\overline{G}_T)} \sum_{i=0}^{\infty}(k_1k_2)^i \\ &\le \frac{2(\xi^2+\eta^2)}{1-k_1k_2}\,\|\widetilde{f}\|_{C(\overline{G}_T)} \le \frac{1+\widetilde{k}^2}{1-k_1k_2}\,T^2\|\widetilde{f}\|_{C(\overline{G}_T)}, \quad \widetilde{k} := \max\{\widetilde{k}_1,\widetilde{k}_2\}, \end{aligned}$$

whence in its turn it follows that

$$\|\widetilde{L}_0^{-1}\|_{C(\overline{G}_T)\to C(\overline{G}_T)} \le \frac{1+\widetilde{k}^2}{1-k_1k_2}\,T^2. \tag{3.11}$$

Lemma 3.1. *The function $v \in C(\overline{G}_T)$ is a strong generalized solution of problem (3.1_λ), (3.2_λ) of the class C in the domain G_T, if and only if this function is a continuous solution of the following nonlinear Volterra type integral equation*

$$v(\xi,\eta) + \lambda(\widetilde{L}_0^{-1}K\,v)(\xi,\eta) = (\widetilde{L}_0^{-1}\widetilde{f})(\xi,\eta), \quad (\xi,\eta) \in G_T. \tag{3.12}$$

Proof. Indeed, let $v \in C(\overline{G}_T)$ be a solution of Eq. (3.12). Since $\widetilde{f} \in C(\overline{G}_T)$, and the space $C^2(\overline{G}_T)$ is dense in $C(\overline{G}_T)$ (see, e.g., [16, p. 37]), there exists a sequence of functions $\widetilde{f}_n \in C^2(\overline{G}_T)$ such that $\widetilde{f}_n \to \widetilde{f}$ in the space $C(\overline{G}_T)$, as $n \to \infty$. Analogously, since $v \in C(\overline{G}_T)$, there exists a sequence of functions $w_n \in C^2(\overline{G}_T)$ such that $w_n \to v$ in the space $C(\overline{G}_T)$, as $n \to \infty$. Assume $v_n := -\lambda\widetilde{L}_0^{-1}Kw_n + \widetilde{L}_0^{-1}\widetilde{f}_n$, $n = 1,2,\ldots$. Taking into account (3.5), (3.6), (3.9) and (3.10), it is easy to see that $v_n \in C^2(\overline{G}_T)$, and $v_n|_{\gamma_{i,T}} = 0$, $i = 1,2$. On the one hand, by virtue of estimate (3.1_λ) and equality (3.12), we have $v_n \to -\lambda\widetilde{L}_0^{-1}Kv + \widetilde{L}_0^{-1}\widetilde{f} = v$ in the space $C(\overline{G}_T)$, as $n \to \infty$, i.e., $v_n \to v$ in $C(\overline{G}_T)$, as $n \to \infty$. On the other hand, $\widetilde{L}_0 v_n = -\lambda K w_n + \widetilde{f}_n$, but since $\lim_{n\to\infty}\|v_n - v\|_{C(\overline{G}_T)} = 0$, $\lim_{n\to\infty}\|w_n - v\|_{C(\overline{G}_T)} = 0$ and $\lim_{n\to\infty}\|\widetilde{f}_n - \widetilde{f}\|_{C(\overline{G}_T)} = 0$, in view of (2.3) we have $\widetilde{L}_\lambda v_n = \widetilde{L}_0 v_n + \lambda K v_n = -\lambda K w_n + \widetilde{f}_n + \lambda K v_n = -\lambda(Kw_n - Kv) + \lambda(Kv_n - Kv) + \widetilde{f}_n \to \widetilde{f}$ in the space $C(\overline{G}_T)$, as $n \to \infty$. Thus, the function $v \in C(\overline{G}_T)$ is a strong generalized solution of problem (3.1_λ), (3.2_λ) of the class C in the domain G_T. The converse is obvious. □

