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On thick subgroups of uncountable σ-compact locally compact commutative groups

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ABSTRACT

We show that, for any uncountable commutative group $(G,+)$, there exists a countable covering $\{G_j: j \in J\}$ where each G_j is a subgroup of G satisfying the equality $card(G/G_j) = card(G)$. This purely algebraic fact is used in certain constructions of thick and nonmeasurable subgroups of an uncountable σ-compact locally compact commutative group equipped with the completion of its Haar measure.

There are several constructions of nonmeasurable sets with respect to the completion of the left Haar measure on a locally compact topological group. Some of those constructions yield even nonmeasurable subgroups of an initial group (see, for instance, [1] and [4]). Let us briefly recall the main technical tool for obtaining nonmeasurable subgroups.

Let G be a σ-compact locally compact topological group and let μ denote the completion of the left Haar measure on G. If H is a dense subgroup of G, then the disjunction of the following two relations holds true:

(1) H is of μ-measure zero;
(2) H is μ-thick in G, i.e., $\mu_*(G \setminus H) = 0$, where μ_* denotes the inner measure associated with μ.

The proof of this disjunction relies on the two fundamental properties of μ: the metrical transitivity (or ergodicity) and the Steinhaus property (cf. [1,4,14]).

Consequently, if H is a dense subgroup of G distinct from G and, in addition, H is not of μ-measure zero, then H turns out to be thick and nonmeasurable with respect to μ.

Some of μ-nonmeasurable subsets of G can be used for getting proper left-invariant extensions of μ. Indeed, it is known that there are various left-invariant extensions of μ (see, e.g., [1,4–9,12,13,16,18–21]). However, it should be noticed that

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the above-mentioned two fundamental properties of μ may fail to be true for such extensions. In particular, it was proved that:

(a) there exists a translation-invariant extension ν' of the Lebesgue measure λ on the real line **R**, which is metrically transitive but does not possess the Steinhaus property (see [14]);
(b) there exists a translation-invariant extension ν'' of the same λ, which is not metrically transitive but has the Steinhaus property (see [12,14,18]).

It easily follows from (a) and (b) that there exists a translation-invariant extension ν of the two-dimensional Lebesgue measure λ_2 on the plane $\mathbf{R}^2$, which is not metrically transitive and does not possess the Steinhaus property. Indeed, it suffices to take as ν the product measure $\nu' \otimes \nu''$.

These facts show that, for obtaining nonmeasurable subgroups with respect to left-invariant extensions of the left Haar measure, a new method is needed exploiting an essential different idea.

In this paper we consider the case of an uncountable σ-compact locally compact commutative group $(G, +)$ equipped with its Haar measure and suggest an approach based on purely algebraic properties of G. The main result is formulated in Theorem 2.

First, we will establish the existence of certain types of nonmeasurable subgroups of an uncountable commutative group $(G, +)$, where G is not assumed to be endowed with any topology but only is equipped with a nonzero σ-finite G-invariant measure μ. The above-mentioned subgroups turn out to be helpful in an appropriate construction of G-invariant extensions of μ.

In order to describe such subgroups, some auxiliary facts from the general theory of commutative groups should be recalled. The algebraic structure of an arbitrary commutative group is investigated more or less thoroughly. The following statement yields a description of this structure and will be crucial in our further constructions (below, the symbol ω denotes, as usual, the least infinite cardinal number).

Kulikov's theorem. *Any commutative group $(G, +)$ admits a representation $G = \bigcup\{G_n\colon n < \omega\}$, where $\{G_n\colon n < \omega\}$ is an increasing (by inclusion) sequence of subgroups of G and, for each $n < \omega$, the group G_n is a direct sum of cyclic groups (finite or infinite).*

For the proof of this important statement, see, e.g., [3] or [17, p. 148]. As an immediate consequence of Kulikov's theorem, we get two lemmas, which are similar to each other.

