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COMBINATORIAL PROPERTIES OF FAMILIES OF SETS AND EULER-VENN DIAGRAMS

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As a rule, any course of set theory (or mathematical logic) oriented to beginners contains certain information on Euler-Venn diagrams. These diagrams help students to see visually various kinds of sets, inclusion relations between them and the standard set-theoretical operations such as the union, intersection, difference and symmetric difference of two given sets. At first sight, the material about Euler-Venn diagrams looks as very easy and not problematic. But, as turns out, such diagrams have interesting connections with purely combinatorial (sometimes, rather difficult) problems, with discrete geometry, with the theory of knots and many other topics. In this note, we would like to touch upon several questions of this sort.

In the sequel, a version of the precise definition of an Euler-Venn diagram for a given finite family of subsets of a universal set is presented. Certain geometrical properties of such diagrams are discussed and close connections with purely combinatorial problems and the theory of convex sets (e.g., Helly's type theorems and related topics) are indicated. In particular, some geometrical realizations of uncountable independent families of sets are considered.

Let U (respectively, V) be a nonempty universal set, $\{X_1, X_2, \dots, X_n\}$ (respectively, $\{Y_1, Y_2, \dots, Y_n\}$) be a finite family of subsets of U (respectively, of V).

We shall say that these two families are combinatorially isomorphic (or combinatorially equivalent) if, for each subset I of $\{1, 2, \dots, n\}$ and for any function $f : I \to \{0, 1\}$, we have

$$\cap\{X_i^{f(i)} : i \in I\} \neq \emptyset \Leftrightarrow \cap\{Y_i^{f(i)} : i \in I\} \neq \emptyset,$$

where $X_i^{f(i)}$ (respectively, $Y_i^{f(i)}$) coincides with X_i (respectively, with Y_i) if $f(i) = 0$ and coincides with $U \setminus X_i$ (respectively, with $V \setminus Y_i$) if $f(i) = 1$.

It is not hard to see that the families $\{X_1, X_2, \dots, X_n\}$ and $\{Y_1, Y_2, \dots, Y_n\}$ are combinatorially isomorphic if and only if

$$\cap\{X_i^{f(i)} : i \in \{1, 2, \dots, n\}\} \neq \emptyset \Leftrightarrow \cap\{Y_i^{f(i)} : i \in \{1, 2, \dots, n\}\} \neq \emptyset$$

for any function $f : \{1, 2, \dots, n\} \to \{0, 1\}$ (where $X_i^{f(i)}$ and $Y_i^{f(i)}$ are defined as above).

Remark 1. Obviously, the same notion of the combinatorial isomorphism can be introduced for any two (not necessarily finite) families $\{X_j : j \in J\}$ and $\{Y_j : j \in J\}$ of subsets of the universal sets U and V. In this more general case, it is required that

$$\cap\{X_i^{f(i)} : i \in I\} \neq \emptyset \Leftrightarrow \cap\{Y_i^{f(i)} : i \in I\} \neq \emptyset$$

for each finite subset I of J and for every function $f : I \to \{0, 1\}$.

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Let D be a geometric figure in the Euclidean plane $\mathbf{R}^2$ and let $D_1, D_2, \ldots, D_n$ be geometric figures all of which are contained in D.

We shall say that $\{D_1, D_2, \ldots, D_n\}$ is an Euler-Venn diagram of the family $\{X_1, X_2, \ldots, X_n\}$ (with respect to the pair (U, D)) if $\{D_1, D_2, \ldots, D_n\}$ is combinatorially isomorphic to $\{X_1, X_2, \ldots, X_n\}$. In this case, we shall also say that the family $\{D_1, D_2, \ldots, D_n\}$ is a geometrical realization of $\{X_1, X_2, \ldots, X_n\}$ by figures $D_1, D_2, \ldots, D_n$.

For extensive information about Euler-Venn diagrams, see e.g. [3], [7], [8], [10], [11], where some of their applications are also presented. Of course, there are many other works devoted to such diagrams and their combinatorial or purely geometrical properties. As a rule, the authors of those works do not formulate the precise description of Euler-Venn diagrams and, in fact, they are considered intuitively, without any formal definition. Notice that our definition differs from the one given in [3].

