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ON A THEOREM OF LUZIN AND SIERPINSKI

In this report three classical constructions of Lebesgue nonmeasurable sets on the real line **R** are envisaged from the point of view of the thickness of those sets with respect to the standard Lebesgue measure λ on **R**.

Very soon after Lebesgue's invention of λ, nontrivial constructions of extra-ordinary point sets in **R** have followed. They were done, respectively, by Vitali [16], Hamel [3], and Bernstein [1]. An important by-product of each of these constructions is the statement of the existence of a Lebesgue nonmeasurable subset of **R**. In this connection, it is reasonable to stress that the above-mentioned three constructions differ essentially from each other. Also, it is needless to say that these constructions are based on appropriate uncountable forms of the Axiom of Choice (**AC**), which were radically rejected by Lebesgue in that time. Many years later, it was demonstrated by Solovay [15] that some uncountable version of **AC** is absolutely necessary for obtaining Lebesgue nonmeasurable point sets in **R**.

Denote by **c** the cardinality of the continuum. By using the method of transfinite recursion, Luzin and Sierpiński [10] extended Bernstein's construction for obtaining a partition of the unit interval $[0, 1]$ (or, equivalently, of **R**) into continuum many Lebesgue nonmeasurable sets. Actually, they have proved the following statement.

Theorem 1. *The real line* **R** *admits a partition* $\{B_i : i \in I\}$ *such that:*
(1) $\mathrm{card}(I) = \mathbf{c}$;
(2) *every set* B_i $(i \in I)$ *meets any nonempty perfect subset of* **R**;
In particular, all B_i $(i \in I)$ *are Bernstein subsets of* **R** *and, consequently, are nonmeasurable in the Lebesgue sense.*

Further generalization of Bernstein's construction looks as follows (see, e.g., [7], [11]).

Theorem 2. *There exists a covering* $\{B_j : j \in J\}$ *of the real line* **R** *with its subsets, satisfying these three conditions:*
(1) $\mathrm{card}(J) > \mathbf{c}$;

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(2) *every set* B_j $(j \in J)$ *meets each nonempty perfect set in* $\mathbf{R}$*;*

(3) *the family* $\{B_j : j \in J\}$ *is almost disjoint, i.e.,* $\mathrm{card}(B_j \cap B_{j'}) < \mathbf{c}$ *for any two distinct indices* $j \in J$ *and* $j' \in J$.

The conditions (2) and (3) of Theorem 2 readily imply that every set B_j $(j \in J)$ is a Bernstein subset of $\mathbf{R}$.

The role of Bernstein sets in general topology, the theory of Boolean algebras, and measure theory is well known (see, for instance, [9], [11], [12]). In classical measure theory, the significance of these sets is primarily caused by providing various counterexamples for seemingly valid statements in real analysis and by constructions of measures lacking various regularity properties (see, e.g., [6], [7]).

Let E be a ground set and let μ be a measure defined on some σ-algebra of subsets of E.

Recall that μ is said to be diffused (or continuous) if all singletons in E belong to the domain of μ and μ vanishes at all of them.

A set $Z \subset E$ is said to be μ-thick in E if the equality $\mu_*(E \setminus Z) = 0$ holds true, where μ_* denotes the inner measure associated with μ.

Example 1. Let $\mathcal{M}$ denote the class of the completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$. It is easy to show that if B is any Bernstein set in $\mathbf{R}$ and μ is any measure from the class $\mathcal{M}$, then both B and $\mathbf{R} \setminus B$ are μ-thick subsets of $\mathbf{R}$ and, consequently, they are nonmeasurable with respect to μ. In fact, this property completely characterizes Bernstein sets in $\mathbf{R}$ (see, e.g., [2], [7]).