4. The case of global solvability of problem (1.1), (1.2) in the class of continuous functions

Lemma 4.1. *The operator $\widetilde{L}_0^{-1}$ defined by formula (3.10) is the linear continuous operator acting from the space $C(\overline{G}_T)$ to the space $C^1(\overline{G}_T)$.*

Proof. To prove the lemma, we first show that for $\widetilde{f} \in C(\overline{G}_T)$, the series in the right-hand side of (3.10) differentiated formally with respect to ξ and to η converges uniformly on the set $\overline{G}_T$. Indeed, as it can be easily verified, we have

$$\frac{\partial}{\partial\xi}\bigg[\sum_{i=0}^{\infty}(-1)^i\int_{P_{i+1}M_iP_iN_i}\widetilde{f}\,d\xi_1\,d\eta_1\bigg]=\sum_{n=0}^{\infty}\bigg[(k_1k_2)^n\int_{N_{2n}P_{2n}}\widetilde{f}\,d\eta_1+(k_1k_2)^{n+1}\int_{P_{2n+2}M_{2n+1}}\widetilde{f}\,d\eta_1 - (k_1k_2)^nk_1\int_{M_{2n+1}N_{2n}}\widetilde{f}\,d\xi_1\bigg], \tag{4.1}$$

$$\frac{\partial}{\partial\eta}\bigg[\sum_{i=0}^{\infty}(-1)^i\int_{P_{i+1}M_iP_iN_i}\widetilde{f}\,d\xi_1\,d\eta_1\bigg]=\sum_{n=0}^{\infty}\bigg[(k_1k_2)^n\int_{M_{2n}P_{2n}}\widetilde{f}\,d\xi_1+(k_1k_2)^{n+1}\int_{P_{2n+2}N_{2n+1}}\widetilde{f}\,d\xi_1 - (k_1k_2)^nk_2\int_{N_{2n+1}M_{2n}}\widetilde{f}\,d\eta_1\bigg]. \tag{4.2}$$

By virtue of (3.6), the equalities

$$|N_{2n}P_{2n}|=(k_1k_2)^n(\eta-k_1\xi),\quad |P_{2n+2}M_{2n+1}|=(k_1k_2)^nk_1(\xi-k_2\eta),\quad |M_{2n+1}N_{2n}|=(k_1k_2)^n(1-k_1k_2)\xi,$$
$$|M_{2n}P_{2n}|=(k_1k_2)^n(\xi-k_2\eta),\quad |P_{2n+2}N_{2n+1}|=(k_1k_2)^nk_2(\eta-k_1\xi),\quad |N_{2n+1}M_{2n}|=(k_1k_2)^n(1-k_1k_2)\eta,$$

hold, hence with regard for (3.5), it follows that the series (4.1) and (4.2) converge uniformly and absolutely, and we have the estimate

$$\max\bigg\{\Big\|\frac{\partial}{\partial\xi}\,(\widetilde{L}_0^{-1}\widetilde{f})\Big\|_{C(\overline{G}_T)},\ \Big\|\frac{\partial}{\partial\eta}\,(\widetilde{L}_0^{-1}\widetilde{f})\Big\|_{C(\overline{G}_T)}\bigg\}\le\frac{3}{1-(k_1k_2)^2}\,T\,\|\widetilde{f}\|_{C(\overline{G}_T)}.$$

Thus by virtue of 3.1 and the fact that $\|v\|_{C^1} := \max\{\|v\|_C,\ \|v_\xi\|_C,\ \|v_\eta\|_C\}$, we obtain the assertion of Lemma 4.1. □

Remark 4.1. Since the space $C^1(\overline{G}_T)$ is compactly embedded into $C(\overline{G}_T)$ (see, e.g., [17, p. 135]), the operator $\widetilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ in view of (3.1_λ) and Lemma 4.1 is linear and compact one.

