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SOME ASPECTS OF THE MEASURE EXTENSION PROBLEM

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Abstract. The general measure extension problem is discussed in the present report. Among various aspects of this problem the following three are especially underlined: purely set-theoretical, algebraic and topological. Also, several constructions of extensions of the classical Lebesgue measure on the real line are considered and compared to each other.

რეზიუმე. განხილულია ზომის გაგრძელების ზოგადი ამოცანა. გამოყოფილია ამ ამოცანის შემდეგი სამი ასპექტი: წმინდა სიმრავლურ-თეორიული, ალგებრული და ტოპოლოგიური. გარდა ამისა, აგებულია ლებეგის კლასიკური ზომის რამდენიმე საკუთრივი გაგრძელება და ეს გაგრძელებული ზომები შედარებულია ერთმანეთთან მათი თვისებების მიხედვით.

Let E be a set, $\mathcal{S}$ be a σ-algebra of subsets of E containing all one-element subsets (singletons) of E and let μ be a nonzero σ-finite continuous (i.e. vanishing at all singletons) measure on $\mathcal{S}$. The general measure extension problem requires to extend μ onto a wider σ-algebra of subsets of E. This problem was originally formulated within classical real analysis and, as well known, was partially solved by Lebesgue and Carathéodory. Afterwards, this problem found important applications in many other domains of mathematics: axiomatic set theory, general topology, functional analysis, probability and stochastic processes, etc.

A purely set-theoretical aspect of the above-mentioned problem was first considered by Banach and Kuratowski, Ulam, Sierpinski, and Marczewski. In particular, according to Ulam's famous theorem (see, for instance, [1], [2] and [5]), it is consistent with the axioms of set theory that the domain of any extension μ' of μ cannot coincide with the power set of E. Consequently, there always exists a set $X \subset E$ such that $X \notin \mathrm{dom}(\mu')$. An easy argument shows that μ' can be extended to a measure μ'' so that X becomes μ''-measurable and, in general, there are various possibilities to construct such

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an extension μ''. Thus, by starting with the assumption that there are no large cardinals, one can infer that there are no maximal extensions of the original measure μ.

In this context, let us also recall a similar result, which states that if $\{X_i : i \in I\}$ is an arbitrary partition of E, then there exists a measure ν on E extending μ and satisfying the relation

$$\{X_i : i \in I\} \subset \mathrm{dom}(\nu).$$

For the proof of this result, see [3] or [4]. In particular, having any finite family of subsets of E, we can always extend μ to a measure which measures all these subsets. On the other hand, it is well known that an analogous assertion fails to be true for arbitrary countable families of subsets of E. Moreover, the existence of a Luzin set on the real line $\mathbf{R}$ (see, e.g., [2], [5]) implies that there is a countably generated σ-algebra of subsets of $\mathbf{R}$ containing all singletons of $\mathbf{R}$ and not admitting a nonzero σ-finite continuous measure. Recall that the existence of a Luzin set needs additional set-theoretical axioms. It should be underlined, in this context, that the existence of other small subsets of $\mathbf{R}$ having cardinality ω_1 can be established within the theory **ZFC** (see, e.g., [5] or [10]).

Another aspect of the measure extension problem has an algebraic (in fact, group-theoretical) flavour. Namely, suppose that an uncountable group $(G, \cdot)$ is given and suppose that it is equipped with a nonzero σ-finite left G-invariant (more generally, left G-quasiinvariant) measure μ. As shown by Erdös and Mauldin [6] and Kharazishvili [7], the domain of such a μ cannot be identical with the family of all subsets of G. Notice that this result does not need additional set-theoretical assumptions. So it is natural to ask whether there exists a left G-invariant (left G-quasiinvariant) measure μ' on G properly extending μ. For some sufficiently wide classes of uncountable groups this question has a positive answer in terms of certain subgroups of the original group. For instance, if $(G, \cdot)$ is an arbitrary uncountable solvable group, then there always exists a μ-nonmeasurable subgroup H of G such that μ can be extended to a left G-invariant (left G-quasiinvariant) measure μ' for which we have $H \in \mathrm{dom}(\mu')$ and $\mu'(H) = 0$. A more detailed information about this relatively recent result can be found in [12].

The third aspect of the measure extension problem is purely topological. Indeed, most measures considered in mathematical analysis and general topology are regular in an appropriate sense. For example, if a σ-finite measure μ is given on a topological space E and, for each set $X \in \mathrm{dom}(\mu)$, the equality

$$\mu(X) = \sup\{\mu(K) \ : \ K \in \mathrm{dom}(\mu),\ K \subset X,\ K \ \text{ is compact}\}$$

holds true, then μ is said to be a Radon measure. Radon measures on locally compact spaces play an important role in various questions of analysis and probability theory. A more general class constitute the so-called perfect measures (in the sense of Gnedenko and Kolmogorov). There are some deep results concerning extensions of measures which preserve the regularity property. In particular, it is well known that any finite Radon measure defined on a σ-subalgebra of the Borel σ-algebra of a Hausdorff topological space can be extended to a Radon measure defined on the whole Borel σ-algebra. Here the method of extending a given measure is essentially based on the A. D. Alexandrov theorem stating that any finite and finitely additive Radon measure is countably additive.

