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Original article

An approximate solution of one class of singular integro-differential equations

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Abstract

The problem of definition of mechanical field in a homogeneous plate supported by finite inhomogeneous inclusion is considered. The contact between the plate and inclusion is realized by a thin glue layer. The problem is reduced to the boundary value problem for singular integro-differential equations. Asymptotic analysis is carried out. Using the method of orthogonal polynomials, the problem is reduced to the solution of an infinite system of linear algebraic equations. The obtained system is investigated for regularity.

Keywords: Elastic inclusion; Integro-differential equation; Orthogonal polynomials

1. Statement of the Problem and its Reduction to a Singular Integro-Differential Equation (SIDE)

Let an elastic plane with the modulus of elasticity E_2 and the Poisson coefficient ν_2 on a finite interval $[-1, 1]$ of the ox-axis be reinforced by an inclusion in the form of a cover plate of small thickness $h_1(x)$, with the modulus of elasticity $E_1(x)$ and the Poisson coefficient ν_1, loaded by tangential force of intensity $\tau_0(x)$, and the plate at infinity towards to the ox and oy-axes be subjected to uniformly stretching forces of intensities p and q, respectively.

Under the conditions of plane deformation we are required to determine contact stresses acting in the interval of the inclusion and plate joint. An inclusion will be assumed to be a thin plate free from bending rigidity, and the contact between the plate and inclusion is realized by a thin glue layer with thickness h_0 and modulus of shear G_0.

Equation of equilibrium of differential element of inclusion has the form [1]

$$\frac{d}{dx}\left(E(x)\frac{du_1(x)}{dx}\right) = \tau_-(x) - \tau_+(x) - \tau_0(x), \quad |x| < 1, \tag{1}$$

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where $\tau_{\pm}(x)$ are unknown tangential contact stresses at the upper and lower contours of the inclusion, $u_1(x)$ is horizontal displacement of inclusion points towards the ox-axis, $E(x) = \frac{E_1(x)h_1(x)}{1-\nu_1^2}$. Introducing the notation $\tau(x) := \tau_-(x) - \tau_+(x)$ and based on Eq. (1), deformation of points of inclusion can be expressed as

$$\varepsilon_x^{(1)} := \frac{du_1(x)}{dx} = \frac{1}{E(x)} \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \quad |x| < 1. \tag{2}$$

The condition of equilibrium of the inclusion has the form

$$\int_{-1}^{1} [\tau(t) - \tau_0(t)]dt = 0. \tag{3}$$

Assuming that every element of the glue layer is under the conditions of pure shear, the contact condition has the form [2]

$$u_1(x) - u_2(x, 0) = k_0\tau(x), \quad |x| \le 1, \tag{4}$$

where $u_2(x, y)$ are displacement of the plate points along the ox-axis, $k_0 := h_0/G_0$.

On the basis of the well-known results (see, e.g., [3]), the deformation $\varepsilon_x^{(2)} := \frac{du_2(x,0)}{dx}$ of the plane point along the ox-axis caused by the force factors $\tau(x)$, p and q is represented in the form

$$\varepsilon_x^{(2)} = \frac{\aleph}{2\pi\mu_2(1+\aleph)} \int_{-1}^{1} \frac{\tau(t)dt}{t-x} + \frac{\aleph+1}{8\mu_2} p + \frac{\aleph-3}{8\mu_2} q, \tag{5}$$

where $\aleph = 3 - 4\nu_2$, while λ_2 and μ_2 are the Lamé parameters.