We thus conclude that Bernstein's construction directly yields the partition $\{B, \mathbf{R} \setminus B\}$ of $\mathbf{R}$ into two λ-thick subsets. In this connection, let us demonstrate that Hamel's construction directly leads to a partitions of $\mathbf{R}$ into countably many λ-thick subsets of $\mathbf{R}$. For this purpose, consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers. Let $\{e_i : i \in I\}$ be a Hamel basis for this space containing 1, i.e., $e_{i_0} = 1$ for some index $i_0 \in I$. Denote by V the vector space over $\mathbf{Q}$ generated by $\{e_i : i \in I \setminus \{i_0\}\}$. It is not difficult to check that V is a special kind of a Vitali set in $\mathbf{R}$. Actually, V is a selector of $\mathbf{R}/\mathbf{Q}$ but the choice of this selector is done so carefully that V turns out to be able to carry the vector structure over $\mathbf{Q}$ induced by $\mathbf{R}$. We now assert that V is λ-thick in $\mathbf{R}$. Indeed, suppose otherwise, i.e., there exists a λ-measurable set $C \subset \mathbf{R}$ such that

$$\lambda(C) > 0, \quad C \cap V = \emptyset.$$

It is easy to see that V is everywhere dense in $\mathbf{R}$ (because any uncountable subgroup of $(\mathbf{R}, +)$ is necessarily everywhere dense in $\mathbf{R}$). So we may take a countable family $\{v_i : i \in I\} \subset V$ which is everywhere dense in $\mathbf{R}$, too. Obviously, for this family, we may write

$$V \cap (\{v_i : i \in I\} + C) = \emptyset.$$

Taking into account the metrical transitivity (ergodicity) of λ with respect to any everywhere dense subset of $\mathbf{R}$, we get

$$\lambda(\mathbf{R} \setminus (\{v_i : i \in I\} + C)) = 0.$$

Therefore, $\lambda(V) = 0$, which is impossible in view of the translation invariance of λ and of the relations

$$\mathbf{R} = \mathbf{Q} + V = \cup\{q + V : q \in \mathbf{Q}\}, \quad \lambda(\mathbf{R}) = +\infty.$$

The obtained contradiction yields the desired result. We thus come to the countable partition $\{q + V : q \in \mathbf{Q}\}$ of $\mathbf{R}$ into λ-thick sets. It follows from this fact that, for any natural number $n \geq 2$, there exists a partition $\{A_1, A_2, \ldots, A_n\}$ of $\mathbf{R}$ into λ-thick sets, and so all A_k $(1 \leq k \leq n)$ are nonmeasurable with respect to λ.

Remark 1. In general, Vitali's construction does not lead to a λ-thick subset of $\mathbf{R}$. Indeed, fix a real $\varepsilon > 0$ and take an arbitrary nonempty open interval Δ in $\mathbf{R}$ with $\lambda(\Delta) < \varepsilon$. For any $x \in \mathbf{R}$, the set $x + \mathbf{Q}$ is everywhere dense in $\mathbf{R}$, so has nonempty intersection with Δ. This circumstance immediately implies that there exists a Vitali set W entirely contained in Δ and, consequently, $\lambda^*(W) < \varepsilon$, where λ^* denotes the outer measure associated with λ. We thus see that there are Vitali sets in $\mathbf{R}$ with arbitrarily small outer Lebesgue measure. Some other unexpected and extra-ordinary properties of Vitali sets are discussed in [8].

Our goal now is to obtain (within a certain weak fragment of set theory) a partition of $\mathbf{R}$ into continuum many λ-thick sets, by starting with a partition $\{A, A'\}$ of $\mathbf{R}$ consisting of two λ-thick sets. As shown above, Bernstein's and Hamel's constructions give such a partition $\{A, A'\}$.

We need the following two auxiliary propositions which both belong to **ZF** & **DC** theory, where **DC** stands, as usual, for the Principle of Dependent Choices (see [4], [5], [15]). This principle is stronger than the Axiom of Countable Choice (**CC**) and much weaker than **AC**. Moreover, according to Solovay's famous result [15], under the assumption of the existence of a strongly inaccessible cardinal there is a model of **ZF** & **DC**, in which all subsets of $\mathbf{R}$ are measurable in the Lebesgue sense.

