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One nonlocal problem in time for a semilinear multidimensional wave equation

Sergo Kharibegashvili [a,b] **and Bidzina Midodashvili** [c,d]

[a] A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, 6, Tamarashvili str., Tbilisi 0177, Georgia
[b] Department of Mathematics, Georgian Technical University, 77, M. Kostava str., Tbilisi 0175, Georgia
[c] Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, 13, University str., Tbilisi 0143, Georgia
[d] Faculty of Education, Exact and Natural Sciences, Gori Teaching University, 5, I. Chavchavadze str., Gori, Georgia
(e-mail: kharibegashvili@yahoo.com; bidmid@hotmail.com)

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Abstract. We consider a nonlocal problem in time for semilinear multidimensional wave equations and prove theorems on existence, uniqueness, and nonexistence of solutions.

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1 Introduction

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t, in the cylindrical domain $D_T = \Omega \times (0, T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, we consider a nonlocal problem of finding a solution $u(x, t)$ of the equation

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda f(x, t, u) = F(x, t), \quad (x, t) \in D_T, \tag{1.1}$$

satisfying the Dirichlet homogeneous boundary condition

$$u|_\Gamma = 0 \tag{1.2}$$

on the lateral face $\Gamma := \partial\Omega \times (0, T)$ of the cylinder D_T and the homogeneous nonlocal conditions

$$K_\mu u := u(x, 0) - \mu u(x, T) = 0, \quad x \in \Omega, \tag{1.3}$$

$$K_\mu u_t := u_t(x, 0) - \mu u_t(x, T) = 0, \quad x \in \Omega, \tag{1.4}$$

where f and F are given functions, λ and μ are given nonzero constants, and $n \geqslant 2$.

Remark 1. Many papers are devoted to nonlocal problems for partial differential equations. Nonlocal problems posed for abstract evolution equations and hyperbolic partial differential equations are considered in the works [1, 2, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 25] and the references therein. Note that, for $|\mu| \neq 1$, it suffices to consider the case $|\mu| < 1$ since the case $|\mu| > 1$ can be reduced to the latter by passing from variable t to variable $t' = T - t$. The case $|\mu| = 1$ is considered at the end of the work. Particularly, when $\mu = 1$, problem (1.1)–(1.4) can be considered as a periodic problem.

We further impose the following requirements on the function $f = f(x,t,u)$:

$$f \in C(\overline{D}_T \times \mathbb{R}), \qquad \big|f(x,t,u)\big| \leqslant M_1 + M_2|u|^\alpha, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{1.5}$$

where

$$0 \leqslant \alpha = \text{const} < \frac{n+1}{n-1}. \tag{1.6}$$

We consider the following functional spaces:

$$\overset{\circ}{C}{}^2_\mu(\overline{D}_T) := \big\{v \in C^2(\overline{D}_T)\colon\ v|_\Gamma = 0,\ K_\mu v = 0,\ K_\mu v_t = 0\big\}, \tag{1.7}$$

$$\overset{\circ}{W}{}^1_{2,\mu}(D_T) := \big\{v \in W^1_2(D_T)\colon\ v|_\Gamma = 0,\ K_\mu v = 0\big\}, \tag{1.8}$$

where $W^1_2(D_T)$ represents the known Sobolev space, and the equalities $v|_\Gamma = 0$, $K_\mu v = 0$ must be understood in the sense of the trace theory [19].

Remark 2. The embedding operator $I : \overset{\circ}{W}{}^1_2(D_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < 2(n+1)/(n-1)$ when $n > 1$ [19]. At the same time, the Nemitski operator $N : L_q(D_T) \to L_2(D_T)$, acting by the formula $Nu = f(x,t,u)$, is continuous by (1.5) and bounded if $q \geqslant 2\alpha$ [18]. Thus, since by (1.6) we have $2\alpha < 2(n+1)/(n-1)$, there exists a number q such that $1 < q < 2(n+1)/(n-1)$ and $q \geqslant 2\alpha$. Therefore, in this case the operator

$$N_0 = NI\colon \quad \overset{\circ}{W}{}^1_{2,\mu}(D_T) \to L_2(D_T) \tag{1.9}$$

is continuous and compact. Besides, from $u \in \overset{\circ}{W}2,\mu^1(D_T)$ it follows that $f(x,t,u) \in L_2(D_T)$ and that if $u_m \to u$ in the space $\overset{\circ}{W}2,\mu^1(D_T)$, then $f(x,t,u_m) \to f(x,t,u)$ in the space $L_2(D_T)$.

DEFINITION 1. Let function f satisfy conditions (1.5) and (1.6), and $F \in L_2(D_T)$. We call a function u a generalized solution of problem (1.1)–(1.4) if $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0, \qquad \lim_{m\to\infty} \|L_\lambda u_m - F\|_{L_2(D_T)} = 0. \tag{1.10}$$

Note that this definition of a generalized solution of problem (1.1)–(1.4) also remains in the linear case, that is, for $\lambda = 0$.

It is obvious that a classical solution $u \in C^2(\overline{D}_T)$ of problem (1.1)–(1.4) represents a generalized solution of this problem. It is easy to verify that a generalized solution of problem (1.1)–(1.4) is a solution of Eq. (1.1) in the sense of the theory of distributions. Indeed, let $F_m := L_\lambda u_m$. Multiplying both sides of the equality $L_\lambda u_m = F_m$ by a test function $w \in V_\mu := \{v \in W^1_2(D_T)\colon\ v|_\Gamma = 0,\ v(x,T) - \mu v(x,0) = 0,\ x \in \Omega\}$ and integrating in the domain D_T, after simple transformations connected with integration by parts and the equality $w|_\Gamma = 0$, we get

$$\int_\Omega \big[u_{mt}(x,T)w(x,T) - u_{mt}(x,0)w(x,0)\big]\,\mathrm{d}x + \int_{D_T}\left[-u_{mt}w_t + \sum_{i=1}^n u_{mx_i}w_{x_i} + \lambda f(x,t,u_m)w\right]\mathrm{d}x\,\mathrm{d}t = \int_{D_T} F_m w\,\mathrm{d}x\,\mathrm{d}t \quad \forall w \in V_\mu. \tag{1.11}$$

Since $K_\mu u_{mt} = 0$ and $w(x,T) - \mu w(x,0) = 0$, $x \in \Omega$, it is easy to see that $u_{mt}(x,T)w(x,T) - u_{mt}(x,0) \times w(x,0) = u_{mt}(x,T)(w(x,T) - \mu w(x,0)) - w(x,0)(u_{mt}(x,0) - \mu u_{mt}(x,T)) = 0$. Therefore, Eq. (1.11) takes the form

$$\int_{D_T}\left[-u_{mt}w_t + \sum_{i=1}^n u_{mx_i}w_{x_i} + \lambda f(x,t,u_m)w\right]\mathrm{d}x\,\mathrm{d}t = \int_{D_T} F_m w\,\mathrm{d}x\,\mathrm{d}t \quad \forall w \in V_\mu. \tag{1.12}$$

In view of (1.5), (1.6), and Remark 2, we have $f(x,t,u_m) \to f(x,t,u)$ in the space $L_2(D_T)$ as $u_m \to u$ in the space $\mathring{W}^1_{2,\mu}(D_T)$. Therefore, by (1.10), passing to the limit in Eq. (1.12) as $m \to \infty$, we get

$$\int_{D_T}\left[-u_t w_t + \sum_{i=1}^n u_{x_i}w_{x_i} + \lambda f(x,t,u)w\right]\mathrm{d}x\,\mathrm{d}t = \int_{D_T} F w\,\mathrm{d}x\,\mathrm{d}t \quad \forall w \in V_\mu. \tag{1.13}$$

Since $C_0^\infty(D_T) \subset V_\mu$, from (1.13), integrating by parts, we have

$$\int_{D_T} u\,\Box w\,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{D_T} f(x,t,u)w\,\mathrm{d}x\,\mathrm{d}t = \int_{D_T} F w\,\mathrm{d}x\,\mathrm{d}t \quad \forall w \in C_0^\infty(D_T), \tag{1.14}$$

where $\Box := \partial^2/\partial t^2 - \sum_{i=1}^n \partial^2/\partial x_i^2$, and $C_0^\infty(D_T)$ is the space of finite infinitely differentiable functions in D_T. Equality (1.14), which is valid for any $w \in C_0^\infty(D_T)$, means that a generalized solution u of problem (1.1)–(1.4) is a solution of Eq. (1.1) in the sense of the theory of distributions. Besides, since the trace operators $u \to u|_{t=0}$ and $u \to u|_{t=T}$ are continuous operators acting from the space $W_2^1(D_T)$ into the spaces $L_2(\Omega \times \{t=0\})$ and $L_2(\Omega \times \{t=T\})$, respectively, then by (1.10) the generalized solution u of problem (1.1)–(1.4) satisfies the nonlocal condition (1.3) in the sense of the trace theory. As for the nonlocal condition (1.4), it is taken into account in the integral sense in Eq. (1.13), which is valid for all $w \in V_\mu$. Note also that if a generalized solution u belongs to the class $C^2(\overline{D}_T)$, then by the standard reasoning, combined with the integral identity (1.13) [19], we have that u is a classical solution of problem (1.1)–(1.4), satisfying pointwise Eq. (1.1), the boundary condition (1.2), and the nonlocal conditions (1.3) and (1.4).

