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ON SETS WITH HOMOGENEOUS LINEAR SECTIONS

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In this report we will use the following fairly standard notation:
$\mathbf{R} =$ the real line;
$\mathbf{c} =$ the cardinality of the continuum;
$\lambda_1 =$ the linear Lebesgue measure on $\mathbf{R}$.

Let a and b be any two positive real numbers. It is easy to indicate a subset Z of the Euclidean plane $\mathbf{R}^2$, such that all horizontal sections of Z are line segments of length a and all vertical sections of Z are line segments of length b. In fact, Z can be taken as a strip in $\mathbf{R}^2$ whose boundary lines are expressible in the form of the equations

$$y = (b/a)x + c_1, \quad y = (b/a)x + c_2,$$

where $|c_1 - c_2| = b$. This example is absolutely elementary and visual. The natural question arises whether it is possible to construct a bounded set with analogous properties of its linear (horizontal and vertical) sections.

More precisely, suppose that $0 \le a \le 1$ and $0 \le b \le 1$. Then one may ask whether there exists a set $W \subset [0,1]^2$ such that:

(1) all horizontal sections $([0,1] \times \{y\}) \cap W$, where $y \in [0,1]$, are of linear Lebesgue measure a;

(2) all vertical sections $(\{x\} \times [0,1]) \cap W$, where $x \in [0,1]$, are of linear Lebesgue measure b.

In the sequel, we shall say that $W \subset [0,1]^2$ is an (a,b)-homogeneous set in the unit square $[0,1]^2$ if both relations (1) and (2) are satisfied for W.

Notice that if $a = b$, then a set W with the above-mentioned property can be constructed effectively, i.e., without the aid of the Axiom of Choice. The main idea of such a construction is as follows. We first represent the given number $a \in [0,1]$ in the form

$$a = 1/2^{n_1} + 1/2^{n_2} + \cdots + 1/2^{n_k} + \cdots,$$

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where $(n_1, n_2, \dots, n_k, \dots)$ is a strictly increasing sequence of positive integers, and then we define by recursion a sequence $(W_1, W_2, \dots, W_k, \dots)$ of subsets of $[0,1]^2$, which increases by the inclusion relation and, for each natural number $k > 0$, the horizontal and vertical sections of W_k by the line segments

$$[0,1] \times \{y\}, \quad [0,1] \times \{x\} \quad (x \in [0,1],\ y \in [0,1])$$

are of linear Lebesgue measure $1/2^{n_1} + 1/2^{n_2} + \dots + 1/2^{n_k}$. Finally, we put

$$W = \cup\{W_k : 0 < k < \omega\}.$$

If $a \neq b$, then no effective construction of the required set W is possible, because according to the classical Fubini theorem, such a W must be nonmeasurable with respect to the two-dimensional Lebesgue measure λ_2 on the plane $\mathbf{R}^2$. Moreover, as follows from one result of Friedman [3], a set W with the desired properties cannot be constructed even within the Zermelo-Fraenkel set theory. However, by starting with the classical Sierpiński decomposition of the unit square $[0,1]^2$ (see [6], [7]), it becomes possible to establish the following statement.

Theorem 1. *Suppose that all subsets of* $\mathbf{R}$ *whose cardinalities are less than* $\mathbf{c}$ *have* λ_1*-measure zero. Then there exists an* (a,b)*-homogeneous subset of the square* $[0,1]^2$.

For any (a,b)-homogeneous set $W \subset [0,1]^2$, denote by χ_W the characteristic function of W. It is easy to see that there exist iterated integrals

$$\int_0^1 \left(\int_0^1 \chi_W(x,y)dx \right) dy = a, \quad \int_0^1 \left(\int_0^1 \chi_W(x,y)dy \right) dx = b.$$

Clearly, these iterated integrals are equal to each other if and only if $a = b$.

Some situations, where the equality of the iterated integrals is fulfilled for those subsets Z of $[0,1]^2$ which are not *a priori* assumed to be λ_2-measurable, are discussed in the old paper by Pkhakadze [5] (cf. also [1]–[4]).