Taking into account (2) and (5), from the contact conditions (4), we get

$$\frac{1}{E(x)} \int_{-1}^{x} [\tau(t)dt - \tau_0(t)]dt - \frac{\aleph}{2\pi\mu_2(1+\aleph)} \int_{-1}^{1} \frac{\tau(t)dt}{t-x} - \frac{\aleph+1}{8\mu_2} p - \frac{\aleph-3}{8\mu_2} q = k_0\tau'(x), \quad |x| < 1. \tag{6}$$

In the notations

$$\varphi(x) = \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \quad \lambda = \frac{\aleph}{2\mu_2(1+\aleph)},$$
$$g(x) = \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\tau_0(t)dt}{t-x} + k_0\tau_0'(x) + \frac{\aleph+1}{8\mu_2} p + \frac{\aleph-3}{8\mu_2} q,$$

we rewrite Eq. (6) in the form

$$\frac{\varphi(x)}{E(x)} - \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) = g(x), \quad |x| < 1. \tag{7}$$

Thus the equilibrium condition (3) takes the form

$$\varphi(1) = 0. \tag{8}$$

Thus the above posed boundary contact problem is reduced to the solution of SIDE (7) with the condition (8). From the symmetry of the problem, we assume, that function $E(x)$ is even and external load $\tau_0(x)$ is uneven, the solution of Eq. (7) under the condition (8) can be sought in the class of even functions. Moreover, we assume that the function is continuous and has a continuous first order derivative on the interval $[-1, 1]$.

2. Asymptotic investigation

Under the assumption that

$$E(x) = (1 - x^2)^{\gamma} b_0(x), \quad \gamma \ge 0, \quad b_0(x) = b_0(-x), \quad b_0 \in C([-1, 1]), \tag{9}$$
$$b_0(x) \ge c_0 = \text{const} > 0$$

a solution of problem (7), (8) will be sought in the class of even functions whose derivatives are representable in the form

$$\varphi'(x) = (1-x^2)^{\alpha} g_0(x), \quad \alpha > -1, \tag{10}$$

where $g_0(x) = -g_0(-x)$, $g_0 \in C'([-1,1])$, $g_0(x) \neq 0$, $x \in [-1,1]$.

Taking into account the following asymptotic formulas [4], for $-1 < \alpha < 0$, we have

$$\int_{-1}^{1} \frac{(1-t^2)^{\alpha} g_0(t)dt}{t-x} = \mp\pi \ \operatorname{ctg} \pi\alpha \, g_0(\mp 1) 2^{\alpha} (1 \pm x)^{\alpha} + \Phi_{\mp}(x), \quad x \to \mp 1,$$

where $\Phi_{\mp}(x) = \Phi^*_{\mp}(x)(1 \pm x)^{\alpha_{\mp}}$, $\Phi^*_{\mp}$ belongs to the class H in the neighbourhoods of the points $x = \mp 1$, $\alpha_{\mp} = \text{const} > \alpha$;

If $\alpha = 0$, we have

$$\int_{-1}^{1} \frac{g_0(t)dt}{t-x} = \mp g_0(\mp 1) \ln(1 \pm x) + \widetilde{\Phi}_{\mp}(x), \quad x \to \mp 1,$$

where $\widetilde{\Phi}_{\pm}(x)$ satisfies the H condition in the neighbourhoods of the points $x = \mp 1$, respectively.

If $\alpha > 0$, the function $\Phi_0(x) := \int_{-1}^{1} \frac{(1-t^2)^{\alpha} g_0(t)dt}{t-x}$ belongs to the class H in the neighbourhoods of the points $x = \pm 1$. Moreover, we have [5]

$$\int_{-1}^{x} (1-t^2)^{\alpha} g_0(t)dt = \frac{2^{\alpha}(1 \pm x)^{\alpha+1}}{\alpha+1} g_0(\mp 1) F(\alpha+1, \ -\alpha, \ 2+\alpha, \ (1 \pm x)/2) + G_{\mp}(x), \quad x \to \mp 1,$$

where $F(a,b,c,x)$ is the Gaussian hypergeometric function, $\lim_{x \to \mp 1} G_{\mp}(x)(1 \pm x)^{\alpha+1} = 0$.