Lemma 1. *Let E_1 and E_2 be two Polish spaces, let μ_1 be a Borel probability diffused measure on E_1, and let μ_2 be a Borel probability diffused measure on E_2. Then there exists a Borel isomorphism $\phi : E_1 \rightarrow E_2$ which is simultaneously an isomorphism between μ_1 and μ_2, i.e., we have $\mu_2(\phi(X)) = \mu_1(X)$ for every Borel subset X of E_1.*

This lemma is well known (for the proof, within **ZF** & **DC** theory, see e.g. [2] or [6]).

Lemma 2. *Let* $\{E_n : n = 1, 2, \dots, n, \dots\}$ *be a countable family of separable metric spaces and let, for each natural number* $n \geq 1$*, the space* E_n *be equipped with a probability Borel measure* μ_n*. Further, let us denote:*

$$E = \prod\{E_n : n = 1, 2, \dots, n, \dots\}, \quad \mu = \otimes\{\mu_n : n = 1, 2, \dots, n, \dots\}.$$

Suppose also that a sequence of sets $X_n \subset E_n$ $(n = 1, 2, \dots, n, \dots)$ *is given.*

Then the following two assertions are equivalent:

(1) *the product set* $X = \prod\{X_n : n = 1, 2, \dots, n, \dots\}$ *is* μ*-thick in* E*;*

(2) *the set* X_n *is* μ_n*-thick in* E_n *for each index* $n = 1, 2, \dots$.

Remark 2. In Lemma 2, the assumption that all spaces E_n are separable and metrizable is not necessary. The conclusion of this lemma remains valid under much weaker assumptions, but the above formulation suffices for our further purposes.

Remark 3. Preserving the notation of Lemma 2, let Z be an arbitrary μ-thick set in E. Then it is easy to verify that, for every natural number $n \geq 1$, the set $\mathrm{pr}_n(Z)$ is μ_n-thick in E_n. The converse assertion is not true, in general. Indeed, simple examples show that the equalities $\mathrm{pr}_n(Z) = E_n$ may be valid simultaneously for all natural numbers $n \geq 1$ but, at the same time, the set Z may be of μ-measure zero.

Remark 4. Let $k \geq 1$ be a natural number, $\{E_n : n = 1, 2, \dots, k\}$ be a finite family of ground sets and let, for each natural number $n \in \{1, 2, \dots, k\}$, the set E_n be equipped with a probability measure μ_n. Further, let us denote:

$$E = \prod\{E_n : n = 1, 2, \dots, k\}, \quad \mu = \otimes\{\mu_n : n = 1, 2, \dots, k\}.$$

Suppose also that a finite sequence of sets $X_n \subset E_n$ $(n = 1, 2, \dots, k)$ is given. Then the following two assertions are equivalent:

(a) the product set $X = \prod\{X_n : n = 1, 2, \dots, k\}$ is μ-thick in E;

(b) the set X_n is μ_n-thick in E_n for each index $n \in \{1, 2, \dots, k\}$.

We thus see that in the case of a finite sequence of probability measure spaces (or, more generally, of nonzero σ-finite measure spaces) the analogue of Lemma 2 is valid in **ZF** & **DC** theory without assuming any regularity properties of the measures.

In what follows we denote by the same symbol λ the restriction of the Lebesgue measure to the unit interval $[0, 1]$. Using Lemmas 1 and 2, we obtain

Theorem 3. *Working in* **ZF** & **DC** *theory, suppose that there exists a partition* $\{A, A'\}$ *of the unit interval* $[0, 1]$ *into two subsets such that*

$$\lambda^*(A) = \lambda^*(A') = 1.$$

Then there exists a partition $\{Z_i : i \in I\}$ *of the same interval, which satisfies the following two relations:*

(1) $\mathrm{card}(I) = \mathbf{c}$*;*

(2) $\lambda^*(Z_i) = 1$ *for each index* $i \in I$*.*

Nontrivial (i.e., discontinuous) endomorphisms of the additive group $(\mathbf{R}, +)$ were first exhibited in [3] and all of them turned out to be non-measurable in the Lebesgue sense. In connection with this fact, it is worth noticing that some of such endomorphisms can be measurable with respect to certain measures belonging to the class $\mathcal{M}$ introduced in Example 1.