Remark 3. Note that even in the linear case, that is, for $\lambda = 0$, problem (1.1)–(1.4) is not always well posed. For example, when $\lambda = 0$ and $|\mu| = 1$, the corresponding to (1.1)–(1.4) homogeneous problem may have an infinite number of linearly independent solutions (see Remark 6).

The work is organized in the following way. In Section 2, we single out the class of semilinear equations (1.1) when, for $|\mu| < 1$, an a priori estimate for the generalized solution of problem (1.1)–(1.4) is valid. In Section 3, on the basis of the a priori estimate obtained in the previous section, we prove the solvability of problem (1.1)–(1.4). In Section 4, we consider the conditions imposed on the data of the problem that ensure the uniqueness of the solution of this problem. In Section 5, using the method of test functions, we show that when the conditions imposed on the nonlinear term in Eq. (1.1) are violated, problem (1.1)–(1.4) may not have a solution. Finally, in the last section, we consider the case $|\mu| = 1$ as an application of the results obtained in the previous sections.

2 A priori estimate of the solution of problem (1.1)–(1.4)

Let

$$g(x,t,u) = \int\limits_0^u f(x,t,s)\,\mathrm{d}s, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}. \tag{2.1}$$

Consider the following conditions imposed on the function $g = g(x,t,u)$:

$$g(x,t,u) \geqslant 0, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{2.2}$$

$$g_t \in C(\overline{D}_T \times \mathbb{R}), \qquad g_t(x,t,u) \leqslant M_3, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{2.3}$$

$$g(x,0,\mu u) \leqslant \mu^2 g(x,T,u), \quad (x,u) \in \overline{\Omega} \times \mathbb{R}, \tag{2.4}$$

where $M_3 = \text{const} \geqslant 0$, and μ is the fixed constant from (1.3)– (1.4).

Remark 4. Let us consider the class of functions f from (1.1) satisfying conditions (1.5), (2.2), (2.3), and (2.4). For $\alpha = \beta + 1$, consider the function $f = f_0(t)|u|^\beta u$, where $f_0 \in C^1([0,T])$, $f_0 \geqslant 0$, $\mathrm{d}f_0/\mathrm{d}t \leqslant 0$, $f_0(0)\mu^\beta \leqslant f_0(T)$, $\beta \geqslant 0$, and $\mu > 0$ is the fixed constant from (1.3)–(1.4). In particular, these conditions are satisfied if $f_0 = \text{const} > 0$ and $0 < \mu \leqslant 1$. Indeed, with these conditions, by (2.1) we have: $g = f_0(t)|u|^{\beta+2}/(\beta+2)$, $g \geqslant 0$, $g_t \leqslant 0$, and $g(x,0,\mu v) = f_0(0)|\mu v|^{\beta+2}/(\beta+2) = \mu^2(f_0(0)\mu^\beta)|v|^{\beta+2}/(\beta+2) \leqslant \mu^2 f_0(T)(|v|^{\beta+2})/(\beta+2) = \mu^2 g(x,T,v)$.

Lemma 1. *Let $\lambda > 0$, $|\mu| < 1$, $f \in C(\overline{D}_T \times \mathbb{R})$, $F \in L_2(D_T)$, and conditions* (2.2)–(2.4) *be satisfied. Then, for a generalized solution u of problem* (1.1)–(1.4)*, we have the a priori estimate*

$$\|u\|_{\mathring{W}^1_{2,\mu}(D_T)} \leqslant c_1\|F\|_{L_2(D_T)} + c_2 \tag{2.5}$$

with nonnegative constants $c_i = c_i(\lambda,\mu,\Omega,T,M_1,M_2,M_3)$ not depending on u and $F, c_1 > 0$, whereas in the linear case ($\lambda = 0$), the constant $c_2 = 0$, and in this case, by (2.5) *we have the uniqueness of the generalized solution of problem* (1.1)–(1.4)*.*

Proof. Let u be a generalized solution of problem (1.1)–(1.4). By Definition 1 there exists a sequence of functions $u_m \in \mathring{C}^2_\mu(D_T)$ such that the limit equalities (1.10) are satisfied.

Set

$$L_\lambda u_m = F_m, \quad (x,t) \in D_T. \tag{2.6}$$

Multiplying both sides of Eq. (2.6) by $2u_{mt}$ and integrating in the domain $D_\tau := D_T \cap \{t < \tau\}$, $0 < \tau \leqslant T$, by (2.1) we obtain

$$\begin{aligned}&\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - 2\int\limits_{D_\tau}\sum_{i=1}^n \frac{\partial^2 u_m}{\partial x_i^2}\frac{\partial u_m}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\lambda\int\limits_{D_\tau}\frac{\mathrm{d}}{\mathrm{d}t}\big(g(x,t,u_m(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t - 2\lambda\int\limits_{D_\tau} g_t\big(x,t,u_m(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int\limits_{D_\tau} F_m\frac{\partial u_m}{\partial t}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.7}$$

Let $\omega_\tau := \{(x,t) \in \overline{D}_T\colon\ x \in \Omega,\ t = \tau\}$, $0 \leqslant \tau \leqslant T$, where ω_0 and ω_T are upper and lower bases of the cylindrical domain D_T, respectively. Denote by $\nu := (\nu_{x_1}, \nu_{x_2}, \dots, \nu_{x_n}, \nu_t)$ the unit vector of the outer normal to ∂D_τ. Since

$$\nu_{x_i}|_{\omega_\tau \cup \omega_0} = 0, \quad i = 1, \dots, n,$$
$$\nu_t|_{\Gamma_\tau := \Gamma \cap \{t \leqslant \tau\}} = 0, \qquad \nu_t|_{\omega_\tau} = 1, \qquad \nu_t|_{\omega_0} = -1,$$

taking into account that $u_m \in \mathring{C}^2_\mu(D_T)$ and, therefore, by (1.7)

$$u_m|_\Gamma = 0, \qquad K_\mu u_m = 0, \qquad K_\mu u_{mt} = 0, \tag{2.8}$$

integrating by parts, we obtain

$$\begin{aligned}
\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t &= \int\limits_{\partial D_\tau} \left(\frac{\partial u_m}{\partial t}\right)^2 \nu_t\,\mathrm{d}s \\
&= \int\limits_{\omega_\tau} u_{mt}^2\,\mathrm{d}x - \int\limits_{\omega_0} u_{mt}^2\,\mathrm{d}x - 2\int\limits_{D_\tau} \frac{\partial^2 u_m}{\partial x_i^2}\frac{\partial u_m}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \qquad (2.9) \\
&= \int\limits_{D_\tau} \left[\left(u_{mx_i}^2\right)_t - 2(u_{mx_i}u_{mt})_{x_i}\right] \mathrm{d}x\,\mathrm{d}t \\
&= \int\limits_{\omega_\tau} u_{mx_i}^2\,\mathrm{d}x - \int\limits_{\omega_0} u_{mx_i}^2\,\mathrm{d}x, \quad i = 1, \dots, n, \qquad (2.10)
\end{aligned}$$

$$\begin{aligned}
2\lambda \int\limits_{D_\tau} \frac{d}{dt}\big(g(x,t,u_m(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t &= 2\lambda \int\limits_{\partial D_\tau} g\big(x,t,u_m(x,t)\big)\nu_t\,\mathrm{d}s \\
&= 2\lambda \int\limits_{\omega_\tau} g\big(x,t,u_m(x,t)\big)\,\mathrm{d}x - 2\lambda \int\limits_{\omega_0} g\big(x,t,u_m(x,t)\big)\,\mathrm{d}x. \qquad (2.11)
\end{aligned}$$

In view of (2.9)–(2.11), from (2.7) we get

$$\begin{aligned}
\int\limits_{\omega_\tau} \left[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\right] \mathrm{d}x &= \int\limits_{\omega_0} \left[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\right] \mathrm{d}x \\
&\quad - 2\lambda \int\limits_{\omega_\tau} g\big(x,t,u_m(x,t)\big)\,\mathrm{d}x + 2\lambda \int\limits_{\omega_0} g\big(x,t,u_m(x,t)\big)\,\mathrm{d}x \\
&\quad + 2\lambda \int\limits_{D_\tau} g_t\big(x,t,u_m(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t + 2\int\limits_{D_\tau} F_m u_{mt}\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.12)
\end{aligned}$$