In the case of the condition $-1 < \alpha < 0$, Eq. (7) in the neighbourhoods of the points $x = -1$ takes the form

$$\begin{aligned} &\lambda \operatorname{ctg} \pi\alpha \, g_0(-1) 2^{\alpha} (1+x)^{\alpha} - \frac{\lambda}{\pi} \Phi_-(x) + \frac{2^{\alpha}(1+x)^{\alpha+1} g_0(-1)}{2^{\gamma}(\alpha+1)(1+x)^{\gamma} b_0(-1)} + G_-(x) \\ &\quad - k_0 2^{\alpha} (1+x)^{\alpha-1} \tilde{g}_0(-1) = g(-1), \quad \tilde{g}_0(x) = (1-x^2) g_0'(x) - 2x g_0(x) \end{aligned}$$

which in the neighbourhoods of the points $x = -1$ is not satisfied. In the condition $-1 < \alpha < 0$, Eq. (7) has no solutions. Note, that the negative value of the index α contradicts the physical meaning of condition (4).

Let $0 \le \alpha \le 1$, then we have

$$\begin{aligned} &\frac{\lambda}{\pi} g_0(-1) \ln(1+x) - \frac{\lambda}{\pi} \widetilde{\Phi}_-(x) + \frac{(1+x) g_0(-1)}{2^{\gamma}(1+x)^{\gamma} b_0(-1)} + G_-(x) \\ &\quad - k_0 (1+x)^{-1} \tilde{g}_0(-1) = g(-1), \end{aligned} \tag{11}$$

for $\alpha = 0$, and

$$-\frac{\lambda}{\pi} \Phi_0(x) + \frac{2^{\alpha}(1+x)^{\alpha+1} g_0(-1)}{2^{\gamma}(\alpha+1)(1+x)^{\gamma} b_0(-1)} + G_-(x) - k_0 2^{\alpha} (1+x)^{\alpha-1} \tilde{g}_0(-1) = g(-1) \tag{12}$$

for $0 < \alpha \le 1$.

Multiplying now both sides of relations (11) $(1+x)^{1+\varepsilon}$ and (12) by $(1+x)^{1+\varepsilon-\alpha}$ (ε is an arbitrarily small positive number), we obtain

$$\begin{aligned} &\lambda g_0(-1)(1+x)^{1+\varepsilon} \ln(1+x) - \frac{\lambda}{\pi} (1+x)^{1+\varepsilon} \widetilde{\Phi}_-(x) \\ &\quad + \frac{(1+x)^{2+\varepsilon} g_0(-1)}{2^{\gamma}(1+x)^{\gamma} b_0(-1)} + G_-(x)(1+x)^{1+\varepsilon} - k_0 2^{\alpha} (1+x)^{\varepsilon} \tilde{g}_0(-1) \\ &= g(-1)(1+x)^{1+\varepsilon} \end{aligned}$$

and

$$-\frac{\lambda}{\pi}(1+x)^{1+\varepsilon-\alpha}\Phi_0(x)+\frac{2^{\alpha}(1+x)^{2+\varepsilon}g_0(-1)}{2^{\gamma}(\alpha+1)(1+x)^{\gamma}b_0(-1)}+G_-(x)(1+x)^{1+\varepsilon-\alpha}$$
$$-k_0 2^{\alpha}(1+x)^{\varepsilon}\tilde{g}_0(-1)=g(-1)(1+x)^{1+\varepsilon-\alpha}.$$

When passing to the limit $x\to -1$, analysis of the obtained equalities shows that the inequality $2+\varepsilon>\gamma$, i.e. $\gamma\le 2$, needs to be fulfilled.

If $\alpha>1$, then from relation (12) it follows that $\alpha=\gamma-1$.

Analogous result is obtained in the neighbourhoods of the points $x=1$.

Thus we have proved the following statement: When fulfilling condition (9), if problem (7), (8) has a solution whose derivative is representable in the form (10), then we have: if $\gamma>2$, then $\alpha=\gamma-1$, $(\alpha>1)$; if $\gamma\le 2$, then $0\le\alpha\le 1$.