Let us briefly recall several typical methods of extending measures and touch upon certain nonseparable extensions of the classical Lebesgue measure on the real line $\mathbf{R}$.

All measures considered below are assumed to be continuous (i.e., vanishing at singletons). For any σ-finite measure space $(E, \mathcal{S}, \mu)$, we denote by $\mathcal{I}(\mu)$ the σ-ideal generated by the family of all μ-measure zero sets (in short, μ-null-sets). The standard method of extending a given measure μ is based on adding to $\mathcal{I}(\mu)$ some new sets, which are nonmeasurable with respect to μ and whose inner μ-measure is equal to zero. Proceeding in this way, we come to a σ-ideal $\mathcal{I}'$ which properly contains $\mathcal{I}(\mu)$ and whose elements are of inner μ-measure zero. This property of $\mathcal{I}'$ enables us to extend μ onto the σ-algebra generated by $\mathcal{S} \cup \mathcal{I}'$. This extension is unique if $\mathrm{card}(E)$ is not a real-valued measurable cardinal in the sense of Ulam.

The above-mentioned method of extending measures was first considered by Marczewski (see, for instance, [11]). However, it has a weak side. Indeed, from the viewpoint of the theory of Boolean algebras, μ and its extension μ' are the same, because the corresponding quotient Boolean algebras coincide. Some modification of the method can distinguish these Boolean algebras, but does not essentially change the metrical structure of μ. Indeed, the metric spaces associated with a measure and its extension, respectively, have the same topological weight. Therefore, if the original measure μ is separable (i.e., its metric space is separable), then the extended measure is separable, too.

We say that μ' is an essential extension of μ if the Hilbert space $L_2(\mu')$ is not isomorphic to the Hilbert space $L_2(\mu)$. In particular, any nonseparable extension of a separable measure μ is an essential extension of μ.

Kakutani and Oxtoby [8] presented a construction of a nonseparable extension of the Lebesgue measure λ on the real line $\mathbf{R}$. Another construction of this kind was given by Kodaira and Kakutani [9]. It is remarkable that both obtained extensions turn out to be invariant under the group of all isometries of $\mathbf{R}$. In addition, it should be noticed that the extension of λ constructed by Kakutani and Oxtoby necessarily yields new null-sets, i.e., there always appear null-sets which are not of Lebesgue measure zero. By applying the method of Kodaira and Kakutani, one can also obtain new null-sets (see, for instance, Theorem 2 below). In this connection, the following question is of interest: does there exist a nonseparable extension of λ whose all null-sets are exactly λ-null-sets? Under the Continuum Hypothesis, the answer to this question is positive (see Theorem 1 below). The construction of such an extension may be regarded as a certain combination of the method of Kodaira and Kakutani [9] with the method of Sierpiński by means of which he proved the existence of his set on the real line (see, e.g., [2] and [5]).

Let $(E_1, \mathcal{S}_1, \mu_1)$ and $(E_2, \mathcal{S}_2, \mu_2)$ be two measure spaces, such that μ_1 is σ-finite and μ_2 is a probability measure. Let $f : E_1 \to E_2$ be a mapping and let $G(f)$ denote the graph of f. Suppose that $G(f)$ is $(\mu_1 \times \mu_2)$-thick in the product space $E_1 \times E_2$, i.e. suppose that

$$(\mu_1 \times \mu_2)_*((E_1 \times E_2) \setminus G(f)) = 0.$$

For any set $Z \in \mathrm{dom}(\mu_1 \times \mu_2)$, let us define

$$Z' = \{x \in E_1 : (x, f(x)) \in Z\}.$$

Further, introduce a class of sets

$$\mathcal{S}'_1 = \{Z' : Z \in \mathrm{dom}(\mu_1 \times \mu_2)\}$$

and define a functional

$$\mu'_1(Z') = (\mu_1 \times \mu_2)(Z) \quad (Z \in \mathrm{dom}(\mu_1 \times \mu_2)).$$

Then $\mathcal{S}'_1$ is a σ-algebra of subsets of E_1 and the funcional μ'_1 is a measure on $\mathcal{S}'_1$ extending μ_1 (cf. [9], [10]). In addition, the mapping f turns out to be measurable with respect to the σ-algebras $\mathcal{S}'_1$ and $\mathcal{S}_2$, i.e., for any set $Y \in \mathcal{S}_2$, we have $f^{-1}(Y) \in \mathcal{S}'_1$.