Example 2. There exists a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the following three conditions:

(a) the range $\mathrm{ran}(f)$ of f is contained in the field $\mathbf{Q}$ (consequently, $\mathrm{ran}(f)$ is at most countable);

(b) f is measurable with respect to some measure from the class $\mathcal{M}$;

(c) f is a nontrivial endomorphism of the additive group $(\mathbf{R}, +)$.

To obtain such an f, consider a nonempty perfect subset P of $\mathbf{R}$ linearly independent over the field $\mathbf{Q}$ (the existence of P is a well-known fact of classical point set theory; cf. [6], [11]). Let $\{e_i : i \in I\}$ stand for some Hamel basis of $\mathbf{R}$ containing P. We define $f : \mathbf{R} \to \mathbf{Q}$ as follows. Every real number x admits a unique representation in the form

$$x = q_{i_1}e_{i_1} + q_{i_2}e_{i_2} + \cdots + q_{i_n}e_{i_n},$$

where $n = n(x)$ is a natural number, $\{i_1, i_2, \ldots, i_n\}$ is a finite injective family of indices from I, and $\{q_{i_1}, q_{i_2}, \ldots, q_{i_n}\}$ is a finite family of nonzero rational numbers. We put

$$f(x) = q_{i_1} + q_{i_2} + \cdots + q_{i_n}.$$

Obviously, f is an additive function acting from $\mathbf{R}$ into $\mathbf{Q}$, so conditions (a) and (c) are valid. Further, the restriction $f|P$ is identically equal to 1. Let μ be a Borel diffused probability measure on $\mathbf{R}$ whose support is P, i.e., $\mu(\mathbf{R} \setminus P) = 0$, and let μ' denote the completion of μ. It is clear that $\mu' \in \mathcal{M}$ and f turns out to be μ'-measurable. Thus condition (b) is satisfied, too.

Remark 5. It can be shown that:

(a) there exists a subset of $\mathbf{R}$ which is simultaneously a Vitali set and a Bernstein set;

(b) there exists a subset of $\mathbf{R}$ which is simultaneously a Hamel basis and a Bernstein set;

(c) there exists no subset of $\mathbf{R}$ which is simultaneously a Hamel basis and a Vitali set.

Remark 6. Let μ be an arbitrary measure from the class $\mathcal{M}$. By using Lemma 1, it is not difficult to prove within **ZF** & **DC** theory that if there exists a μ-nonmeasurable subset of $\mathbf{R}$, then there exists a partition of $\mathbf{R}$ into two μ-thick subsets. So, taking into account Lemma 1 and Theorem 3, we may conclude that the following four assertions are equivalent in **ZF** & **DC** theory:

(a) there exists a μ-nonmeasurable subset of $\mathbf{R}$;

(b) there exists a partition of $\mathbf{R}$ into two μ-thick subsets;

(c) there exists a partition of $\mathbf{R}$ into continuum many μ-thick subsets;

(d) there exists a function $g : \mathbf{R} \to \mathbf{R}$ such that $\mathrm{ran}(g|X) = \mathbf{R}$ for every μ-measurable set X with $\mu(X) > 0$.

In this context, the transfinite construction given in [10] becomes superfluous. At the same time, it seems that the natural analogue of Theorem 2 cannot be deduced within **ZF** & **DC** theory by assuming that there exists a λ-nonmeasurable subset of $\mathbf{R}$.

Remark 7. Consider the theory **ZF** & **DC** & $(\omega_1 \leq \mathbf{c})$, where ω_1 denotes, as usual, the least uncountable cardinal. It was proved in this theory that there exists a λ-nonmeasurable subset of $\mathbf{R}$ (see [13] and [14]). Consequently, within the same theory, there exists a partition of $\mathbf{R}$ into continuum many λ-thick subsets.

Remark 8. Supposing that $\mathbf{c}$ is a regular cardinal number, the assertion of Theorem 1 readily follows from the assertion of Theorem 2.

Theorem 4. *Assume the Continuum Hypothesis. Then:*

(1) *there exists a partition of* $\mathbf{R}$ *into continuum many Sierpiński sets all of which are* λ*-thick;*

(2) *there exists a partition of* $\mathbf{R}$ *into continuum many Luzin sets all of which are thick in the sense of category.*

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