From the relation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{(1-t)^{\alpha}(1+t)^{\beta}P_m^{(\alpha,\beta)}(t)dt}{t-x}=\operatorname{ctg}\pi\alpha(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)$$
$$-\frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+m+1)}{\pi\Gamma(\alpha+\beta+m+1)}F(m+1,\ -\alpha-\beta-m,\ 1-\alpha,\ (1-x)/2)$$

obtained by Tricomi [6] for orthogonal Jacobi polynomials $P_m^{(\alpha,\beta)}$ and from the well-known equality (see, e.g., [7])

$$m!P_m^{(\alpha,\beta)}(1-2x)=\frac{\Gamma(\alpha+m+1)}{\Gamma(1+\alpha)}F(\alpha+\beta+m+1,\ -m,\ 1+\alpha,\ x)$$

we get the following spectral relation for the Hilbert singular operator

$$\int_{-1}^{1}\frac{(1-t^2)^{n-1/2}P_m^{(n-1/2,n-1/2)}(t)dt}{t-x}=-2^{2n-1}\Gamma(n-1/2)\Gamma(3/2-n)P_{m+2n-1}^{(1/2-n,1/2-n)}(x),\tag{13}$$

where $\Gamma(z)$ is the known Gamma function.

If the inclusion rigidity varies by the law

$$E(x)=(1-x^2)^{n+\frac{1}{2}}b_0(x),$$

where $b_0(x)>0$ for $|x|\le 1$, $b_0(x)=b_0(-x)$, $n\ge 0$ is integer, then following from the above asymptotic analysis, we obtain $\alpha=n-\frac{1}{2}$ for $n=2,3,\ldots$ and $0<\alpha<1$ for $n=0$ or $n=1$ (the same result is obtained for $E(x)=b_0(x)>0$, or $E(x)=\text{const}$, $|x|\le 1$).

3. An approximate solution of SIDE (7)

On the basis of the above asymptotic analysis performed in the cases $n=0$, $n=1$, $E(x)=b_0(x)>0$, $E(x)=\text{const}$, $|x|\le 1$ a solution of Eq. (7) will be sought in the form

$$\varphi'(x)=\sqrt{1-x^2}\sum_{k=1}^{\infty}X_kP_k^{(1/2,1/2)}(x),\tag{14}$$

where the numbers X_k have to be defined, $k=1,2,\ldots$.

Using the relations arising from (13) and from the Rodrigue formula (see [8, p. 107]), for the orthogonal Jacobi polynomials, we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}P_k^{(1/2,1/2)}(t)dt}{t-x}=-2\pi P_{k+1}^{(-1/2,-1/2)}(x),$$

$$\varphi(x)=-(1-x^2)^{3/2}\sum_{k=1}^{\infty}\frac{X_k}{2k}P_{k-1}^{(3/2,3/2)}(x),\qquad \varphi''(x)=-2(1-x^2)^{-1/2}\sum_{k=1}^{\infty}kX_kP_{k+1}^{(-1/2,-1/2)}(x).\tag{15}$$

Substituting relations (14), (15) into Eq. (7), we have

$$-\frac{(1-x^2)^{3/2}}{E(x)}\sum_{r=1}^{\infty}\frac{X_k}{2k}P_{k-1}^{(3/2,3/2)}(x)-2\lambda\sum_{k=1}^{\infty}X_kP_{k+1}^{(-1/2,-1/2)}(x)$$
$$+2k_0(1-x^2)^{-1/2}\sum_{k=1}^{\infty}kX_kP_{k+1}^{(-1/2,-1/2)}(x)=g(x),\quad |x|\le 1. \tag{16}$$