We need the following three simple lemmas.

Lemma 1. *If μ_1 is nonzero and μ_2 is nonseparable, then the measure μ'_1 is nonseparable, too.*

Let $\mathbf{c}$ denote the cardinality of the continuum and let J be a set with $\mathrm{card}(J) = \mathbf{c}$. For any index $j \in J$, denote by ν_j the restriction of the Lebesgue measure λ to the Borel σ-algebra of $[0, 1]$. Thus, ν_j is a Borel probability measure on $[0, 1]$. Let ν stand for the product measure $\prod_{j \in J} \nu_j$.

Lemma 2. *The cardinality of* $\mathrm{dom}(\nu)$ *is equal to* $\mathbf{c}$ *and the topological weight of the metric space associated with* ν *is also equal to* $\mathbf{c}$*. In particular, ν is a nonseparable measure.*

Lemma 3. *Let α be an infinite cardinal number satisfying the equality $\alpha^\omega = \alpha$, let $(E_1, \mathcal{S}_1)$ and $(E_2, \mathcal{S}_2)$ be two measurable spaces such that* $\mathrm{card}(\mathcal{S}_1) \leq \alpha$ *and* $\mathrm{card}(\mathcal{S}_2) \leq \alpha$*. Then the cardinality of the product σ-algebra $\mathcal{S}_1 \otimes \mathcal{S}_2$ does not exceed α, either.*

The next auxiliary proposition is crucial.

Lemma 4. *Assume the Continuum Hypothesis. There exists a mapping*

$$f : \mathbf{R} \to [0,1]^J$$

satisfying the following relations:

(1) *the graph of f is thick with respect to the product measure $\lambda \times \nu$;*

(2) *for any $(\lambda \times \nu)$-measure zero zet Z, the set $Z' = \{x : (x, f(x)) \in Z\}$ is of λ-measure zero.*

Applying the above-mentioned lemmas, we obtain the following theorem.

Theorem 1. *Under the Continuum Hypothesis, there exists a nonseparable measure on $\mathbf{R}$ which extends λ and whose null-sets coincide with λ-null-sets.*

On the other hand, the following statement is valid.

Theorem 2. *Suppose that $(E, \mathcal{S}, \mu)$ is a probability space such that* $\mathrm{card}(\mathcal{S}) \leq \mathbf{c}$. *Then there exists a mapping $h : \mathbf{R} \to E$ satisfying the following conditions:*

(1) *the graph of h is thick in the product space $\mathbf{R} \times E$, so h determines a certain extension λ' of the Lebesgue measure λ;*

(2) *there is a set in $\mathcal{I}(\lambda')$ which is of full outer λ-measure; in particular, we have $\mathcal{I}(\lambda') \neq \mathcal{I}(\lambda)$.*

The proof of Theorem 2 is based on the existence of a Bernstein subset of $\mathbf{R}$. Information on Bernstein sets and their properties can be found, e.g., in [2] and [5].

At this moment it is unknown whether a nonseparable extension of λ in Theorem 1 can be chosen to be invariant under all isometries of $\mathbf{R}$. Also, the following problem remains unsolved.

Problem. Let $(G, \cdot)$ be an uncountable group and let μ be a nonzero σ-finite separable left G-invariant (left G-quasiinvariant) measure on G. Does there exist a nonseparable left G-invariant (left G-quasiinvariant) extension of μ?

In connection with the above-mentioned problem, we can formulate the following result.

Theorem 3. *Assume the Continuum Hypothesis. Let $(G, \cdot)$ be a group with* $\mathrm{card}(G) = \mathbf{c}$ *and let μ be a nonzero σ-finite left G-invariant metrically transitive measure on G. Then there exists a nonseparable left G-invariant extension μ' of μ. More precisely, the Hilbert dimension of the space $L_2(\mu')$ is equal to $2^{\mathbf{c}}$.*

Actually, Theorem 3 can be generalized to the case when we are given a set E with $\mathrm{card}(E) = \mathbf{c}$, a group G of transformations of E with $\mathrm{card}(G) \leq \mathbf{c}$ and a nonzero σ-finite G-invariant metrically transitive measure μ on E.

Note that Theorem 3 essentially strengthens the corresponding result for the left (right) Haar measure on an infinite compact metrizable topological group (see [13], Chapter 4). The proof of this theorem is based on some properties of Ulam's transfinite matrix (see, e.g., [2]) and does not need any concept from the theory of topological groups. Since the left (right) Haar measure on a σ-compact locally compact topological group is metrically transitive, the assertion of Theorem 3 immediately implies the result from [13].

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