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ON NONMEASURABLE UNIONS OF MEASURE ZERO SECTIONS OF PLANE SETS

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Let λ denote, as usual, the standard Lebesgue measure on the real line $\mathbf{R}$. There are many works devoted to various constructions of λ-nonmeasurable subsets of $\mathbf{R}$ (see, for instance, [1], [2], [3], [5], [6], [7], and [10])). Of course, the list of references can be significantly continued and expanded. Here we would like to make some remarks in connection with the recent paper by I. Reclaw [7], in which the following statement was established.

Theorem 1. *Assume Martin's Axiom and let B be a Borel subset of the Euclidean plane $\mathbf{R}^2$ such that:*

(1) *for each $y \in \mathbf{R}^2$, the section $B(y) = \{x : (x,y) \in B\}$ is of λ-measure zero;*

(2) $\lambda(pr_1(B)) = \lambda(\cup\{B(y) : y \in \mathbf{R}\}) > 0$.

Then there exists a set $Y \subset \mathbf{R}$ for which $\cup\{B(y) : y \in Y\}$ is not measurable with respect to λ.

In particular, under Martin's Axiom this result yields a positive solution to one problem formulated by J. Cichon (for more details, see [7]). The proof of Theorem 1 is based on the Luzin-Jankov-von Neumann theorem concerning the existence of measurable selectors (see, e.g., [4]) and on the next simple (probably, well-known) fact.

Lemma 1. *Assume Martin's Axiom. Let λ_2 denote the standard two-dimensional Lebesgue measure on the plane $\mathbf{R}^2$ and let Z be a λ_2-measure zero subset of $\mathbf{R}^2$. Then there exist two sets $X_1 \subset \mathbf{R}$ and $X_2 \subset \mathbf{R}$ such that:*

(1) *both X_1 and X_2 are λ-thick in $\mathbf{R}$, i.e., we have*

$$\lambda_*(\mathbf{R} \setminus X_1) = \lambda_*(\mathbf{R} \setminus X_2) = 0;$$

(2) $(X_1 \times X_2) \cap Z = \emptyset$.

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Notice that the required sets X_1 and X_2 can readily be constructed by using the method of transfinite recursion and utilizing the Fubini theorem at each step of the recursion. Actually, Lemma 1 does not need the full power of Martin's Axiom and it suffices to suppose that the covering number of the σ-ideal of all λ-measure zero sets is equal to $\mathbf{c}$, where $\mathbf{c}$ denotes, as usual, the cardinality of the continuum. In other words, it suffices to assume that $\mathbf{c}$ coincides with the smallest cardinality of a covering of $\mathbf{R}$ by λ-measure zero sets.

It should be emphasized that an abstract analogue of Lemma 1 holds under the Continuum Hypothesis. In order to formulate this analogue, let us recall that a pseudo-base for a given measure μ is any family $\mathcal{U} \subset dom(\mu)$ satisfying the following conditions:

(i) every set from $\mathcal{U}$ is of strictly positive μ-measure;

(ii) for any set $X \in dom(\mu)$ with $\mu(X) > 0$, there exists a set $Y \in \mathcal{U}$ such that $Y \subset X$.

Lemma 2. *Assume* **CH**. *Let μ be a σ-finite measure given on a set E and having a pseudo-base whose cardinality does not exceed* $\mathbf{c}$. *Further, let Z be a subset of $E \times E$ such that μ-almost all horizontal sections and μ-almost all vertical sections of Z are of μ-measure zero. Then there exist two μ-thick subsets X_1 and X_2 of E such that $(X_1 \times X_2) \cap Z = \emptyset$.*

Notice that a measure μ of Lemma 2 may be nonseparable (or, equivalently, the Hilbert space $L_2(\mu)$ of all μ-square-integrable real-valued functions may be nonseparable). Notice also that, in general, a set Z of the same lemma is not measurable with respect to the completion of the product measure $\mu \otimes \mu$. Moreover, a classical example due to Sierpiński shows that Z even may be $(\mu \otimes \mu)$-thick in the product space $E \times E$.