We rewrite Eq. (3.12) in the form

$$v = A\,v := \widetilde{L}_0^{-1}(-\lambda\,K\,v+\widetilde{f}), \tag{4.3}$$

where the operator $A : C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous and compact, since the nonlinear operator $K : C(\overline{G}_T) \to C(\overline{G}_T)$, acting by formula (3.3), is bounded and continuous, whereas the linear operator $\widetilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is, according to Remark 4.1, compact. At the same time, by Lemmas 2.1 and 3.1, and by equalities (2.17), for an arbitrary parameter $\tau \in [0, 1]$ and for any solution $v \in C(\overline{G}_T)$ of equation $v = \tau A v$, the a priori estimate $\|v\|_{C(\overline{G}_T)} \le c_1\|\widetilde{f}\|_{C(\overline{G}_T)} + c_2$ with the same nonnegative constants c_1 and c_2 as in (2.1), not depending on v, τ and $\widetilde{f}$, is valid. Therefore, by the Leray–Schauder's theorem (see, e.g., [18, p. 375]), Eq. (4.3) under the condition of Lemma 2.1 has at least one solution $v \in C(\overline{G}_T)$. Thus, owing to Lemma 3.1, we have proved the following.

Theorem 4.1. *Let $\alpha, \beta \in C([0, T])$, $g \in C(\overline{D}_T \times \mathbb{R}^2)$, $f \in C(\overline{D}_T)$ and condition (2.1) be fulfilled. Then problem (1.1), (1.2) has at least one strong generalized solution of the class C in the domain D_T in the sense of Definition 1.1.*

Corollary 4.1. *Let $\alpha, \beta \in C([0, \infty])$, $g \in C(\overline{D}_\infty \times \mathbb{R}^2)$, $f \in C(\overline{D}_\infty)$ and condition (2.1) for $(x, t) \in \overline{D}_\infty$ be fulfilled. Then problem (1.1), (1.2) is globally solvable in the class C in the sense of Definition 1.2.*

5. The smoothness and uniqueness of a solution of problem (1.1), (1.2). The existence of a global solution in D_∞

From equalities (3.12), (4.1), (4.2), by Lemmas 3.1 and 4.1 we immediately have

Lemma 5.1. *Let u be a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T in the sense of Definition 1.1. Then if $\alpha, \beta \in C^1([0, T])$, $g \in C^1(\overline{D}_T \times \mathbb{R}^2)$ and $f \in C^1(\overline{D}_T)$, then $u \in C^2(\overline{D}_T)$.*

Lemma 5.2. *For $g \in C^1(\overline{D}_T \times \mathbb{R}^2)$, problem (1.1), (1.2) fails to have more than one strong generalized solution of the class C in the domain D_T.*

Proof. Indeed, assume that problem (1.1), (1.2) has two possible different strong generalized solutions u^1 and u^2 of the class C in the domain D_T. By Definition 1.1, there exists a sequence of functions $u_n^i \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, $i = 1, 2$, such that

$$\lim_{n\to\infty} \|u_n^i - u^i\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_n^i - f\|_{C(\overline{D}_T)} = 0, \quad i = 1, 2. \tag{5.1}$$

Assume $v_n := u_n^2 - u_n^1$. It can be easily seen that the function $v_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a classical solution of the problem

$$L_0\, w_n + \lambda\, g_n^1\, v_n + \lambda\, g_n^2 \int_{\alpha(t)}^{\beta(t)} v_n dx = f_n, \tag{5.2}$$

$$v_n\big|_{\Gamma_T} = 0. \tag{5.3}$$

Here,

$$\begin{aligned} g_n^1 &:= \int_0^1 g_{s_1}\left[x, t, u_n^1 + s(u_n^2 - u_n^1), \int_{\alpha(t)}^{\beta(t)} u_n^1 dx\right] ds, \\ g_n^2 &:= \int_0^1 g_{s_2}\left[x, t, u_n^2, \int_{\alpha(t)}^{\beta(t)} u_n^1 dx + s\int_{\alpha(t)}^{\beta(t)} (u_n^2 - u_n^1) dx\right] ds, \end{aligned} \tag{5.4}$$

$$f_n := L_\lambda\, u_n^2 - L_\lambda\, u_n^1, \tag{5.5}$$

where we have used the following obvious equality

$$\begin{aligned} \varphi(x_2, y_2) - \varphi(x_1, y_1) = {}& (x_2 - x_1) \int_0^1 \varphi_x\big[x_1 + s(x_2 - x_1), y_1\big] ds \\ & + (y_2 - y_1) \int_0^1 \varphi_y\big[x_2, y_1 + s(y_2 - y_1)\big] ds \end{aligned}$$

for the function $\varphi(x, y)$.