Lemma 1. *Let $(G, +)$ be an uncountable commutative group. Then G admits a representation in the form $G = \bigcup\{G_n\colon n < \omega\}$, where $\{G_n\colon n < \omega\}$ is an increasing sequence of subgroups of G satisfying the following conditions:*

(a) *$card(G_n) > \omega$ for any $n < \omega$;*
(b) *each G_n is a direct sum of cyclic groups.*

Lemma 2. *Let $(G, +)$ be a commutative group whose cardinality is not cofinal with ω. Then G admits a representation $G = \bigcup\{G_n\colon n < \omega\}$, where $\{G_n\colon n < \omega\}$ is an increasing sequence of subgroups of G satisfying the following conditions:*

(a) *$card(G_n) = card(G)$ for any $n < \omega$;*
(b) *each G_n is a direct sum of cyclic groups.*

Indeed, to see the validity of Lemma 1, take an uncountable commutative group $(G, +)$ and its representation $G = \bigcup\{G_n\colon n < \omega\}$ as in Kulikov's theorem. Then at least one subgroup G_m must be uncountable, too, so the sequence $\{G_n\colon m \leqslant n < \omega\}$ of subgroups of G leads to the required result.

Analogously, if the cardinality of a commutative group $(G, +)$ is not cofinal with ω and $G = \bigcup\{G_n\colon n < \omega\}$ is a representation of G as in Kulikov's theorem, then, for some $m < \omega$, we must have $card(G_m) = card(G)$, and once again the sequence $\{G_n\colon m \leqslant n < \omega\}$ yields the desired result.

Lemma 3. *Any uncountable commutative group $(G, +)$ admits a representation*

$$G = \bigcup\{G_j\colon j \in J\},$$

where J is a countable set, G_j is a subgroup of G for each $j \in J$, and the inequality $card(G/G_j) > \omega$ holds true.

This lemma is completely sufficient for the proof of Theorem 1 below, which is algebraic and measure-theoretic in its nature. However, for the proof of Theorem 2, in the formulation of which the topological structure plays an essential role, we will need the following strengthened version of Lemma 3.

Lemma 4. *Let $(G,+)$ be an arbitrary uncountable commutative group. Then G admits a representation*

$$G = \bigcup\{G_j\colon\ j \in J\},$$

where J is a countable set, G_j is a subgroup of G for each $j \in J$, and the relation $card(G/G_j) = card(G) > \omega$ holds true.

Proof. Only two cases are possible.

1. $card(G)$ is cofinal with ω. This means that $G = \bigcup\{X_j\colon\ j \in J\}$, where J is a countable set of indices and $card(X_j) < card(G)$ for all $j \in J$. Let us denote:

$$G_j = \text{the subgroup of } G \text{ generated by } X_j.$$

Then, for each $j \in J$, we have $card(G_j) \leqslant card(X_j) + \omega < card(G)$ and

$$G = \bigcup\{G_j\colon\ j \in J\}, \qquad card(G/G_j) = card(G),$$

which gives us the required result.

2. $card(G)$ is not cofinal with ω. In this case, consider a representation of G as in Kulikov's theorem, i.e., write

$$G = \bigcup\{G_n\colon\ n < \omega\}.$$

Then, in view of Lemma 2, we may assume, without loss of generality, that all G_n in the above-mentioned representation have the same cardinality as G. Let us express G_n in the form of a direct sum

$$G_n = \sum_{i \in I_n} G_i,$$

where all G_i are cyclic groups (finite or infinite) and

$$card(G_i) \geqslant 2 \quad (i \in I).$$

According to our assumption, $card(I_n) = card(G)$. Further, let us represent I_n in the form

$$I_n = \bigcup\{I_{n,k}\colon\ k < \omega\},$$

where $\{I_{n,k}\colon\ k < \omega\}$ is a partition of I_n and $card(I_{n,k}) = card(I_n)$ for each $k < \omega$. Finally, let us put

$$G_{n,k} = \sum_{i \in I_{n,0} \cup I_{n,1} \cup \cdots \cup I_{n,k}} G_i$$

and consider the countable family

$$\{G_j\colon\ j \in J\} = \{G_{n,k}\colon\ n < \omega,\ k < \omega\}$$

of subgroups of G. It is not difficult to verify the validity of the following two relations:

(1) $card(G_n/G_{n,k}) = card(G)$ for any $n < \omega$ and $k < \omega$;
(2) $\bigcup\{G_{n,k}\colon\ k < \omega\} = G_n$ for every $n < \omega$.