Multiplying both parts of equality (16) by $P_{m+1}^{(-1/2,-1/2)}(x)$ and integrating in the interval $(-1,1)$, we obtain an infinite system of linear algebraic equations of the type

$$k_0m\left(\frac{\Gamma(m+3/2)}{\Gamma(m+2)}\right)^2X_m-\sum_{k=1}^{\infty}\left(R_{mk}^{(1)}+\frac{R_{mk}^{(2)}}{k}\right)X_k=g_m,\quad m=1,2,\ldots, \tag{17}$$

where

$$R_{mk}^{(1)}=-2\lambda\int_{-1}^{1}P_{k+1}^{(-1/2,-1/2)}(x)P_{m+1}^{(-1/2,-1/2)}(x)dx,$$
$$R_{mk}^{(2)}=\frac{1}{2}\int_{-1}^{1}\frac{(1-x^2)^{3/2}}{E(x)}P_{k-1}^{(3/2,3/2)}(x)P_{m+1}^{(-1/2,-1/2)}(x)dx,$$
$$g_m=\int_{-1}^{1}g(x)P_{m+1}^{(-1/2,-1/2)}(x)dx.$$

Investigating system (17) for regularity in the class of bounded sequences and using the known relations for the Chebyshev first order polynomials and for the function $\Gamma(z)$ (see [5, pp. 584, 83]),

$$P_m^{(-1/2,-1/2)}(x)=\frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}T_m(x),\qquad T_m(\cos\theta)=\cos m\theta$$
$$\lim_{m\to\infty}m^{b-a}\frac{\Gamma(m+a)}{\Gamma(m+b)}=1,$$

we obtain

$$R_{mk}^{(1)}=-\frac{2\lambda\alpha(k)\beta(m)}{\pi\sqrt{(k+1)m+1}}\int_0^{\pi}\cos(k+1)\theta\cos(m+1)\theta\sin\theta d\theta$$
$$=-\frac{2\lambda\alpha(k)\beta(m)}{\pi\sqrt{(k+1)(m+1)}}$$
$$\times\begin{cases}1-\dfrac{1}{(2m+3)(2m+1)}, & k=m,\\ -\dfrac{(-1)^{k+m}+1}{2}\left[\dfrac{1}{(k+m+3)(k+m+1)}+\dfrac{1}{(k-m+1)(k-m-1)}\right], & k\ne m,\end{cases}$$
$$=\begin{cases}O(m^{-1}), & k=m,\quad m\to\infty,\\ O(m^{-5/2}),\quad O(k^{-5/2}), & k\ne m,\quad m\to\infty,\quad k\to\infty,\end{cases}$$

$\alpha(k),\beta(m)\to 1$ for $k,m\to\infty$. Introducing the notation $\widetilde{X}_m=\omega_mX_m$, where $\omega_m=m\left(\frac{\Gamma(m+3/2)}{\Gamma(m+2)}\right)^2\to 1, m\to\infty$, system (17) will take the form

$$k_0\widetilde{X}_m-\sum_{k=1}^{\infty}\left(\frac{R_{mk}^{(1)}}{\omega_k}+\frac{R_{mk}^{(2)}}{\omega_kk}\right)\widetilde{X}_k=g_m,\quad m=1,2,\ldots. \tag{18}$$

By virtue of the Darboux asymptotic formula (see [8, p. 175]), we obtain analogous estimates likewise for $R^{(2)}_{mk}$, and the right-hand side g_m of Eq. (18) satisfies at least the estimate

$$g_m = O(m^{1/2}), \quad m \to \infty.$$

However, if $n = 2$, a solution of Eq. (7) will be sought in the form

$$\varphi'(x) = (1 - x^2)^{3/2} \sum_{k=1}^{\infty} Y_k P_k^{(3/2,3/2)}(x), \tag{19}$$

where the numbers Y_k are to be defined, $k = 1, 2, \ldots$.