Theorem 1 admits an extension to the case of an analytic (i.e., Suslin) subset A of $\mathbf{R}^2$. Namely, the following statement is valid.

Theorem 2. *Suppose that the covering number of the σ-ideal of all λ-measure zero sets is equal to* $\mathbf{c}$. *Let A be an analytic subset of the Euclidean plane $\mathbf{R}^2$ such that:*

(1) for each $y \in \mathbf{R}^2$, the section $A(y) = \{x : (x, y) \in A\}$ is of λ-measure zero;

(2) $\lambda(pr_1(A)) = \lambda(\cup\{A(y) : y \in \mathbf{R}\}) > 0$.

Then there exists a set $Y \subset \mathbf{R}$ for which $\cup\{A(y) : y \in Y\}$ is not measurable with respect to λ.

Proof. The argument is quite similar to that of [7]. Only a few technical details occur. According to the Luzin-Jankov-von Neumann theorem, there exists a λ-measurable function $f : pr_1(A) \to \mathbf{R}$ whose graph is contained in A. Further, there is a Borel subset T of $pr_1(A)$ such that:

(a) $\lambda(pr_1(A) \setminus T) = 0$;

(b) the restriction $f|T$ is a Borel function.

Let us consider the product set $T \times \mathbf{R}$ and let us define a mapping

$$\Phi : T \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$$

by the formula

$$\Phi(x, y) = (y, f(x)) \qquad (x \in T,\ y \in \mathbf{R}).$$

Also, let us put $Z = \Phi^{-1}(A)$. Since Φ is a Borel mapping and A is an analytic set, Z is analytic, too. Consequently, Z is λ_2-measurable (and, more generally, Z is universally measurable). For any $x \in T$, we have

$$Z(x) = \{y : (x, y) \in Z\} = \{y : (y, f(x)) \in A\} = A(f(x)).$$

This relation shows that all x-sections of Z are of λ-measure zero, from which it follows (in view of the λ_2-measurability of Z) that $\lambda_2(Z) = 0$. Now, applying Lemma 1, we can find two sets $X_1 \subset \mathbf{R}$ and $X_2 \subset \mathbf{R}$ such that

$$\lambda_*(\mathbf{R} \setminus X_1) = \lambda_*(\mathbf{R} \setminus X_2) = 0, \quad (X_1 \times X_2) \cap Z = \emptyset.$$

We are going to verify that the set $Y = f(X_1 \cap T)$ is the required one, i.e., the union $\cup\{A(y) : y \in Y\}$ is nonmeasurable with respect to λ. First, let us check the inclusion

$$X_1 \cap T \subset \cup\{A(y) : y \in Y\}.$$

Indeed, take an arbitrary $x_1 \in X_1 \cap T$ and denote $y = f(x_1)$. Then $y \in Y$ and $(x_1, y) = (x_1, f(x_1)) \in A$. Therefore, $x_1 \in A(y)$, which yields the desired result. On the other hand, let us verify that

$$(X_2 \cap T) \cap (\cup\{A(y) : y \in Y\}) = \emptyset.$$

Indeed, take an arbitrary $x_2 \in X_2 \cap T$ and suppose to the contrary that $x_2 \in \cup\{A(y) : y \in Y\}$. This means that there exists $x_1 \in X_1 \cap T$ for which

$$x_2 \in A(f(x_1)), \quad (x_2, f(x_1)) \in A, \quad (x_1, x_2) \in Z,$$

which contradicts the equality $(X_1 \times X_2) \cap Z = \emptyset$.

Thus, the set $\cup\{A(y) : y \in Y\}$ is almost contained in T, contains $X_1 \cap T$ and does not intersect $X_2 \cap T$. By virtue of the equalities

$$\lambda^*(X_1 \cap T) = \lambda^*(X_2 \cap T) = \lambda(T),$$

we conclude that $\cup\{A(y) : y \in Y\}$ is not λ-measurable. □

From Theorem 2 one can readily infer the next proposition (cf. [2], [7]).