Relation (1) implies that

$$(\forall n < \omega)(\forall k < \omega)\big(card(G/G_{n,k}) = card(G)\big).$$

Relation (2) implies at once that

$$G = \bigcup\{G_{n,k}\colon\ n < \omega,\ k < \omega\}.$$

We thus see that the required result is immediate from relations (1) and (2), which ends the proof. □

Concerning Lemmas 3 and 4, let us underline that a countable family $\{G_j\colon\ j \in J\}$ of subgroups of G, the existence of which is stated by these lemmas, is not, in general, increasing by inclusion.

Suppose now that an uncountable σ-compact locally compact commutative group $(G,+)$ is given. Obviously, we may speak of the Haar measure on G and its translation-invariant extensions (see, for instance, [1,4,5,8,9,12,16]). For our further purposes, we need only one topological property of G, which is formulated in the following well-known statement essentially due to Kakutani (see [1,11]).

Lemma 5. *If $(G,+)$ is an uncountable σ-compact locally compact commutative group, then $card(G) = 2^{\alpha}$, where α denotes the topological weight of G.*

Since for every infinite cardinal α, the inequality $cf(2^{\alpha}) > \alpha$ is valid, $card(G)$ is not cofinal with ω, so Lemma 2 can be applied to G.

Before formulating the next lemma, let us recall that a measure μ on a commutative group $(G, +)$ is G-quasi-invariant if $dom(\mu)$ is a G-invariant σ-algebra of subsets of G and

$$(\forall g \in G)\big(\forall X \in dom(\mu)\big)\big(\mu(g + X) = 0 \Leftrightarrow \mu(X) = 0\big).$$

Clearly, this property of μ is much weaker than the G-invariance property.

Lemma 6. *Let $(G, +)$ be an uncountable commutative group, μ be a σ-finite G-invariant (G-quasi-invariant) measure on G and let H be a subgroup of G such that $card(G/H) > \omega$. Then there exists a G-invariant (G-quasi-invariant) measure μ' on G, which extends μ and satisfies the equality $\mu'(H) = 0$.*

Proof. We use a fairly standard argument based on Marczewski's method of extending σ-finite invariant (quasi-invariant) measures (cf. [6,14,18,19]). Namely, let us consider the following family of sets in G:

$$\mathcal{I} = \{X\colon\ X \text{can be covered by countably many translates of } H\}.$$

It can readily be shown that $\mathcal{I}$ is a G-invariant σ-ideal of subsets of G. Moreover, if $X \in \mathcal{I}$, then, by definition, for some countable family $\{g_i\colon\ i \in I\} \subset G$, we must have

$$X \subset \bigcup\{g_i + H\colon\ i \in I\}.$$

Therefore, $X \cap (g + X) = \emptyset$ for each element $g \in G$ not belonging to the set

$$\bigcup\{g_i - g_j + H\colon\ i \in I,\ j \in I\}.$$

This implies (by easy transfinite induction) that there exists an uncountable family $\{g_\xi\colon\ \xi < \omega_1\}$ of elements of G such that the family

$$\{g_\xi + X\colon\ \xi < \omega_1\}$$

consists of pairwise disjoint sets. We thus conclude that $\mu_*(X) = 0$ (in view of the σ-finiteness and G-quasi-invariance of μ).