It should be noticed that, for any $a \in [0,1]$, there exists an (a,a)-homogeneous set $W \subset [0,1]^2$ nonmeasurable with respect to λ_2. Obviously, the iterated integrals of χ_W do exist and are equal to each other.

Now, it is natural to extend the above considerations to the case of the three-dimensional Euclidean space $\mathbf{R}^3$. Here two possibilities must be taken into account. On the one hand, we may consider again the linear sections of a given set $W \subset \mathbf{R}^3$ and, on the other hand, we may also consider its sections by those planes which are parallel to the three coordinate planes xOy, yOz, and zOx.

First of all, let us remark that if any three positive numbers a, b, and c are given, then there exists a set $P \subset \mathbf{R}^3$ such that:

($*$) all sections of P by the planes parallel to xOy are triangles of area a;
($**$) all sections of P by the planes parallel to yOz are triangles of area b;
($***$) all sections of P by the planes parallel to zOx are triangles of area c.

Moreover, an elementary argument shows that, similarly to the case of $\mathbf{R}^2$, some unbounded prism can be taken as P.

If we want to obtain an analogous result for bounded sets in $\mathbf{R}^3$, then we again need to use some delicate set-theoretical techniques inspired by the Sierpiński decomposition of $[0,1]^2$.

Suppose that

$$0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad 0 \leq c \leq 1.$$

We shall say that a set $W \subset [0,1]^3$ is (a,b,c)-homogeneous with respect to its two-dimensional sections if the following conditions are satisfied:

(i) all sections of W by the planes $\{x\} \times \mathbf{R} \times \mathbf{R}$, where $x \in [0,1]$, have λ_2-measure a;

(ii) all sections of W by the planes $\mathbf{R} \times \{y\} \times \mathbf{R}$, where $y \in [0,1]$, have λ_2-measure b;

(iii) all sections of W by the planes $\mathbf{R} \times \mathbf{R} \times \{z\}$, where $z \in [0,1]$, have λ_2-measure c.

In terms of this definition, we can formulate and prove the statement analogous to Theorem 1.

Theorem 2. *Suppose that all subsets of* $\mathbf{R}$ *whose cardinalities are less than* $\mathbf{c}$ *have* λ_1*-measure zero. Then there exists an* (a,b,c)*-homogeneous set in* $[0,1]^3$ *with respect to its two-dimensional sections.*

Obviously, if the disjunction

$$a \neq b \quad \vee \quad b \neq c \quad \vee \quad c \neq a$$

holds true, then the required set W is not measurable with respect to the three-dimensional Lebesgue measure λ_3 on $\mathbf{R}^3$.

Let a, b, and c be any three positive real numbers. It is easy to see that there exists a subset S of $\mathbf{R}^3$ such that all linear sections of S by the lines parallel to the axis Oz are segments of length a, all linear sections of S by the lines parallel to the axis Ox are segments of length b, and all linear sections of S by the lines parallel to the axis Oy are segments of length c.

In fact, the role of S can be played by the set of all points lying between certain two parallel planes in $\mathbf{R}^3$.

Taking this circumstance into account, we may introduce another notion of homogeneity of subsets of $[0,1]^3$. Namely, we shall say that a set $W \subset [0,1]^3$ is (a,b,c)-homogeneous with respect to its linear sections if the following three conditions are satisfied:

(i') all sections of W by the lines $\{x\} \times \{y\} \times \mathbf{R}$, where $x \in [0,1]$ and $y \in [0,1]$, have λ_1-measure a;

(ii') all sections of W by the lines $\mathbf{R} \times \{y\} \times \{z\}$, where $y \in [0,1]$ and $z \in [0,1]$, have λ_1-measure b;

(iii') all sections of W by the lines $\{x\} \times \mathbf{R} \times \{z\}$, where $x \in [0,1]$ and $z \in [0,1]$, have λ_1-measure c.

By applying the method similar to Sierpiński's construction [6], it is possible to prove the next statement.