Let

$$w_m(\tau) := \int\limits_{\omega_\tau} \left[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 + 2\lambda g\big(x,t,u_m(x,t)\big)\right] \mathrm{d}x. \tag{2.13}$$

Since $2F_m u_{mt} \leqslant \epsilon^{-1} F_m^2 + \epsilon u_{mt}^2$ for any $\epsilon = \text{const} > 0$ and since $\lambda > 0$, by (2.3) and (2.13) from (2.12) it follows that

$$\begin{aligned} w_m(\tau) &= w_m(0) + 2\lambda \int_{D_\tau} g_t\big(x, t, u_m(x,t)\big)\, \mathrm{d}x\, \mathrm{d}t + 2 \int_{D_\tau} F_m u_{mt}\, \mathrm{d}x\, \mathrm{d}t \\ &\leqslant w_0(0) + 2\lambda M_3 \tau \operatorname{mes} \Omega + \epsilon \int_{D_\tau} u_{mt}^2\, \mathrm{d}x\, \mathrm{d}t + \epsilon^{-1} \int_{D_\tau} F_m^2\, \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{2.14}$$

Since $\lambda > 0$, taking into account and (2.2) and the inequality

$$\begin{aligned} \int_{D_\tau} u_{mt}^2\, \mathrm{d}x\, \mathrm{d}t &= \int_0^\tau \bigg[\int_{\omega_s} u_{mt}^2\, \mathrm{d}x \bigg] \mathrm{d}s \\ &\leqslant \int_0^\tau \bigg[\int_{\omega_s} \Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 + 2\lambda g\big(x, t, u_m(x,t)\big) \Big] \mathrm{d}x \bigg] \mathrm{d}s \\ &= \int_0^\tau w_m(s)\, \mathrm{d}s, \end{aligned}$$

from (2.14) we obtain

$$w_m(\tau) \leqslant \epsilon \int_0^\tau w_m(s)\, \mathrm{d}s + w_m(0) + 2\lambda M_3 \tau \operatorname{mes} \Omega + \epsilon^{-1} \int_{D_\tau} F_m^2\, \mathrm{d}x\, \mathrm{d}t, \quad 0 < \tau \leqslant T. \tag{2.15}$$

Because of $D_\tau \subset D_T$, $0 < \tau \leqslant T$, the right-hand side of inequality (2.15) is a nondecreasing function of variable τ, and by Gronwall's lemma [3] from (2.15) it follows that

$$w_m(\tau) \leqslant \bigg[w_m(0) + 2\lambda M_3 T \operatorname{mes} \Omega + \epsilon^{-1} \int_{D_T} F_m^2\, \mathrm{d}x\, \mathrm{d}t \bigg] \mathrm{e}^{\epsilon \tau}, \quad 0 < \tau \leqslant T. \tag{2.16}$$

In view of $\lambda > 0$, by (2.4) and (2.8) from (2.13) it follows that

$$\begin{aligned} w_m(0) &= \int_\Omega \bigg[u_{mt}^2(x,0) + \sum_{i=1}^n u_{mx_i}^2(x,0) + 2\lambda g\big(x, 0, u_m(x,0)\big) \bigg] \mathrm{d}x \\ &= \int_\Omega \bigg[\mu^2 u_{mt}^2(x,T) + \mu^2 \sum_{i=1}^n u_{mx_i}^2(x,T) + 2\lambda g\big(x, 0, \mu u_m(x,T)\big) \bigg] \mathrm{d}x \\ &\leqslant \mu^2 \int_\Omega \bigg[u_{mt}^2(x,T) + \sum_{i=1}^n u_{mx_i}^2(x,T) + 2\lambda g\big(x, T, u_m(x,T)\big) \bigg] \mathrm{d}x \\ &= \mu^2 w_m(T). \end{aligned} \tag{2.17}$$

Using inequality (2.16) for $\tau = T$, from (2.17) we obtain

$$w_m(0) \leqslant \mu^2 w_m(T) \leqslant \mu^2 \Biggl[w_m(0) + 2\lambda M_3 T \operatorname{mes} \Omega + \epsilon^{-1} \int\limits_{D_T} F_m^2 \,\mathrm{d}x\,\mathrm{d}t \Biggr] \mathrm{e}^{\epsilon T}$$
$$= \mu^2 \mathrm{e}^{\epsilon T} w_m(0) + M_4 + \mu^2 \epsilon^{-1} \mathrm{e}^{\epsilon T} \|F_m\|^2_{L_2(D_T)}, \tag{2.18}$$

where

$$M_4 := \mu^2 2\lambda M_3 T \mathrm{e}^{\epsilon T} \operatorname{mes} \Omega. \tag{2.19}$$

Since $|\mu| < 1$, a positive constant $\epsilon = \epsilon(\mu, T)$ can be chosen small enough so that

$$\mu_1 = \mu^2 \mathrm{e}^{\epsilon T} < 1. \tag{2.20}$$

For example, we can set $\epsilon = (1/T)\ln(1/|\mu|)$.
By (2.20) from (2.18) we have

$$w(0) \leqslant (1-\mu_1)^{-1} M_4 + (1-\mu_1)^{-1} \mu^2 \epsilon^{-1} \mathrm{e}^{\epsilon T} \|F_m\|^2_{L_2(D_T)}. \tag{2.21}$$

From (2.16) and (2.21) it follows that

$$w_m(\tau) \leqslant \bigl[(1-\mu_1)^{-1} M_4 + (1-\mu_1)^{-1} \mu^2 \epsilon^{-1} \mathrm{e}^{\epsilon T} \|F_m\|^2_{L_2(D_T)}$$
$$+ 2\lambda M_3 T \operatorname{mes} \Omega + \epsilon^{-1} \|F\|^2_{L_2(D_T)} \bigr] \mathrm{e}^{\epsilon T}$$
$$\leqslant \sigma_1 \|F_m\|^2_{L_2(D_T)} + \sigma_2, \quad 0 < \tau \leqslant T, \tag{2.22}$$

where

$$\sigma_1 = \bigl[(1-\mu_1)^{-1} \mu^2 \mathrm{e}^{\epsilon T} + 1 \bigr] \epsilon^{-1} \mathrm{e}^{\epsilon T}, \qquad \sigma_2 = \bigl[(1-\mu_1)^{-1} M_4 + 2\lambda M_3 T \operatorname{mes} \Omega \bigr] \mathrm{e}^{\epsilon T}. \tag{2.23}$$

Since, for fixed τ, the function $u_m(x,\tau)$ belongs to the space $\mathring{W}2^1(\Omega) := \{ v \in \mathring{W}2^1(\Omega) : v|_{\partial\Omega} = 0 \}$, by the Friedrichs inequality [19], taking into account (2.2) and $\lambda > 0$, we have

$$\int\limits_{\omega_\tau} \Biggl[u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \Biggr] \mathrm{d}x \leqslant c_0 \int\limits_{\omega_\tau} \Biggl[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \Biggr] \mathrm{d}x$$
$$\leqslant c_0 \int\limits_{\omega_\tau} \Biggl[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 + \lambda g\bigl(x,t,u_m(x,t)\bigr) \Biggr] \mathrm{d}x$$
$$= c_0 w_m(\tau), \tag{2.24}$$

where the positive constant $c_0 = c_0(\Omega)$ does not depend on u_m.
From (2.22) and (2.24) it follows

$$\|u_m\|^2_{\mathring{W}2,\mu^1(D_T)} = \int\limits_0^T \Biggl[\int\limits_{\omega_\tau} \Biggl(u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \Biggr) \mathrm{d}x \Biggr] \mathrm{d}\tau \leqslant c_0 \int\limits_0^T w_m(\tau)\,\mathrm{d}\tau$$
$$\leqslant c_0 \int\limits_0^T \bigl[\sigma_1 \|F\|^2_{L_2(D_T)} + \sigma_2 \bigr] \mathrm{d}\tau = c_0 \sigma_1 T \|F_m\|^2_{L_2(D_T)} + c_0 \sigma_2 T. \tag{2.25}$$

Taking the square root from the both sides of inequality (2.25) and using the inequality $(a^2+b^2)^{1/2} \leqslant |a|+|b|$, we get

$$\|u_m\|_{\overset{\circ}{W}2,\mu^1(D_T)} \leqslant c_1\|F_m\|_{L_2(D_T)} + c_2, \tag{2.26}$$

where

$$\begin{aligned} c_1 &= \left(c_0T\left[(1-\mu_1)^{-1}\mu^2 \mathrm{e}^{\epsilon T}+1\right]\epsilon^{-1}\mathrm{e}^{\epsilon T}\right)^{1/2},\\ c_2 &= \left(c_0T\left[(1-\mu_1)^{-1}\mu^2 2\lambda M_3 T\mathrm{e}^{\epsilon T}\operatorname{mes}\Omega + 2\lambda M_3 T\operatorname{mes}\Omega\right]\mathrm{e}^{\epsilon T}\right)^{1/2}. \end{aligned} \tag{2.27}$$

In view of the limit equalities (1.10), passing to the limit in inequality (2.26) as $m\to\infty$, we obtain (2.5). This proves Lemma 1. □

3 The existence of the solution of problem (1.1)–(1.4)

For the existence of the solution of problem (1.1)–(1.4) in the case $|\mu|<1$, we will use the well-known facts on the solvability of the following linear mixed problem [19]:

$$L_0u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \quad (x,t)\in D_T, \tag{3.1}$$

$$u|_\Gamma = 0, \qquad u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \quad x\in\Omega, \tag{3.2}$$

where F,φ, and ψ are given functions.