Now, applying the above-mentioned properties of this σ-ideal $\mathcal{I}$, we can extend a given measure μ in the following manner. Namely, we suppose (without loss of generality) that μ is complete and introduce the σ-algebra

$$\mathcal{S}' = \big\{Y \triangle X\colon\ Y \in dom(\mu),\ X \in \mathcal{I}\big\}.$$

Further, we put

$$\mu'(Y \triangle X) = \mu(Y)$$

for any set $Y \triangle X \in \mathcal{S}'$ (here $Y \in dom(\mu)$ and $X \in \mathcal{I}$). A straightforward verification shows that μ' is well defined and is a complete G-invariant (G-quasi-invariant) measure extending μ. According to the definition, $\mu'(X) = 0$ for all members X of the σ-ideal $\mathcal{I}$. In particular, we have $\mu'(H) = 0$. This ends the proof of the lemma. □

Remark 1. In fact, the preceding argument establishes that H is a G-absolutely negligible subset of G (for the definition and various properties of such subsets of G, see [13,14,18]). Observe that this argument essentially uses the commutativity of a given group G.

Theorem 1. *Let $(G, +)$ be an uncountable commutative group. There exists a countable family $\{G_j\colon\ j \in J\}$ of subgroups of G such that*:

(1) *for any nonzero σ-finite G-invariant (G-quasi-invariant) measure μ on G, at least one subgroup G_j is nonmeasurable with respect to μ*;
(2) *if G_j is nonmeasurable with respect to μ, then there exists a G-invariant (G-quasi-invariant) measure μ' on G extending μ and satisfying the relation $\mu'(G_j) = 0$.*

Proof. Let $\{G_j\colon\ j \in J\}$ be as in Lemma 3 and let μ be an arbitrary nonzero σ-finite G-invariant (G-quasi-invariant) measure on G. By virtue of the equality

$$G = \bigcup\{G_j\colon\ j \in J\},$$

we may write

$$0 < \mu(G) \leqslant \sum_{j \in J} \mu^*(G_j),$$

where μ^* denotes the outer measure associated with μ. Consequently, there exists an index $j \in J$ such that $\mu^*(G_j) > 0$. In view of the relation

$$card(G/G_j) > \omega$$

and of the G-quasi-invariance of μ, we easily infer that G_j must be nonmeasurable with respect to μ. Finally, applying Lemma 6, we conclude that, for the same G_j, there exists a G-invariant (G-quasi-invariant) measure μ' on G extending μ and satisfying the equality $\mu'(G_j) = 0$. This completes the proof. □

Remark 2. In view of Theorem 1, every nonzero σ-finite G-invariant (respectively, G-quasi-invariant) measure on G can be strictly extended by using some subgroup of G, which belongs to a fixed countable family of subgroups of G.

In Theorem 1 a group $(G, +)$ is not endowed with any topology. Now, we turn our attention to the case of an uncountable σ-compact locally compact commutative group $(G, +)$. In this case Theorem 1 can be significantly sharpened.

Theorem 2. *Let $(G, +)$ be an arbitrary uncountable σ-compact locally compact commutative group and let μ denote the completion of the Haar measure on G. There exists a countable family $\{H_j\colon j \in J_0\}$ of subgroups of G such that*:

(1) *$card(G/H_j) = card(G)$ for any $j \in J_0$*;
(2) *every H_j is μ-thick (and, consequently, dense) in G*;
(3) $\mu(G \setminus \bigcup\{H_j\colon j \in J_0\}) = 0$;
(4) *for any G-invariant extension μ' of μ, there exists an index $j \in J_0$ such that H_j is nonmeasurable with respect to μ'*;
(5) *if H_j is nonmeasurable with respect to μ', then there exists a G-invariant extension μ'' of μ' for which $\mu''(H_j) = 0$.*

Proof. Obviously, we may apply Lemma 4 to G. Let $G = \bigcup\{G_j\colon j \in J\}$ be a representation of G as in that lemma. Taking into account the equality $card(G) = 2^{\alpha}$, where α is the topological weight of G, we may choose a dense subset (even subgroup) D of G with $card(D) = \alpha$. Now, for each $j \in J$, let us denote:

$H_j =$ the group generated by $G_j \cup D$.