Theorem 3. *Suppose again that the covering number of the σ-ideal of all λ-measure zero sets is equal to* $\mathbf{c}$. *Let P_1 and P_2 be any two analytic sets in* $\mathbf{R}$, *let P_1 be of λ-measure zero, and let the algebraic sum*

$$P_1 + P_2 = \{p_1 + p_2 : p_1 \in P_1,\ p_2 \in P_2\}$$

have strictly positive λ-measure. Then there exists a subset Q_2 of P_2 such that the algebraic sum $P_1 + Q_2$ is not λ-measurable.

Proof. It suffices to consider the analytic set

$$A = \{(x, y) : x - y \in P_1,\ y \in P_2\}$$

in the plane $\mathbf{R}^2$ and to apply Theorem 1 to this A. □

Theorem 4. *Suppose that Martin's Axiom and the negation of the Continuum Hypothesis hold. Let A be a Σ^1_2-subset of $\mathbf{R}^2$ satisfying the relations:*

(1) *for each $y \in \mathbf{R}^2$, the section $A(y) = \{x : (x, y) \in A\}$ is of λ-measure zero;*

(2) $\lambda(pr_1(A)) = \lambda(\cup\{A(y) : y \in \mathbf{R}\}) > 0$.

Then there exists a set $Y \subset \mathbf{R}$ for which $\cup\{A(y) : y \in Y\}$ is not measurable with respect to λ.

The proof is carried out similarly to the above argument. We only need to take into account the following two well-known facts:

(*) every Σ^1_2-subset of the plane admits a Σ^1_2-uniformization (a consequence of Kondo's classical theorem);

(**) under **MA** & $\neg$**CH**, every Σ^1_2-subset of the real line (of the Euclidean plane) is Lebesgue measurable and, moreover, is universally measurable.

On the other hand, in Gödel's Constructible Universe there are Σ^1_2-subsets of the plane, for which the assertion of Theorem 3 fails to be true. In particular, Sierpński's classical decomposition of $\mathbf{R}^2$ carried out in the Constructible Universe leads to subsets of such a kind (cf. [8] and [9]). In this context, the following three examples should be mentioned.

Example 1. Suppose that all those subsets of $\mathbf{R}$ which have cardinality strictly less than $\mathbf{c}$ are of λ-measure zero. Let $\preceq$ be an arbitrary well-ordering of $\mathbf{R}$ isomorphic to the smallest ordinal of cardinality $\mathbf{c}$. Denote

$$S = \{(x, y) : x \preceq y\}.$$

By virtue of the Fubini theorem, S is not λ_2-measurable. Further, for any $y \in \mathbf{R}$, the section $S(y) = \{x : x \preceq y\}$ is of λ-measure zero. At the same time, it can readily be verified that, for each set $Y \subset \mathbf{R}$, the corresponding union $\cup\{S(y) : y \in Y\}$ is either of λ-measure zero or coincides with the whole real line $\mathbf{R}$.

Actually, Example 1 copies Sierpiński's construction [8], in which the Continuum Hypothesis is used instead of the assumption formulated above.

Example 2. Suppose again that all those subsets of $\mathbf{R}$ which have cardinality strictly less than $\mathbf{c}$ are of λ-measure zero. Let $C \subset \mathbf{R}$ be a set of cardinality continuum and with $\lambda(C) = 0$ (e.g., the role of C can be played by the classical Cantor set). Let α denote the least ordinal of cardinality continuum. Fix two enumerations $\mathbf{R} = \{x_\xi : \xi < \alpha\}$ and $C = \{y_\zeta : \zeta < \alpha\}$. Further, define the set

$$G = \{(x_\xi, y_\zeta) : \xi < \zeta\}.$$

Again, all y-sections of G are of λ-measure zero. Furthermore, since $G \subset \mathbf{R} \times C$, we deduce that G is of λ_2-measure zero. At the same time, as in Example 1, for each set $Y \subset \mathbf{R}$, the corresponding union $\cup\{G(y) : y \in Y\}$ is either of λ-measure zero or coincides with the whole $\mathbf{R}$.