In connection with the introduced notion, the following natural question arises: what kind of figures can be taken in geometrical realizations of various finite families of subsets of a universal set U?

First, let us give two simple examples which show that, in general, one cannot guarantee the existence of an Euler-Venn diagram whose figures satisfy rather natural geometrical conditions.

Example 1. Take any four subsets X_1, X_2, X_3, X_4 of U such that

$$X_1 \cap X_2 \cap X_3 \neq \emptyset, \quad X_1 \cap X_3 \cap X_4 \neq \emptyset, \quad X_1 \cap X_2 \cap X_4 \neq \emptyset, \quad X_2 \cap X_3 \cap X_4 \neq \emptyset,$$
$$X_1 \cap X_2 \cap X_3 \cap X_4 = \emptyset.$$

A simple geometrical argument leads to the conclusion that there exists no Euler-Venn diagram of the family $\{X_1, X_2, X_3, X_4\}$, consisting of convex subsets D_1, D_2, D_3, D_4 of the plane. This fact easily follows from the well-known Helly theorem on intersections of convex sets (see, for instance, [4]) but can also be proved directly, without reference to the above-mentioned theorem.

Example 2. Take any three subsets X_1, X_2, X_3 of U such that

$$X_1^{f(1)} \cap X_2^{f(2)} \cap X_3^{f(3)} \neq \emptyset$$

for every function $f : \{1, 2, 3\} \to \{0, 1\}$. Let D_1, D_2, D_3 be three closed circles (i.e. discs) in the plane with the same radius r. One can assert that:

(a) if the centers of D_1, D_2, D_3 are collinear, then these circles never yield an Euler-Venn diagram of $\{X_1, X_2, X_3\}$;

(b) if the centers of D_1, D_2, D_3 are the vertices of a triangle all whose angles are strictly less than $\pi/2$, then for some $r > 0$, these circles yield an Euler-Venn diagram of $\{X_1, X_2, X_3\}$.

In connection with Example 2, let us mention that if three given points of the plane are collinear, then no three circles with centers in these points form an Euler-Venn diagram of $\{X_1, X_2, X_3\}$. On the other hand, if three given points of the plane are not collinear, then there exist three circles in the same plane, which form an Euler-Venn diagram of $\{X_1, X_2, X_3\}$ and whose centers coincide, respectively, with these points. By using a separation theorem for any two disjoint subsets of the set of all vertices of a multi-dimensional simplex, the last fact can be easily extended to the Euclidean space $\mathbf{R}^k$ (cf. [1], [9]).

Every set representable in the form $[a, b[\times[c, d[$, where $\{a, b, c, d\} \subset \mathbf{R}$, is called a (half-open) rectangle in the plane $\mathbf{R}^2$.

A set $X \subset \mathbf{R}^2$ is rectangular if X can be represented as the union of a finite family of rectangles (note that this family can always be chosen to be disjoint).

It is well known that if D is a nonempty rectangular set, then the family of all rectangular subsets of D is an algebra of sets.

The following statement is valid (which can be proved by induction on n).

Theorem 1. *Let D be a nonempty rectangular set on the plane $\mathbf{R}^2$. For any finite family $\{X_1, X_2, \ldots, X_n\}$ of subsets of U, there exists an Euler-Venn diagram $\{D_1, D_2, \ldots, D_n\}$ of $\{X_1, X_2, \ldots, X_n\}$ such that all D_i $(1 \leq i \leq n)$ are rectangular subsets of D.*

Remark 2. Actually, the argument applied in the proof of Theorem 1 does not depend on the dimension of the plane. The same argument works for Euclidean space of any nonzero dimension. In particular, in the case of the real line $\mathbf{R}$, we can assert that an arbitrary finite family $\{X_1, X_2, \ldots, X_n\}$ of subsets of U admits a geometrical realization $\{D_1, D_2, \ldots, D_n\}$ such that each set D_i is representable as the union of a finite family of half-open subintervals of D, where $D = [a, b[$ is a fixed nonempty half-open interval in $\mathbf{R}$.