For $F\in L_2(D_T)$, $\varphi\in \overset{\circ}{W}{}_2^1(\Omega)$, and $\psi\in L_2(\Omega)$, the unique generalized solution u of problem (3.1), (3.2) (in the sense of the integral identity

$$-\int_\Omega \psi w(x,0)\,\mathrm{d}x + \int_{D_T}\left[-u_tw_t + \sum_{i=1}^n u_{x_i}w_{x_i}\right]\mathrm{d}x\,\mathrm{d}t = \int_{D_T} Fw\,\mathrm{d}x\,\mathrm{d}t \quad \forall w\in V_0,$$

where $V_0 := \{v\in W_2^1(D_T)\colon v|_\Gamma=0,\ v(x,T)=0,\ x\in\Omega\}$ and $u|_{t=0}=\varphi$) from the space $E_{2,1}(D_T)$ with the norm

$$\|v\|^2_{E_{2,1}(D_T)} = \sup_{0\leqslant\tau\leqslant T}\int_{\omega_\tau}\left[v^2+v_t^2+\sum_{i=1}^n v_{x_i}^2\right]\mathrm{d}x$$

is given by the formula [19]

$$u = \sum_{k=1}^\infty\left(\tilde{a}_k\cos\mu_k t + \tilde{b}_k\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_k(\tau)\sin\mu_k(t-\tau)\,\mathrm{d}\tau\right)\varphi_k(x), \tag{3.3}$$

where $\tilde\lambda_k = -\mu_k^2$ ($0<\mu_1\leqslant\mu_2\leqslant\cdots$, $\lim_{k\to\infty}\mu_k=\infty$) and $\varphi_k\in\overset{\circ}{W}{}_2^1(\Omega)$ are the eigenvalues and corresponding eigenfunctions of the spectral problem $\Delta w = \tilde\lambda w$, $w|_{\partial\Omega}=0$ in the domain Ω ($\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$), simultaneously forming an orthonormal basis in $L_2(\Omega)$ and an orthogonal basis in $\overset{\circ}{W}{}_2^1(\Omega)$ with respect to the scalar product $(v,w)_{\overset{\circ}{W}{}_2^1(\Omega)} = \int_\Omega\sum_{i=1}^n v_{x_i}w_{x_i}\,\mathrm{d}x$ [19], that is,

$$(\varphi_k,\varphi_l)_{L_2(\Omega)} = \delta_k^l, \quad (\varphi_k,\varphi_l)_{\overset{\circ}{W}{}_2^1(\Omega)} = -\tilde\lambda_k\delta_k^l, \quad \delta_k^l = \begin{cases}1, & l=k,\\ 0, & l\neq k.\end{cases} \tag{3.4}$$

Here

$$\tilde{a}_k = (\varphi, \varphi_k)_{L_2(\Omega)}, \quad \tilde{b}_k = \mu_k^{-1}(\psi, \varphi_k)_{L_2(\Omega)}, \quad k = 1, 2, \dots, \tag{3.5}$$

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t)\varphi_k(x), \qquad F_k(t) = (F, \varphi_k)_{L_2(\omega_t)}, \qquad \omega_\tau := D_T \cap \{t = \tau\}. \tag{3.6}$$

Besides, for the solution u from (3.3), we have the following estimate

$$\|u\|_{E_{2,1}(D_T)} \leqslant \gamma\big(\|F\|_{L_2(D_T)} + \|\varphi\|_{\mathring{W}2^1(\Omega)} + \|\psi\|_{L_2(\Omega)}\big) \tag{3.7}$$

with positive constant γ independent of F, φ, and ψ [19, 21].

Let us consider the linear problem corresponding to (1.1)–(1.4), that is, the case $\lambda = 0$:

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \quad (x,t) \in D_T, \tag{3.8}$$

$$u|_\Gamma = 0, \tag{3.9}$$

$$u(x,0) - \mu u(x,T) = 0, \quad u_t(x,0) - \mu u_t(x,T) = 0, \quad x \in \Omega. \tag{3.10}$$

Let us show that when $|\mu| < 1$, for any $F \in L_2(D_T)$, there exists a unique generalized solution of problem (3.8)–(3.10). Indeed, since the space of finite infinitely differentiable functions $C_0^\infty(D_T)$ is dense in the space $L_2(D_T)$, for $F \in L_2(D_T)$ and any natural number m, there exists a function $F_m \in C_0^\infty(D_T)$ such that

$$\|F_m - F\|_{L_2(D_T)} < \frac{1}{m}. \tag{3.11}$$

On the other hand, for a function F_m in the space $L_2(D_T)$, we have the following expansion [19]:

$$F_m(x,t) = \sum_{k=1}^{\infty} F_{m,k}(t)\varphi_k(x), \qquad F_{m,k}(t) = (F_m, \varphi_k)_{L_2(\Omega)}. \tag{3.12}$$

Therefore, there exists a natural number l_m such that $\lim_{m\to\infty} l_m = \infty$ and, for

$$\tilde{F}_m(x,t) = \sum_{k=1}^{l_m} F_{m,k}(t)\varphi_k(x), \tag{3.13}$$

we have

$$\|\tilde{F}_m - F_m\|_{L_2(D_T)} < \frac{1}{m}. \tag{3.14}$$

From (3.11) and (3.14) it follows

$$\lim_{m\to\infty} \|\tilde{F}_m - F\|_{L_2(D_T)} = 0. \tag{3.15}$$

The solution $u = u_m$ of problem (3.1)–(3.2) for

$$\varphi = \sum_{k=1}^{l_m} \tilde{a}_k \varphi_k, \qquad \psi = \sum_{k=1}^{l_m} \mu_k \tilde{b}_k \varphi_k, \qquad F = \tilde{F}_m,$$

is given by formula (3.3), which by (3.4)–(3.6) and (3.13) can be rewritten as follows:

$$u_m=\sum_{k=1}^{l_m}\left(\tilde a_k\cos\mu_k t+\tilde b_k\sin\mu_k t+\frac{1}{\mu_k}\int_0^t F_{m,k}(\tau)\sin\mu_k(t-\tau)\,\mathrm d\tau\right)\varphi_k(x). \tag{3.16}$$

By construction the function u_m from (3.16) satisfies Eq. (3.8) and the boundary condition (3.9) for $F=\tilde F_m$ from (3.13). Let us define define unknown coefficients $\tilde a_k$ and $\tilde b_k$ so that the function u_m from (3.16) would satisfy the nonlocal conditions (3.10) too. For this purpose, let us substitute the right-hand part of expression (3.16) into Eqs. (3.10). As a result, since the system of functions $\{\varphi_k(x)\}$ forms a basis in $L_2(\Omega)$, for defining the coefficients $\tilde a_k$ and $\tilde b_k$, we have the following system of linear algebraic equations:

$$\begin{aligned}&(1-\mu\cos\mu_kT)\tilde a_k-(\mu\sin\mu_kT)\tilde b_k=\frac{\mu}{\mu_k}\int_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,\mathrm d\tau,\\ &(\mu\mu_k\sin\mu_kT)\tilde a_k+\mu_k(1-\mu\cos\mu_kT)\tilde b_k=\mu\int_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,\mathrm d\tau,\end{aligned} \tag{3.17}$$

$k=1,2,\dots,l_m$. Its solution is

$$\tilde a_k=\left[d_{1k}\mu\mu_k\sin\mu_kT-d_{2k}(1-\mu\cos\mu_kT)\right]\Delta_k^{-1},\quad k=1,2,\dots,l_m, \tag{3.18}$$

$$\tilde b_k=\left[d_{2k}(1-\mu\cos\mu_kT)-d_{1k}\mu\mu_k\sin\mu_kT\right]\Delta_k^{-1},\quad k=1,2,\dots,l_m, \tag{3.19}$$

where

$$d_{1k}=\frac{\mu}{\mu_k}\int_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,\mathrm d\tau,$$

$$d_{2k}=\mu\int_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,\mathrm d\tau.$$

Since $|\mu|<1$, for the determinant Δ_k of system (3.17), we have

$$\Delta_k=\mu_k\left[(1-\mu\cos\mu_kT)^2+\mu^2\sin^2\mu_kT\right]\geqslant\mu_k\big(1-|\mu|\big)^2>0. \tag{3.20}$$