In this way we get the countable family $\{H_j\colon j \in J\}$ of subgroups of G and it is clear that $G = \bigcup\{H_j\colon j \in J\}$. Further, let us put

$$I = \{j \in J\colon \mu(H_j) = 0\}, \qquad J_0 = J \setminus I.$$

Then we obtain that

$$\mu\Big(G \setminus \bigcup\{H_j\colon j \in J_0\}\Big) = \mu\Big(\bigcup\{H_j\colon j \in I\}\Big) = 0,$$

i.e., relation (3) holds true.

Since $card(G/G_j) = card(G)$ and $card(D) < card(G)$, we easily get the equality $card(G/H_j) = card(G)$, which yields relation (1).

Further, relation (2) is valid because each H_j ($j \in J_0$) is not of μ-measure zero and is dense in G. By virtue of (1), H_j is a proper subgroup of G and hence turns out to be nonmeasurable with respect to μ.

Relation (4) is satisfied in view of (1) and (3).

Finally, relation (5) is readily implied by Lemma 6, taking into account the fact that $card(G/H_j) = card(G) > \omega$.

This completes the proof of Theorem 2. □

Remark 3. Some constructions of thick nonmeasurable subsets of an uncountable σ-compact locally compact commutative group $(G, +)$ are presented in [1,4] and other works (actually, those constructions develop the classical construction of Bernstein sets in an uncountable Polish space). Theorem 2 yields a much stronger result and shows that, for any G-invariant extension μ' of the Haar measure μ on G, there always exists a μ-thick subgroup of G nonmeasurable with respect to μ'. Moreover, such a subgroup can be found in a certain fixed countable family of μ-thick subgroups of G. In this context, the natural question arises whether the assumption of σ-compactness of G is necessary in Theorem 2. In any case, this theorem fails to be true for all uncountable non-discrete locally compact commutative groups, because there are groups of this type, which do not contain proper dense subgroups (see, e.g., [1]). We thus see that, in general, one cannot guarantee the existence of thick nonmeasurable subgroups. However, it is well known that every non-discrete locally compact commutative group $(\Gamma, +)$ contains an uncountable σ-compact subgroup $(G, +)$, which is open-and-closed in Γ. So Theorem 2 can be trivially applied to such a G.

Remark 4. Let $(G, +)$ be an uncountable compact commutative group equipped with the completion μ of its Haar probability measure. It is well known that any proper dense pseudocompact subgroup of G is thick and nonmeasurable with respect to μ (see, e.g., [1]). In [10] the existence of a large almost disjoint family of dense pseudocompact subgroups of G is established (assuming that G is nonmetrizable and connected). However, all members of that family can be regarded as μ'-measurable sets, where μ' is a certain G-invariant extension of μ. Indeed, the above-mentioned family generates a G-invariant σ-ideal of subsets of G such that the inner μ-measure of any set belonging to this σ-ideal is equal to zero. Therefore, applying Marczewski's method of extending σ-finite invariant measures (cf. the proof of Lemma 6), we come to the required extension μ'. Notice that a certain analogue of the Haar measure extension problem can be formulated in terms of pseudocompact topological group topologies. Namely, the question naturally arises whether any commutative pseudocompact group of uncountable weight admits a strictly larger pseudocompact group topology. This problem is positively solved in [2].

Remark 5. Consider an arbitrary uncountable commutative group $(G,+)$ without assuming that it is endowed with some topology compatible with the algebraic structure of G. Let $M = M(G)$ denote the class of all nonzero σ-finite G-quasi-invariant measures on G (the domains of measures from M may be various G-invariant σ-algebras of subsets of G). It was proved that there exists a set $X \subset G$ absolutely nonmeasurable with respect to M, i.e., X turns out to be nonmeasurable with respect to every measure $\mu \in M$ (see [15] where the same result is established even for uncountable solvable groups). In this connection, it should also be noticed that if H is an arbitrary subgroup of $(G,+)$, then there always exists a nonzero σ-finite G-invariant measure μ on G such that $H \in dom(\mu)$.

So far, it is unknown whether there exist absolutely nonmeasurable sets in any uncountable group.

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