Recall that a family $\{X_1, X_2, \ldots, X_n\}$ of subsets of U is independent (in the purely set-theoretical sense) if

$$X_1^{f(1)} \cap X_2^{f(2)} \cap \cdots \cap X_n^{f(n)} \neq \emptyset$$

for all functions $f : \{1, 2, \ldots, n\} \rightarrow \{0, 1\}$. In such a case $X_1, X_2, \ldots, X_n$ are also called mutually independent subsets of U.

The nonempty sets of the form $X_1^{f(1)} \cap X_2^{f(2)} \cap \cdots \cap X_n^{f(n)}$ are sometimes called atomic components (or constituents) of $\{X_1, X_2, \ldots, X_n\}$ (cf. [6]). Clearly, the number of all atomic components does not exceed 2^n and is equal to 2^n only in the case of an independent family $\{X_1, X_2, \ldots, X_n\}$.

Let us observe that if each set X_r $(r = 1, 2, \ldots, n)$ is represented in the form $X_r = \cup\{X_{i,r} : i = 1, 2, \ldots, k(r)\}$ or in the form $X_r = \cap\{X_{i,r} : i = 1, 2, \ldots, k(r)\}$, then the number of all atomic components of $\{X_1, X_2, \ldots, X_n\}$ does not exceed the number of all atomic components of $\{X_{i,r} : i = 1, 2, \ldots, k(r),\ r = 1, 2, \ldots, n\}$.

In the general case, when an arbitrary family $\{X_j : j \in J\}$ of subsets of U is given, we say that $\{X_j : j \in J\}$ is independent if every finite subfamily of this family is independent in the above-mentioned sense.

Independent families of sets play an important role in many questions of mathematics, especially, in general topology and measure theory (see, e.g., [2], [5], [6], [12]).

Let us return to Euler-Venn diagrams. It is well known that there are no four circles in the plane, which yield an Euler-Venn diagram of an independent family $\{X_1, X_2, X_3, X_4\}$ of subsets of U. A more precise result will be established in our further considerations. First, let us notice that the number of all open connected pairwise disjoint regions (or domains) which are produced by the union of n circumferences in the plane does not exceed $n(n-1)+2$ (it suffices to apply induction on n). In particular, if $n = 4$, then we have at most 14 $(< 16 = 2^4)$ such regions. But for obtaining the required result, only this fact is not sufficient. Indeed, apriori we cannot exclude the possibility that some atomic components of an Euler-Venn diagram of a given family $\{X_1, X_2, X_3, X_4\}$ differ from all above-mentioned regions (for instance, an atomic component apriori may have empty interior). Therefore, a more delicate argument is needed here.

For a given figure $D \subset \mathbf{R}^2$, let the symbol $bd(D)$ denote, as usual, the boundary of D. In particular, if D is a circle, then $bd(D)$ stands for the circumference of D.

The two lemmas formulated below are elementary and their proof is not connected with any difficulty.

Lemma 1. *Let $D_1, D_2, \ldots, D_n$ be some circles in the plane, such that a certain atomic component of the family $\{D_1, D_2, \ldots, D_n\}$ has empty interior. Then this atomic* *component is of the form $\{x\}$ and the disjunction of the following two assertions is satisfied:*

(1) *there are two tangent circles D_i and D_j from the family $\{D_1, D_2, \ldots, D_n\}$, for which the equality $D_i \cap D_j = \{x\}$ holds;*

(2) *there are three circles D_i, D_j, D_k from this family, such that $bd(D_i) \cap bd(D_j) \cap bd(D_k) = \{x\}$.*

Lemma 2. *Let D_1, D_2, D_3, D_4 be four circles in the plane.*

(1) *If two of them are tangent, then the number of atomic components of $\{D_1, D_2, D_3, D_4\}$ does not exceed* 14.

(2) *If the sets $bd(D_1) \cap bd(D_2) \cap bd(D_3)$ and $bd(D_1) \cap bd(D_2) \cap bd(D_4)$ are atomic components of $\{D_1, D_2, D_3, D_4\}$, then either $D_1 \cap D_3 \cap D_4 = \emptyset$ or $D_2 \cap D_3 \cap D_4 = \emptyset$; consequently, the number of atomic components of $\{D_1, D_2, D_3, D_4\}$ does not exceed* 14.