We further assume that the Lipschitz domain Ω is such that the eigenfunctions $\varphi_k\in C^2(\overline\Omega)$, $k\geqslant 1$. For example, this will take place if $\partial\Omega\in C^{[n/2]+3}$ [21]. This fact will also take place in the case of a piece-wise smooth Lipschitz domain, for example, for the parallelepiped $\Omega=\{x\in\mathbb R^n\colon |x_i|<a_i,\ i=1,\dots,n\}$, the corresponding eigenfunctions $\varphi_k\in C^\infty(\overline\Omega)$ [22] (see also Remark 6). Therefore, since $F_m\in C_0^\infty(D_T)$, by (3.12) the function $F_{m,k}\in C^2([0,T])$, and consequently the function u_m from (3.16) belongs to the space $C^2(\overline D_T)$. Further, by construction the function u_m from (3.16) belongs to the space $\mathring C^2_\mu(\overline D_T)$ defined in (1.7), besides,

$$L_0u_m=\tilde F_m,\qquad L_0(u_m-u_k)=\tilde F_m-\tilde F_k. \tag{3.21}$$

From (3.21) and a priori estimate (2.5) we have

$$\|u_m-u_k\|_{\mathring W^1_{2,\mu}(D_T)}\leqslant c_1\|\tilde F_m-\tilde F_k\|_{L_2(D_T)} \tag{3.22}$$

since by Lemma 1 the coefficient $c_2 = 0$ when $\lambda = 0$. In view of (3.15), from (3.22) it follows that the sequence $u_m \in \mathring{C}^2_\mu(\overline{D}_T)$ is fundamental in the complete space $\mathring{W}^1_{2,\mu}(D_T)$. Therefore, there exists a function $u \in \mathring{W}^1_{2,\mu}(D_T)$ such that by (3.15) and (3.21) the limit equalities (1.10) are valid for $\lambda = 0$. This means that the function u is a generalized solution of problem (3.8)–(3.10). The uniqueness of this solution follows from a priori estimate (2.5), where the constant $c_2 = 0$ for $\lambda = 0$, that is,

$$\|u\|_{\mathring{W}^1_{2,\mu}(D_T)} \leqslant c_1 \|F\|_{L_2(D_T)}. \tag{3.23}$$

Therefore, for the solution u of problem (3.8)–(3.10), we have $u = L_0^{-1}(F)$, where $L_0^{-1} : L_2(D_T) \to \mathring{W}^1_{2,\mu}(D_T)$ is a linear continuous operator with norm that by (2.23) can be estimated as follows:

$$\|L_0^{-1}\|_{L_2(D_T) \to \mathring{W}2,\mu^1(D_T)} \leqslant c_1. \tag{3.24}$$

Remark 5. Note that when conditions (1.5) and (1.6) are satisfied and $F \in L_2(D_T)$, by (3.24) and Remark 2 the function $u \in \mathring{W}2,\mu^1(D_T)$ is a generalized solution of problem (1.1)–(1.4) in the sense of Definition 1 if and only if u is a solution of the functional equation

$$u = L_0^{-1}\bigl(-\lambda f(x,t,u)\bigr) + L_0^{-1}(F) \tag{3.25}$$

in the space $\mathring{W}2,\mu^1(D_T)$.

Rewrite Eq. (3.25) in the form

$$u = A_0 u := -\lambda L_0^{-1}(N_0 u) + L_0^{-1}(F), \tag{3.26}$$

where the operator $N_0 : \mathring{W}^1_{2,\mu}(D_T) \to L_2(D_T)$ from (1.9) by Remark 2 is continuous and compact operator. Therefore, by (3.24) the operator $A_0 : \mathring{W}^1_{2,\mu}(D_T) \to \mathring{W}^1_{2,\mu}(D_T)$ from (3.26) is also continuous and compact when $0 \leqslant \alpha < (n+1)/(n-1)$. At the same time, by Lemma 1 and (2.27), for any parameter $\tau \in [0,1]$ and for any solution u of the equation $u = \tau A_0 u$ with the parameter τ, we have the same a priori estimate (2.5) with nonnegative constants c_i independent of u, F, and τ. Therefore, by the Schaefer fixed point theorem [7], Eq. (3.26), and therefore by Remark 5 problem (1.1)–(1.4) has at least one solution $u \in \mathring{W}^1_{2,\mu}(D_T)$. Thus, we have proved the following theorem.

Theorem 1. *Let $\lambda > 0$ and $|\mu| < 1$, and let conditions* (1.5)*,* (1.6)*, and* (2.2)–(2.4) *be satisfied. Then, for any $F \in L_2(D_T)$, problem* (1.1)–(1.4) *has at least one generalized solution $u \in \mathring{W}^1_{2,\mu}(D_T)$ in the sense of Definition* 1.

Remark 6. Note that, for $|\mu| = 1$, even in the linear case, that is, for $f = 0$, the homogeneous problem corresponding to (1.1)–(1.4) may have a finite or even infinite number of linearly independent solutions, whereas for solvability of this problem, the function $F \in L_2(D_T)$ must satisfy a finite or infinite number of conditions of the form $l(F) = 0$, respectively, where l is a continuous functional in $L_2(D_T)$. Indeed, in the case $\mu = 1$, denote by $\Lambda(1)$ the set of those numbers μ_k from (3.3) for which the ratio $\mu_k T/(2\pi)$ is a natural number, that is, $\Lambda(1) = \{\mu_k\colon \mu_k T/(2\pi) \in \mathbb{N}\}$. Formulas (3.18)–(3.19) for determination of unknown coefficients $\tilde{a}_k$ and $\tilde{b}_k$ in representation (3.16) are obtained from the system of linear algebraic equations (3.17). In the case $\Lambda(1) \neq \emptyset$ and $\mu_k \in \Lambda(1)$ with $\mu = 1$, the determinant Δ_k of system (3.17), given by (3.20), equals zero. Moreover, in this case, all coefficients in front of unknown $\tilde{a}_k$ and $\tilde{b}_k$ in the left-hand side part of system (3.17) equal zero. Therefore, by (3.16) the homogeneous problem corresponding to (3.8), (3.9), and (3.10) is satisfied by the function

$$u_k(x,t) = \bigl(C_1 \cos \mu_k t + C_2 \sin \mu_k t\bigr)\varphi_k(x), \tag{3.27}$$

where C_1 and C_2 are arbitrary constant numbers; besides, by (3.17) the necessary conditions of solvability of nonhomogeneous problem (3.8)–(3.10) corresponding $\mu_k \in \Lambda(1)$ are the following:

$$\begin{aligned} l_{k,1}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x)\sin\mu_k(T-t)\,\mathrm{d}x\,\mathrm{d}t = 0, \\ l_{k,2}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x)\cos\mu_k(T-t)\,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned} \tag{3.28}$$

Analogously, in the case $\mu = -1$, denote by $\Lambda(-1)$ the set of points μ_k from (3.3) for which the ratio $\mu_k T/\pi$ is an odd integer number. For $\mu_k \in \Lambda(-1)$ with $\mu = -1$, the function u_k from (3.27) also is a solution of the homogenous problem corresponding to (3.8)–(3.10), and conditions (3.28) are the corresponding necessary conditions for solvability of this problem. For example, when $n = 2$ and $\Omega = (0,1) \times (0,1)$, the eigenvalues and eigenfunctions of the Laplace operator Δ are [22]

$$\lambda_k = -\pi^2\big(k_1^2 + k_2^2\big), \quad \varphi_k(x_1,x_2) = 2\sin k_1\pi x_1 \cdot \sin k_2\pi x_2, \quad k = (k_1,k_2),$$

that is, $\mu_k = \pi(k_1^2 + k_2^2)^{1/2}$. For $k_1 = p^2 - q^2$ and $k_2 = 2pq$, where p and q are any integer numbers, we obtain $\mu_k = \pi(p^2 + q^2)$. In this case, for $T/2 \in \mathbb{N}$, we have $\mu_k T/(2\pi) = (p^2+q^2)T/2 \in \mathbb{N}$, and by the preceding, when $\mu = 1$, the homogeneous problem corresponding to (3.8)–(3.10) has an infinite number of linearly independent solutions

$$u_{p,q}(x,t) = \big[C_1\cos\pi\big(p^2+q^2\big)t + C_2\sin\pi\big(p^2+q^2\big)t\big]\sin\big(p^2-q^2\big)\pi x_1 \cdot \sin 2pq\pi x_2$$

for any integer numbers p and q. Analogously, when $\mu = -1$, the solutions of the homogeneous problem corresponding to (3.8)–(3.10) in the case where p is an even number, whereas q and T odd numbers, are the functions from (3.27).

4 The uniqueness of the solution of problem (1.1)–(1.4)

On the function f in Eq. (1.1), wevus impose the following additional requirements:

$$f, f'_u \in C(\overline{D}_T \times \mathbb{R}), \qquad \big|f'_u(x,t,u)\big| \leqslant a + b|u|^\gamma, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{4.1}$$

where $a, b, \gamma = \mathrm{const} \geqslant 0$.