Theorem 3. *Under the assumption that every subset of* $\mathbf{R}$ *with cardinality less than* $\mathbf{c}$ *is of* λ_1*-measure zero, there exists a set* $W \subset [0,1]^3$ *which is* (a,b,c)*-homogeneous with respect to its linear sections.*

Consequently, all sections of W *by the planes parallel to one of the coordinate planes are, respectively,* (a,b)*-homogeneous,* (b,c)*-homogeneous and* (c,a)*-homogeneous.*

If $0 \le a = b = c \le 1$, then one may pose the question whether there exists an effective construction of a λ_3-measurable set $W \subset [0,1]^3$ which is (a,a,a)-homogeneous with respect to its linear sections. It turns out that the answer to this question is positive and the construction of such a set W is similar to some recursive constructions of classical fractals (e.g., Sierpiński's carpet).

A key role in the construction of W is played by the following auxiliary proposition.

Lemma. *There exist two families*

$$K_1, K_2, \dots, K_9, \quad T_1, T_2, \dots, T_9$$

of cubes in $[0,1]^3$ *such that:*

(a) *all cubes* K_i *and* T_j *have edges of length* $1/3$*, which are parallel to the corresponding edges of* $[0,1]^3$*;*

(b) *the orthogonal projection of the set* $\cup\{K_i : 1 \le i \le 9\}$ *onto any facet of* $[0,1]^3$ *coincides with that facet;*

(c) *the orthogonal projection of the set* $\cup\{T_j : 1 \le j \le 9\}$ *onto any facet of* $[0,1]^3$ *coincides with that facet;*

(d) $int(K_i \cap T_j) = \emptyset$ *for all indices* $i = 1,2,\dots,9$ *and* $j = 1,2,\dots,9$.

Moreover, these two families of cubes can be chosen to be symmetric to each other with respect to the center of $[0,1]^3$.

The proof of this lemma is purely geometric.

As soon as the lemma is proved, the construction of the required W can be done similarly to the two-dimensional case. Namely, we take an arbitrary number a from $[0,1/2]$ and represent it in the form

$$a = 1/3^{n_1} + 1/3^{n_2} + \cdots + 1/3^{n_k} + \cdots ,$$

where $n_1 < n_2 < \cdots < n_k < \cdots$. Then we recursively construct an increasing (by the inclusion relation) sequence $(W_1, W_2, \dots, W_k, \dots)$ of subsets of

$[0,1]^3$ such that all sections of W_k by the segments of the form

$$\{x\} \times \{y\} \times [0,1], \quad [0,1] \times \{y\} \times \{z\}, \quad \{x\} \times [0,1] \times \{z\} \qquad (x,y,z \in [0,1])$$

are of λ_1-measure $1/3^{n_1} + 1/3^{n_2} + \cdots + 1/3^{n_k}$. Further, we put

$$W = \cup\{W_k : 0 < k < \omega\}$$

and so get the required set W for a.

If $1/2 < a \leq 1$, then we take the number $a' = 1 - a$ and a set $W' \subset [0,1]^3$ corresponding to this number. Putting

$$W = [0,1]^3 \setminus W',$$

we come to the set W corresponding to a.

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References

1. K. Ciesielski, Set-theoretic real analysis. *J. Appl. Anal.* **3** (1997), No. 2, 143–190.
2. C. Freiling, Axioms of symmetry: throwing darts at the real number line. *J. Symbolic Logic* **51** (1986), No. 1, 190–200.
3. H. Friedman, A consistent Fubini-Tonelli theorem for nonmeasurable functions. *Illinois J. Math.* **24** (1980), No. 3, 390–395.
4. A. B. Kharazishvili, Strange functions in real analysis, Second edition. *Pure and Applied Mathematics* (*Boca Raton*), 272. *Chapman & Hall/CRC, Boca Raton, FL*, 2006.
5. Sh. S. Pkhakadze, On the iterated integrals. (Russian) *Proc. A. Razmadze Math. Institute* **20** (1954), 167–209.
6. W. Sierpiński, Sur un théoréme équivalent á l'hypothése du continu. *Bull. Intern. Acad. Sci. Cracovie, Ser.* A, 1919, 1–3.
7. J. C. Simms, Sierpinski's theorem. *Simon Stevin* **65** (1991), No. 1–2, 69–163.

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