Using the above-mentioned lemmas, we readily obtain that for any four circles D_1, D_2, D_3, D_4 in the plane, the number of atomic components of the family $\{D_1, D_2, D_3, D_4\}$ does not exceed 14. In particular, no four circles in the plane are mutually independent.

Remark 3. It is easy to show that in the plane $\mathbf{R}^2$ there exist three mutually independent circles D_1, D_2, D_3 such that $bd(D_1) \cap bd(D_2) \cap bd(D_3)$ is a singleton and, simultaneously, is an atomic component of the family $\{D_1, D_2, D_3\}$.

Remark 4. For $\mathbf{R}$ we have a result similar to the case of $\mathbf{R}^2$. Namely, no three nondegenerate closed bounded intervals in $\mathbf{R}$ are mutually independent. More generally, if $k+1$ points of the Euclidean space $\mathbf{R}^k$ are not in general position, then no $k+1$ balls with centers in these points are mutually independent (see again [1] and [9]). This result immediately yields the nonexistence of four mutually independent circles in the plane.

Lemma 3. *Let $C_1, C_2, \ldots, C_n, C_{n+1}$ be pairwise distinct circumferences in the plane and let $X = C_{n+1} \cap (C_1 \cup C_2 \cup \cdots \cup C_n)$. Denote:*

X_0 = the set of all points x from X such that $\{x\} = C_{n+1} \cap C_i$ for some $i \in [1, n]$ and x does not belong to $C_1 \cup C_2 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup \cdots \cup C_n$;

X_1 = the set of all points x from X such that $\{x\}$ is a proper subset of $C_{n+1} \cap C_i$ for some $i \in [1, n]$ and x does not belong to $C_1 \cup C_2 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup \cdots \cup C_n$;

$X_2 = X \setminus (X_0 \cup X_1)$;

$n_0 = \mathrm{card}(X_0)$, $n_1 = \mathrm{card}(X_1)$, $n_2 = \mathrm{card}(X_2)$.

Then the inequality $2n_0 + n_1 + 2n_2 \leq 2n$ is satisfied.

The proof of Lemma 3 is based on a purely combinatorial argument. From this lemma one can infer the following statement (by applying induction on n).

Theorem 2. *For any circles $D_1, D_2, \ldots, D_n$ in the plane, the number of all atomic components of $\{D_1, D_2, \ldots, D_n\}$ does not exceed $n(n-1)+2$. In particular, if $n \geq 4$, then the family $\{D_1, D_2, \ldots, D_n\}$ is not independent.*

By using similar ideas, the next statement can be proved.

Theorem 3. *Let $D_1, D_2, \ldots, D_n$ be figures in $\mathbf{R}^2$ whose boundaries are (irreducible) algebraic curves of a degree $\leq m$, where $m > 0$ is a fixed natural number. Then there exists a natural number $n(m)$ such that, for any $n > n(m)$, the family $\{D_1, D_2, \ldots, D_n\}$ is not independent.*

For convex polygons in $\mathbf{R}^2$, the situation is essentially different. Indeed, it is not difficult to show that there exists an infinite countable family of mutually independent convex polygons in $\mathbf{R}^2$ (and it immediately follows from this fact that, for any natural number $k \geq 3$, there exists an infinite countable independent family of convex polyhedra in the Euclidean space $\mathbf{R}^k$). In view of this circumstance, it is natural to ask whether there exists an uncountable independent family of convex polygons in $\mathbf{R}^2$. It turns out that the answer to this question is negative even without assumption of the convexity of polygons, i.e., there is no uncountable independent family of polygons in the plane $\mathbf{R}^2$ and, similarly, there exists no uncountable independent family of polyhedra in the Euclidean space $\mathbf{R}^k$.

We say that a compact set $Q \subset \mathbf{R}^2$ with nonempty interior is a quasi-polygon if $bd(Q)$ admits a representation in the form of the union of countably many line segments.

In terms of quasi-polygons, the following statement is valid.

Theorem 4. *There exists an uncountable independent family of convex quasi-polygons in* $\mathbf{R}^2$.

The proof of the last statement is essentially non-elementary. Actually, it appeals to the method of transfinite induction up to the first uncountable ordinal number ω_1.

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