It is obvious that from (4.1) we have condition (1.5) for $\alpha = \gamma + 1$, and when $\gamma < 2/(n-1)$, we have $\alpha = \gamma + 1 < (n+1)/(n-1)$.

Theorem 2. *Let* $\lambda > 0$, $|\mu| < 1$, $F \in L_2(D_T)$, *let condition* (4.1) *be satisfied for* $\gamma < 2/(n-1)$, *and let also conditions* (2.2)–(2.4) *be satisfied. Then there exists a positive number* $\lambda_0 = \lambda_0(F, f, \mu, D_T)$ *such that, for* $0 < \lambda < \lambda_0$, *problem* (1.1)–(1.4) *has no more than one generalized solution in the sense of Definition* 1.

Proof. Indeed, suppose that problem (1.1)–(1.4) has two different generalized solutions u_1 and u_2. By Definition 1 there exist sequences of functions $u_{jk} \in \mathrm{C}^2_\mu(D_T)$, $j = 1,2$, such that

$$\lim_{k\to\infty} \|u_{jk} - u_j\|_{\overset{\circ}{W}2,\mu^1(D_T)} = 0, \quad j = 1,2, \qquad \lim_{k\to\infty} \|L_\lambda u_{jk} - F\|_{L_2(D_T)} = 0. \tag{4.2}$$

Let

$$w := u_2 - u_1, \qquad w_k := u_{2k} - u_{1k}, \qquad F_k := L_\lambda u_{2k} - L_\lambda u_{1k}, \tag{4.3}$$

$$g_k := \lambda\big(f(x,t,u_{2k}) - f(x,t,u_{1k})\big). \tag{4.4}$$

From (4.2) and (4.3) it easily follows that

$$\lim_{k\to\infty} \|w_k - w\|_{\overset{\circ}{W}2,\mu^1(D_T)} = 0, \qquad \lim_{k\to\infty} \|F_k\|_{L_2(D_T)} = 0. \tag{4.5}$$

In view of (4.3) and (4.4), the functions $w_k \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ satisfy the following equalities:

$$\frac{\partial^2 w_k}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 w_k}{\partial x_i^2} = \big(F_k + g_k\big)(x,t), \quad (x,t) \in D_T, \tag{4.6}$$

$$w_k|_\Gamma = 0, \qquad w_k(x,0) - \mu w_k(x,T) = 0, \quad w_{kt}(x,0) - \mu w_{kt}(x,T) = 0, \quad x \in \Omega. \tag{4.7}$$

First, let us estimate the function g_k from (4.4). Using the obvious inequality

$$|d_1 + d_2|^\gamma \leqslant 2^\gamma \max\big(|d_1|^\gamma, |d_2|^\gamma\big) \leqslant 2^\gamma\big(|d_1|^\gamma + |d_2|^\gamma\big) \quad \text{for } \gamma \geqslant 0,$$

by (4.1) we have

$$\begin{aligned}
\big|f(x,t,u_{2k}) - f(x,t,u_{1k})\big| &= \left|(u_{2k} - u_{1k}) \int_0^1 f'_u\big(x,t,u_{1k} + \tau(u_{2k} - u_{1k})\big)\,\mathrm{d}\tau\right| \\
&\leqslant |u_{2k} - u_{1k}| \int_0^1 \big(a + b\big|(1-\tau)u_{1k} + \tau u_{2k}\big|^\gamma\big)\,\mathrm{d}\tau \\
&\leqslant a|u_{2k} - u_{1k}| + 2^\gamma b|u_{2k} - u_{1k}|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big) \\
&= a|w_k| + 2^\gamma b|w_k|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big).
\end{aligned} \tag{4.8}$$

By (4.4) from (4.8) we have

$$\begin{aligned}
\|g_k\|_{L_2(D_T)} &\leqslant \lambda a\|w_k\|_{L_2(D_T)} + \lambda 2^\gamma b\big\||w_k|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_2(D_T)} \\
&\leqslant \lambda a\|w_k\|_{L_2(D_T)} + \lambda 2^\gamma b\|w_k\|_{L_p(D_T)}\big\|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_q(D_T)},
\end{aligned} \tag{4.9}$$

where we used the Hölder inequality [24]

$$\|v_1 v_2\|_{L_r(D_T)} \leqslant \|v_1\|_{L_p(D_T)}\|v_2\|_{L_q(D_T)} \quad \left(\frac{1}{p} + \frac{1}{q} = \frac{1}{r},\ p,q,r \geqslant 1\right)$$

with

$$p = 2\frac{n+1}{n-1}, \qquad q = n+1, \qquad r = 2. \tag{4.10}$$

Since $\dim D_T = n+1$, by the Sobolev embedding theorem [19] for $1 \leqslant p \leqslant 2(n+1)/(n-1)$, we have

$$\|v\|_{L_p(D_T)} \leqslant C_p\|v\|_{W_2^1(D_T)} \quad \forall v \in W_2^1(D_T) \tag{4.11}$$

with positive constant C_p independent of $v \in W_2^1(D_T)$.

By the condition of the theorem, $\gamma < 2/(n-1)$, and therefore $\gamma(n+1) < 2(n+1)/(n-1)$. Thus, by (4.10) from (4.11) we have

$$\|w_k\|_{L_p(D_T)} \leqslant C_p\|w_k\|_{W_2^1(D_T)}, \quad p = \frac{2(n+1)}{n-1},\ k \geqslant 1, \tag{4.12}$$

$$\begin{aligned}\big\|\big(|u_{1k}|^\gamma+|u_{2k}|^\gamma\big)\big\|_{L_q(D_T)} &\leqslant \big\||u_{1k}|^\gamma\big\|_{L_q(D_T)}+\big\||u_{2k}|^\gamma\big\|_{L_q(D_T)}\\ &=\|u_{1k}\|^\gamma_{L_{\gamma(n+1)}(D_T)}+\|u_{2k}\|^\gamma_{L_{\gamma(n+1)}(D_T)}\\ &\leqslant C^\gamma_{\gamma(n+1)}\big(\|u_{1k}\|^\gamma_{W_2^1(D_T)}+\|u_{2k}\|^\gamma_{W_2^1(D_T)}\big). \end{aligned}\tag{4.13}$$

By the first equality of (4.2) there exists a natural number k_0 such that, for $k\geqslant k_0$, we have

$$\|u_{ik}\|^\gamma_{W_2^1(D_T)}\leqslant \|u_i\|^\gamma_{W_2^1(D_T)}+1,\quad i=1,2,\ k\geqslant k_0. \tag{4.14}$$

Further, by (4.12), (4.13), and (4.14) from (4.9) we have

$$\begin{aligned}\|g_k\|_{L_2(D_T)} &\leqslant \lambda a\|w_k\|_{L_2(D_T)}+\lambda 2^\gamma bC_pC^\gamma_{\gamma(n+1)}\big(\|u_1\|^\gamma_{W_2^1(D_T)}+\|u_2\|^\gamma_{W_2^1(D_T)}+2\big)\|w_k\|_{W_2^1(D_T)},\\ &\leqslant \lambda M_5\|w_k\|_{W_2^1(D_T)},\end{aligned}\tag{4.15}$$

where we have applied the inequality $\|w_k\|_{L_2(D_T)}\leqslant\|w_k\|_{W_2^1(D_T)}$, and

$$M_5=a+2^\gamma bC_pC^\gamma_{\gamma(n+1)}\big(\|u_1\|^\gamma_{W_2^1(D_T)}+\|u_2\|^\gamma_{W_2^1(D_T)}+2\big),\quad p=2\frac{n+1}{n-1}. \tag{4.16}$$

Since a priori estimate (2.5) is also valid for $\lambda=0$ and since, by (2.27), $c_2=0$ in this estimate, for the solution w_k of problem (4.6)–(4.7), we get the estimate

$$\|w_k\|_{\mathring{W}2,\mu^1(D_T)}\leqslant c_1^0\|F_k+g_k\|_{L_2(D_T)}, \tag{4.17}$$

where the constant c_1^0 does not depend on λ, F_k, and g_k.