Example 3. Assuming Martin's Axiom, there exists a subset D of $\mathbf{R}$ satisfying the following conditions:

(a) $\mathrm{card}(D) = \mathbf{c}$ and D is of λ-measure zero;

(b) D is almost translation-invariant, i.e., $(\forall h \in \mathbf{R})(\mathrm{card}((h + D)\triangle D) < \mathbf{c})$;

(c) D is almost symmetric with respect to the origin.

Actually, the construction of such a set D also goes back to Sierpiński. It is not difficult to check that $D + H = \mathbf{R}$ for every set $H \subset D$ with $\mathrm{card}(H) = \mathbf{c}$. This circumstance directly implies that all algebraic sums of the form

$$D + H \quad (H \subset D)$$

are either of λ-measure zero or coincide with $\mathbf{R}$ (see [2]); hence all of them are λ-measurable.

The presented examples show that some regular descriptive properties of a plane set are necessary for the validity of appropriate analogues of Theorem 2.

We shall say that a class $\mathcal{K}$ of subsets of $\mathbf{R}^2$ is admissible if the following conditions are satisfied:

(i) any set from $\mathcal{K}$ can be uniformized by the graph of a partial function extendable to a λ-measurable function;

(ii) if P is an arbitrary member of $\mathcal{K}$ and $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ is an arbitrary Borel mapping, then the pre-image $\Phi^{-1}(P)$ is λ_2-measurable.

For admissible classes of sets we have a suitable analogue of Theorem 2.

Theorem 5. *Suppose that the covering number of the σ-ideal of all λ-measure zero sets is equal to* $\mathbf{c}$. *Let $\mathcal{K}$ be an admissible class of subsets of $\mathbf{R}^2$ and let a set $P \in \mathcal{K}$ satisfy the relations:*

(1) *for each $y \in \mathbf{R}^2$, the section $P(y) = \{x : (x, y) \in P\}$ is of λ-measure zero;*

(2) $\lambda^*(pr_1(P)) = \lambda^*(\cup\{P(y) : y \in \mathbf{R}\}) > 0$.

Then there exists a set $Y \subset \mathbf{R}$ for which $\cup\{P(y) : y \in Y\}$ is not measurable with respect to λ.

The proof is similar to the proof of Theorem 2. Obviously, under certain set-theoretical assumptions, Theorem 5 can be applied to projective plane sets of higher levels. Moreover, by utilizing Lemma 2, natural analogues of Theorem 2 can be obtained for a wide class of extensions of the Lebesgue measure λ. Note that among those extensions there are some nonseparable measures μ having the property that the σ-ideal of all μ-measure zero sets coincides with the σ-ideal of all λ-measure zero sets (see [6]).

Theorem 6. *Assume* **CH**. *Let $(G, \cdot)$ be a σ-compact locally compact topological group such that the cardinality of the Baire σ-algebra of G does not exceed* **c**. *Denote by μ the completion of the left (right) Haar measure on G. Then, for any nonempty μ-measure zero set Y, there exist two μ-thick sets $X_1 \subset G$ and $X_2 \subset G$ such that $(Y \cdot X_1) \cap X_2 = \emptyset$. In particular, the set $Y \cdot X_1$ is nonmeasurable with respect to μ.*

Proof. The argument is similar to the proof of Theorem 2 but is much easier, because it does not need the existence of measurable selectors. In the product group $G \times G$ consider the set

$$Z = \{(x_1, x_2) \in G \times G : x_2 \in Y \cdot x_1\}.$$

Obviously, all horizontal and all vertical sections of Z are μ-measure zero subsets of G. Further, it is well known that the family of all those sets which belong to the Baire σ-algebra of G and have strictly positive μ-measure forms a pseudo-base for μ. So we may apply Lemma 2 to Z. According to this lemma, there exist μ-thick subsets X_1 and X_2 of G such that $(X_1 \times X_2) \cap Z = \emptyset$. This equality readily implies that $(Y \cdot X_1) \cap X_2 = \emptyset$. Since $Y \neq \emptyset$, we conclude that the set $Y \cdot X_1$ is μ-thick and its complement is μ-thick, too. Consequently, $Y \cdot X_1$ is nonmeasurable with respect to μ. □

It is useful to compare Theorem 6 with the situation described in Example 3.

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