Assume

$$A := \Big\{(x, t, s_1, s_2) \in \overline{D}_T \times \mathbb{R}^2 : (x, t) \in \overline{D}_T, \ \ |s_1| \le c_1 \|f\|_{C(\overline{D}_T)} + c_2, \ \ |s_2| \le 2\,T\,c_1\big(\|f\|_{C(\overline{D}_T)} + c_2\big)\Big\}$$

and

$$B := \max\big\{\|g_{s_1}\|_{C(\overline{A})}, \|g_{s_2}\|_{C(\overline{A})}\big\}. \tag{5.6}$$

Taking into account the a priori estimate (2.2), for the functions u_n^1 and u_n^2, with regard for (5.4)–(5.6), we have

$$\left|g_n^1 v_n + g_n^2 \int_{\alpha(t)}^{\beta(t)} v_n dx\right| \le B\left(|v_n| + \left|\int_{\alpha(t)}^{\beta(t)} v_n dx\right|\right). \tag{5.7}$$

Now, by virtue of (5.7), Lemma 2.1 and Remark 2.1 applied to the case when in inequality (2.1) $a = 0, b = B, c = B$ for the solution v_n of problem (5.2), (5.3) we have the following estimate:

$$\|v_n\|_{C(\overline{D}_T)} \le \sqrt{2T}\, \exp\left(\frac{T}{2}\Big[|\lambda|\big(12\, B^2T^2 + 48\, B^2T^4 + 1\big) + 1\Big]\right) \|f_n\|_{C(\overline{D}_T)}. \tag{5.8}$$

Since owing to (5.1),

$$\|u_2 - u_1\| = \lim_{n\to\infty} \|v_n\|_{C(\overline{D}_T)}, \quad \lim_{n\to\infty} \|f_n\|_{C(\overline{D}_T)} = 0,$$

therefore passing in estimate (5.8) to the limit, as $n \to \infty$, we obtain

$$\left\| u_2 - u_1 \right\|_{C(\overline{D}_T)} \le 0,$$

i.e., $u_1 = u_2$, which contradicts our assumption. Thus Lemma 5.2 is proved. □

Theorem 5.1. *Let $\alpha, \beta \in C^1([0, +\infty))$, $g \in C^1(\overline{D}_\infty \times \mathbb{R}^2)$ and condition (2.1) be fulfilled. Then for any $f \in C^1(\overline{D}_\infty)$, problem (1.1), (1.2) has the unique global classical solution $u \in \overset{\circ}{C}{}^2(\overline{D}_\infty, \Gamma_\infty)$ in the domain D_∞.*

Proof. If $f \in C^1(\overline{D}_\infty)$ and condition (2.1) is fulfilled, then according to Theorem 4.1 and Lemmas 5.1 and 5.2, in the domain D_T for $T = n$ there exists the unique classical solution $u \in \overset{\circ}{C}{}^2(\overline{D}_n, \Gamma_n)$ of problem (1.1), (1.2). Since u_{n+1} is likewise a classical solution of problem (1.1), (1.2) in the domain D_n, by Lemma 5.2, we have $u_{n+1}|_{D_n} = u_n$. Therefore, the function u constructed in the domain D_∞ by the rule $u(x,t) = u_n(x,t)$ for $n = [t] + 1$, where $[t]$ is integer part of the number t, and $(x,t) \in D_\infty$, will be the unique classical solution of problem (1.1), (1.2) in the domain D_∞ of the class $\overset{\circ}{C}{}^2(\overline{D}_\infty, \Gamma_\infty)$. Thus Theorem 5.1 is proved. □

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