Because of $\|w_k\|_{\mathring{W}2,\mu^1(D_T)}=\|w_k\|_{W_2^1(D_T)}$, by (4.15) from (4.17) we have

$$\|w_k\|_{\mathring{W}2,\mu^1(D_T)}\leqslant c_1^0\|F_k\|_{L_2(D_T)}+\lambda c_1^0M_5\|w_k\|_{\mathring{W}2,\mu^1(D_T)}. \tag{4.18}$$

Note that since a priori estimate (2.5) is valid for u_1 and u_2, the constant M_5 in (4.16) depends on F, f, μ, D_T, λ. Moreover, by (2.19), (2.23), and (2.27) the value of M_5 continuously depends on $\lambda\geqslant 0$, and

$$0\leqslant\lim_{\lambda\to 0+}M_5=M_5^0<+\infty. \tag{4.19}$$

By (4.19) there exists a positive number $\lambda_0=\lambda_0(F,f,\mu,D_T)$ such that $\lambda c_1^0M_5<1$ for

$$0<\lambda<\lambda_0. \tag{4.20}$$

Indeed, let us fix an arbitrary positive number ε_1. Then, by (4.19) there exists a positive number λ_1 such that $0\leqslant M_5<M_5^0+\varepsilon_1$ for $0\leqslant\lambda<\lambda_1$. It is obvious that, for $\lambda_0=\min(\lambda_1,(c_1^0(M_5^0+\varepsilon_1))^{-1})$, the condition $\lambda c_1^0M_5<1$ is satisfied fulfilled. Therefore, in case (4.20), from (4.18) we get

$$\|w_k\|_{\mathring{W}2,\mu^1(D_T)}\leqslant c_1^0\big(1-\lambda c_1^0M_5\big)^{-1}\|F_k\|_{L_2(D_T)},\quad k\geqslant k_0. \tag{4.21}$$

From (4.2) and (4.3) it follows that $\lim_{k\to\infty}\|w_k\|_{\mathring{W}2,\mu^1(D_T)}=\|u_2-u_1\|_{\mathring{W}2,\mu^1(D_T)}$. On the other hand, by (4.5) from (4.21) we have $\lim_{k\to\infty}\|w_k\|_{\mathring{W}2,\mu^1(D_T)}=0$. Thus $\|u_2-u_1\|_{\mathring{W}2,\mu^1(D_T)}=0$, that is, $u_2=u_1$, which leads to a contradiction. This proves Theorem 2. □

5 The cases of absence of the solution of problem (1.1)–(1.4)

In this section, using the test-function method [23], we show that when condition (2.2) is violated, problem (1.1)–(1.4) may not have a generalized solution in the sense of Definition 1.

Lemma 2. *Let* u *be a generalized solution of problem* (1.1)–(1.4) *in the sense of Definition* 1, *and let conditions* (1.5) *and* (1.6) *be satisfied. Then*

$$\int\limits_{D_T} u\,\Box v\,\mathrm{d}x\,\mathrm{d}t = -\lambda \int\limits_{D_T} f(x,t,u)v\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{D_T} Fv\,\mathrm{d}x\,\mathrm{d}t \tag{5.1}$$

for every test-function v *satisfying the conditions*

$$v \in C^2(\overline{D}_T), \qquad v|_{\partial D_T} = 0, \qquad \nabla_{x,t} v|_{\partial D_T} = 0, \tag{5.2}$$

where $\Box := \partial^2/\partial t^2 - \sum_{i=1}^n \partial^2/\partial x_i^2$, $\nabla_{x,t} := (\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial t)$.

Proof. By the definition of a generalized solution of problem (1.1)–(1.4) there exists a sequence $u_m \in \mathring{C}^2_\mu(\overline{D}_T)$ such that Eqs. (1.10) and (2.8) are valid. Let us multiply both parts of Eq. (2.6) by the function v and integrate the obtained equality in the domain D_T. By (5.2) integration by parts of the left-hand side of this equation yields

$$\int\limits_{D_T} u_m\,\Box v\,\mathrm{d}x\,\mathrm{d}t + \lambda \int\limits_{D_T} f(x,t,u_m)v\,\mathrm{d}x\,\mathrm{d}t = \int\limits_{D_T} F_m v\,\mathrm{d}x\,\mathrm{d}t. \tag{5.3}$$

Passing in Eq (5.3) to limit as $m \to \infty$ and taking into account (2.6), the limit equalities (1.10), and Remark 2, we obtain Eq. (5.2). Lemma 2 is proved. □

Consider the following condition imposed on the function f:

$$f(x,t,u) \leqslant -|u|^p, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R};\ p = \mathrm{const} > 1. \tag{5.4}$$

Note that when condition (5.4) is satisfied, condition (5.2) is violated.

Let us introduce a function $v_0 = v_0(x,t)$ such that

$$v_0 \in C^2(\overline{D}_T), \qquad v_0|_{D_T} > 0, \qquad v_0|_{\partial D_T} = 0, \qquad \nabla_x v_0|_{\partial D_T} = 0, \tag{5.5}$$

and

$$æ_0 := \int\limits_{D_T} \frac{|\Box v_0|^{p'}}{|v_0|^{p'-1}}\,\mathrm{d}x\,\mathrm{d}t < +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{5.6}$$

We further assume that $\partial\Omega \in C^2$; then there exists a function $\omega \in C^2(\mathbb{R}^n)$ such that $\partial\Omega$: $\omega(x) = 0$, $\nabla_x \omega|_{\partial\Omega} \neq 0$, and $\omega|_\Omega > 0$ [8].

Simple verification shows that, as a function v_0 satisfying conditions (5.5) and (5.6), we can take

$$v_0(x,t) = \big[t(T-t)\omega(x)\big]^k, \quad (x,t) \in D_T,$$

for sufficiently big $k = \mathrm{const} > 0$.

By (5.4) and (5.5) from (5.1), where v_0 is taken instead of v, it follows that, when $\lambda > 0$,

$$\lambda \int\limits_{D_T} |u|^p v_0 \,\mathrm{d}x\,\mathrm{d}t \leqslant \int\limits_{D_T} |u||\Box v_0| \,\mathrm{d}x\,\mathrm{d}t - \int\limits_{D_T} F v_0 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.7}$$

Theorem 3. *Let the function $f \in C(\overline{D}_T \times \mathbb{R})$ satisfy conditions* (1.5), (1.6)*, and* (5.4)*; let $\lambda > 0$, $\partial\Omega \in C^2$; $F^0 \in L_2(D_T)$, $F^0 \geqslant 0$, and $\|F^0\|_{L_2(D_T)} \neq 0$. Then there exists a number $\gamma_0 = \gamma_0(F^0, \alpha, p, \lambda) > 0$ such that, for $\gamma > \gamma_0$, problem* (1.1)–(1.4) *has no generalized solution in the sense of Definition* 1 *for $F = \gamma F^0$.*

Proof. In Young's inequality with the parameter $\varepsilon > 0$

$$ab \leqslant \frac{\varepsilon}{p} a^p + \frac{1}{p' \varepsilon^{p'-1}} b^{p'}; \quad a, b \geqslant 0,\ \frac{1}{p} + \frac{1}{p'} = 1,\ p > 1,$$

let us take $a = |u| v_0^{1/p}$, $b = |\Box v_0|/v^{1/p}$. Then, taking into account the equality $p'/p = p' - 1$, we have

$$|u||\Box v_0| = |u| v_0^{1/p} \frac{|\Box v_0|}{v_0^{1/p}} \leqslant \frac{\varepsilon}{p} |u|^p v_0 + \frac{1}{p' \varepsilon^{p'-1}} \frac{|\Box v_0|^{p'}}{v_0^{p'-1}}. \tag{5.8}$$

Since $F = \gamma F^0$, by (5.8) from (5.7) we have

$$\left(\lambda - \frac{\varepsilon}{p}\right) \int\limits_{D_T} |u|^p v_0 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{p' \varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\Box v_0|^{p'}}{v_0^{p'-1}} \,\mathrm{d}x\,\mathrm{d}t - \gamma \int\limits_{D_T} F^0 v_0 \,\mathrm{d}x\,\mathrm{d}t,$$

whence, for $\varepsilon < \lambda p$, we obtain

$$\int\limits_{D_T} |u|^p v_0 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{p}{(\lambda p - \varepsilon) p' \varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\Box v_0|^{p'}}{v_0^{p'-1}} \,\mathrm{d}x\,\mathrm{d}t - \frac{p\gamma}{\lambda p - \varepsilon} \int\limits_{D_T} F^0 v_0 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.9}$$

Since $p' = p/(p-1)$, $p = p'/(p'-1)$, and

$$\min_{0<\varepsilon<\lambda p} \frac{p}{(\lambda p - \varepsilon) p' \varepsilon^{p'-1}} = \frac{1}{\lambda^{p'}},$$

which is reached at $\varepsilon = \lambda$, from (5.9) it follows that

$$\int\limits_{D_T} |u|^p v_0 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{\lambda^{p'}} \int\limits_{D_T} \frac{|\Box v_0|^{p'}}{v_0^{p'-1}} \,\mathrm{d}x\,\mathrm{d}t - \frac{p'\gamma}{\lambda} \int\limits_{D_T} F^0 v_0 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.10}$$

Because of the conditions imposed on the function F^0 and $v_0|_{D_T} > 0$, we have

$$0 < æ_1 := \int\limits_{D_T} F^0 v_0 \,\mathrm{d}x\,\mathrm{d}t < +\infty. \tag{5.11}$$