Using the relations arising from (13) and from the Rodrigue formula for the orthogonal Jacobi polynomials, we get

$$\frac{1}{\pi}\int_{-1}^{1} \frac{(1-x^2)^{3/2} P_k^{(3/2,3/2)}(t)dt}{t-x} = -2\pi P_{k+1}^{\left(-3/2,-3/2\right)}(x),$$

$$\varphi(x) = -(1-x^2)^{5/2} \sum_{k=1}^{\infty} \frac{Y_k}{2k} P_{k-1}^{(5/2,5/2)}(x), \quad \varphi''(x) = -2(1-x^2)^{1/2} \sum_{k=1}^{\infty} kY_k P_{k+1}^{(1/2,1/2)}(x). \tag{20}$$

Substituting relations (19), (20) into Eq. (7) we obtain

$$-\frac{1}{b_0(x)} \sum_{r=1}^{\infty} \frac{Y_k}{2k} P_{k-1}^{(5/2,5/2)}(x) - \frac{2\lambda \Gamma^2\left(1/2\right)}{\pi} \sum_{k=1}^{\infty} Y_k P_{k+1}^{(-3/2,-3/2)}(x)$$
$$+ 2k_0(1-x^2)^{1/2} \sum_{k=1}^{\infty} kY_k P_{k+1}^{(1/2,1/2)}(x) = g(x), \quad |x| \le 1. \tag{21}$$

Reasoning analogous to that carried out for system (18), from (21) we obtain

$$4k_0 m\left(\frac{\Gamma(m+5/2)}{\Gamma(m+3)}\right)^2 Y_m - \sum_{k=1}^{\infty}\left(R^{(3)}_{mk} + \frac{R^{(4)}_{mk}}{k}\right) Y_k = \tilde{g}_m, \quad m = 1, 2, \ldots, \tag{22}$$

where

$$R^{(3)}_{mk} = -2\lambda \int_{-1}^{1} P_{k+1}^{(-3/2,-3/2)}(x) P_{m+1}^{(1/2,1/2)}(x)dx,$$
$$R^{(4)}_{mk} = \frac{1}{2}\int_{-1}^{1} \frac{1}{b_0(x)} P_{k-1}^{(5/2,5/2)}(x)dx\, P_{m+1}^{(1/2,1/2)}(x)dx,$$
$$\tilde{g}_m = \int_{-1}^{1} g(x) P_{m+1}^{(1/2,1/2)}(x)dx.$$

Introducing the notation $\widetilde{Y}_m = \delta_m Y_m$, where $\delta_m = m\left(\frac{\Gamma(m+5/2)}{\Gamma(m+3)}\right)^2 \to 1$, $m \to \infty$, system (22) will take the form

$$4k_0 \widetilde{Y}_m - \sum_{k=1}^{\infty}\left(\frac{R^{(1)}_{mk}}{\delta_k} + m\frac{R^{(2)}_{mk}}{\delta_k k}\right)\widetilde{Y}_k = \widetilde{g}_m, \quad m = 1, 2, \ldots. \tag{23}$$

Using again the Darboux formula, and the known relation for the Chebyshev second order polynomial (see [5, p. 584])

$$P_m^{(1/2,1/2)}(x) = \frac{\Gamma(m+3/2)}{\sqrt{\pi}\,\Gamma(m+2)} U_m(x), \quad U_m(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

for $R^{(3)}_{mk}$ and $R^{(4)}_{mk}$, we obtain the following estimates:

$$R^{(3)}_{mk} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-5/2}), \quad O(k^{-5/2}), & k \neq m, \quad m \to \infty, \quad k \to \infty, \end{cases}$$

$$R^{(4)}_{mk} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-1/2}), \quad O(k^{-1/2}), & k \neq m, \quad m \to \infty, \quad k \to \infty, \end{cases}$$

and for the right-hand side $\tilde{g}_m$ of Eq. (23) we have at least the estimate

$$\tilde{g}_m = O(m^{-1/2}), \quad m \to \infty.$$

Thus systems (18) and (23) are quasi-completely regular for any positive values of parameters k_0 and λ in the class of bounded sequences.

On the basis of the Hilbert alternatives [9,10], if the determinants of the corresponding finite systems of linear algebraic equations are other than zero, then systems (18) and (23) will have unique solutions in the class of bounded sequences. Therefore, by the equivalence of systems (18), (23) and SIDE (7) the latter has likewise a unique solution.

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