Denoting by $\chi = \chi(\gamma)$ the right-hand side of inequality (5.10), which is a linear function with respect to the parameter γ, by (5.6) and (5.11) we have

$$\chi(\gamma) < 0 \quad \text{for } \gamma > \gamma_0 \quad \text{and} \quad \chi(\gamma) > 0 \quad \text{for } \gamma < \gamma_0, \tag{5.12}$$

where

$$\chi(\gamma) = \frac{æ_0}{\lambda^{p'}} - \frac{p'\gamma}{\lambda}æ_1, \qquad \gamma_0 = \frac{æ_0}{\lambda^{p'-1}p'æ_1}.$$

It remains only to note that the left-hand side of inequality (5.10) is nonnegative, whereas its right-hand side by (5.12) is negative for $\gamma > \gamma_0$. Thus, for $\gamma > \gamma_0$, problem (1.1)–(1.4) has no generalized solution in the sense of Definition 1. Theorem 3 is proved. □

6 The case $|\mu| = 1$

As it was mentioned at the end of Section 3, for $|\mu| = 1$, problem (1.1)–(1.4) may turn out to be ill-posed. We further show that in the presence of additional terms $2au_t$ and cu in the left-hand side of Eq. (1.1) the problem is solvable for any $F \in L_2(D_T)$. Consider the equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + 2au_t + cu + f_1(x,t,u) = F(x,t), \quad (x,t) \in D_T, \tag{6.1}$$

with constant real coefficients a and c, where f_1 and F are given real functions.

For Eq. (6.1), consider the problem of finding u in the domain D_T satisfying the boundary condition (1.2) and nonlocal conditions (1.3)–(1.4) for $|\mu| = 1$. For problem (6.1), (1.2)–(1.4), when $f_1 \in C(\overline{D}_T \times \mathbb{R})$ and $F \in L_2(D_T)$, analogously to Definition 1, let us introduce the notion of a generalized solution $u \in \mathring{W}^1_{2,\mu}(D_T)$.

With respect to the new unknown function

$$v := \sigma^{-1}(t)u, \quad \text{where } \sigma(t) := \exp(-at),\ 0 \leqslant t \leqslant T, \tag{6.2}$$

problem (6.1), (1.2)–(1.4) can be rewritten as follows:

$$\frac{\partial^2 v}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} + \left(c - a^2\right)v + \sigma^{-1}(t)f_1\bigl(x,t,\sigma(t)v(x,t)\bigr)$$
$$= \sigma^{-1}(t)F(x,t), \quad (x,t) \in D_T, \tag{6.3}$$
$$v|_\Gamma = 0, \tag{6.4}$$
$$\bigl(K_{\mu_1}v\bigr)(x) = 0, \quad \bigl(K_{\mu_1}v_t\bigr)(x) = 0, \quad x \in \Omega, \tag{6.5}$$

where $\mu_1 = \mu\sigma(T)$, $|\mu| = 1$.

In the case $a > 0$, from (6.2) and $|\mu| = 1$ it follows that $|\mu_1| < 1$.

It is easy to see that, for $c - a^2 \geqslant 0$, the functions $f(x,t,u) = (c - a^2)u$ and $g(x,t,u) = \int_0^u f(x,t,s)\,\mathrm{d}s = (c - a^2)u^2/2$ satisfy (1.5) and (2.2)–(2.4).

For $f(x,t,u) = \sigma^{-1}(t)f_1(x,t,\sigma(t)u)$, we have

$$g(x,t,u) = \int_0^u f(x,t,s)\,\mathrm{d}s = \int_0^u \sigma^{-1}(t)f_1\bigl(x,t,\sigma(t)s\bigr)\,\mathrm{d}s$$
$$= \sigma^{-1}(t)\int_0^{\sigma(t)u} f_1(x,t,s')\,\mathrm{d}s' = \sigma^{-2}(t)g_1\bigl(x,t,\sigma(t)u\bigr), \tag{6.6}$$

where

$$g_1(x,t,u) = \int\limits_0^u f_1(x,t,s)\,\mathrm{d}s. \tag{6.7}$$

Let us show that if the function $g_1(x,t,u)$ from (6.7) satisfies the condition

$$g_1(x,0,\mu_1 u) \leqslant g_1\big(x,T,|\mu_1|u\big), \quad (x,u) \in \overline{\Omega} \times \mathbb{R}, \tag{6.8}$$

for the fixed constant number μ_1 from (6.5), then the function $g(x,t,u)$ from (6.6) satisfies condition (2.4) for $\mu = \mu_1$. Indeed, by (6.2), (6.6), and (6.8), since $\mu_1 = \mu\sigma(T)$, $|\mu| = 1$, and $\sigma(T) = |\mu_1|$, we have

$$g(x,0,\mu_1 u) = \sigma^{-2}(0) g_1\big(x,0,\sigma(0)\mu_1 u\big) = g_1(x,0,\mu_1 u),$$
$$\mu_1^2 g(x,T,u) = \mu_1^2 \sigma^{-2}(T) g_1\big(x,T,\sigma(T)u\big) = g_1\big(x,T,|\mu_1|u\big),$$

whence, by (6.8), (2.4) follows for $\mu = \mu_1$.

Since $\sigma'(t) = -a\sigma(t)$ and $(\sigma^{-2}(t))' = 2a\sigma^{-2}(t)$, according to (6.6) and supposing that $f_1, f_{1t}, f_{1u} \in C(\overline{D}_T \times \mathbb{R})$, we have

$$g_t(x,t,u) = 2a\sigma^{-2}(t) g_1\big(x,t,\sigma(t)u\big) + \sigma^{-2}(t) g_{1t}\big(x,t,\sigma(t)u\big) - a\sigma^{-1}(t) g_{1u}\big(x,t,\sigma(t)u\big).$$

Therefore, condition (2.3) follows from the condition

$$2a\sigma^{-2}(t) g_1(x,t,\sigma(t)u) + \sigma^{-2}(t) g_{1t}\big(x,t,\sigma(t)u\big) - a\sigma^{-1}(t) g_{1u}\big(x,t,\sigma(t)u\big) \leqslant M_3, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}.$$

Note that, due to (6.6), condition (2.2) follows from the condition

$$g_1(x,t,u) \geqslant 0, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}. \tag{6.9}$$

It is easy to see that if the function $f_1(x,t,u)$ satisfies the condition of type (1.5), that is,

$$\big|f_1(x,t,u)\big| \leqslant \tilde{M}_1 + \tilde{M}_2 |u|^\alpha, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R},\ \tilde{M}_i = \mathrm{const} \geqslant 0, \tag{6.10}$$

then the function $f(x,t,u) = \sigma^{-1}(t) f_1(x,t,\sigma(t)u)$ from the left-hand side of Eq. (6.3) satisfies condition (1.5) for some nonnegative constants M_1 and M_2.

Note that in the concrete case $f_1(x,t,u) = |u|^\beta u$ with $\beta = \mathrm{const} \geqslant 0$, the function $g_1(x,t,u) = |u|^{\beta+2}/(\beta+2)$, and

$$f(x,t,u) = \sigma^{-1}(t) f_1\big(x,t,\sigma(t)u\big) = \sigma^\beta(t)|u|^\beta u, \tag{6.11}$$

$$g(x,t,u) = \int\limits_0^u f(x,t,s)\,\mathrm{d}s = \sigma^\beta(t)\frac{|u|^{\beta+2}}{\beta+2}. \tag{6.12}$$

Therefore, since $\sigma'(t) \leqslant 0$, $g(x,0,\mu_1 u) = |\mu_1|^{\beta+2}|u|^{\beta+2}/(\beta+2)$, $\mu_1^2 g(x,T,u) = \mu_1^2\sigma^\beta(T)|u|^{\beta+2}/(\beta+2)$, and $\sigma(T) = |\mu_1|$, it is easy to see that the functions $f(x,t,u)$ and $g(x,t,u)$ from (6.11) and (6.12) satisfy conditions (1.5) and (2.2)–(2.4) for $\mu = \mu_1$, $\alpha = \beta + 1$, $M_3 = 0$.

Further, since problems (6.1), (1.2)–(1.4) and (6.3)–(6.5) are equivalent, from Theorem 1 there follows the following theorem on the existence of a solution of problem (6.1), (1.2)–(1.4).

Theorem 4. *Let* $|\mu| = 1$, $a > 0$, $c - a^2 \geqslant 0$, *and let the function* $f_1(x,t,u)$ *from the left-hand side of Eq.* (6.1) *and the function* $g_1(x,t,u)$ *from* (6.7) *satisfy the conditions* $f_1, f_{1t}, f_{1u} \in C(\overline{D}_T \times \mathbb{R})$ *and* (6.8)–(6.10). *Then if in condition* (6.10) *the order of nonlinearity* α *satisfies the inequality* $\alpha < (n+1)/(n-1)$, *then, for any* $F \in L_2(D_T)$, *problem* (6.1), (1.2)–(1.4) *has at least one generalized solution.*

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