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WEIGHRED ORLICZ CLASS INEQUALITIES FOR CERTAIN FOURIER OPERATORS

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Abstract. The necessary and sufficient conditions of modular inequalities for Fejer and Abel-Poisson means are derived.

Let $\sigma_n f$ $(n = 1, 2, \dots)$ denote the n-th Fejer mean and $P_r f$ $(0 \le r < 1)$-the Abel Poisson mean of a function f. We will assume that $\sigma_n f \equiv P_r f \equiv \infty$ for $n = 1, 2, \dots$ and $0 \le r < 1$, when $f \notin L^1$. The following theorem is a consequence of the results of Rosenblum [1] and Muckenhoupt [2].

Theorem A. *Let w be a weight on $(-\pi, \pi)$ and $1 < p < \infty$. The following statements are equivalent:*

(i) *There is a constant c, such that for every $f \in L^p_w(-\pi, \pi)$*

$$\int_{-\pi}^{\pi} \left|\sigma_n f(x)\right|^p w(x)\, dx \le c \int_{-\pi}^{\pi} \left|f(x)\right|^p w(x)\, dx, \quad n = 1, 2, \dots,$$

$$\int_{-\pi}^{\pi} \left|P_r f(x)\right|^p w(x)\, dx \int_{-\pi}^{\pi} \left|f(x)\right|^p w(x)\, dx, \quad 0 \le r < 1.$$

(ii) $w \in A_p$.

(For the definition of A_p see e.g., [3]).

Our goal is to investigate the same problem for the classes $\varphi_w(L)$. We need some definitions to formulate our results.

Let Φ denote the set of all functions $\varphi : \mathbf{R^1} \to \mathbf{R^1}$ which are nonnegative, even, and increasing on $(0, \infty)$ such that $\varphi(0+) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

A function ω is called a Young function on $[0,\infty)$ if ω is convex, $\omega(0) = 0$, and $\omega(\infty) = \infty$. A function φ is called quasiconvex if there exist a Young

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function ω and a constant $c > 1$ such that $\omega(t) \le \varphi(t) \le (ct)$, $t \ge 0$. The concept of quasiconvexity, as well as the basic definition of the number $p(\varphi)$, which follows, was introduced by V. Kokilashvili and thoroughly investigated by him and his colleagues (see e.g., [4]–[6]).

For any quasiconvex function φ let us define a number $p(\varphi)$ as

$$\frac{1}{p(\varphi)} = \inf\big\{\beta : \varphi^\beta \ \text{ is quasiconvex}\big\}.$$

We will use some properties of quasiconvex functions.

Lemma 1 ([5]). *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *φ is quasiconvex;*
(ii) *there exists a constant c such that*

$$\varphi\bigg(\frac{1}{|I|}\int\limits_I f(x)\,dx\bigg) \le \frac{c}{|I|}\int\limits_I \varphi\big(cf(x)\big)\,dx,$$

for every interval $I\subset\mathbf{R}$ and nonnegative integrable function f with $\operatorname{supp} f\subset I$.

Lemma 2 ([5]). *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *φ^α is quasiconvex for some α, $0 < \alpha < 1$;*
(ii) *there exists a constant c such that*

$$\int\limits_0^t \frac{\varphi(s)}{s^2}\,ds \le c\frac{\varphi(ct)}{t},$$

for every $t > 0$.

Lemma 3 ([6]). *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *φ is quasiconvex and $w \in A_{p(\varphi)}$;*
(ii) *there exists a constant c such that*

$$\varphi\bigg(\frac{1}{|I|}\int\limits_I f(t)\,dt\bigg) \le \frac{c}{wI}\int\limits_I \varphi\big(cf(x)\big)w(x)\,dx,$$

for every interval $I\subset\mathbf{R}$ and nonnegative integrable function f with $\operatorname{supp} f\subset I$.

Theorem 1. *Let w be a weight and $\varphi \in \Phi$. The following statements are equivalent:*
(i) *φ is quasiconvex and $w \in A_{p(\varphi)}$.*
(ii) *there exists a constant c, such that for every $f \in \varphi_w(L)$*

$$\int\limits_{-\pi}^{\pi} \big(\sigma_n f(x)\big)w(x)\,dx \le c\int\limits_{-\pi}^{\pi} \varphi\big(f(x)\big)w(x)\,dx, \quad n = 1, 2, \dots. \tag{1}$$

Proof. We will start with the proof of (i) $\Rightarrow$ (ii). As first let us consider the case $p(\varphi)=1$. As it is well known,

$$\sigma_n f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)\,dt,$$

where $K_n(t)=\frac{1}{n+1}\sum_{j=0}^{n} D_j(t)$.

By Lemma 1 (for the measure $d\mu(t)=K_n(x-t)dt$),

$$\varphi\big(\sigma_n f(x)\big)=\varphi\Big(\int_{-\pi}^{\pi}\frac{f(t)}{\pi}K_n(x-t)\,dt\Big)\le c_1\int_{-\pi}^{\pi}\varphi\Big(\frac{f(t)}{\pi}\Big)K_n(x-t)\,dt\le$$
$$\le c_2\sigma_n\big(\varphi(c_2 f)\big)(x). \tag{2}$$

Hence, as $w\in A_1$, by Theorem A we get

$$\int_{-\pi}^{\pi}\big(\sigma_n f(x)\big)w(x)\,dx\le c_2\int_{-\pi}^{\pi}\sigma_n\big(\varphi(c_2 f)\big)(x)w(x)\,dt\le$$
$$\le c_3\int_{-\pi}^{\pi}\varphi\big(c_3 f(x)\big)w(x)\,dx. \tag{3}$$

Now, let $p(\varphi)>1$. There exists $p<p(\varphi)$, such that $w\in A_p$. From the definition of the number $p(\varphi)$ follows that, $\varphi^{\frac{1}{p}}$ is quasiconvex. According to (2),

$$\varphi^{\frac{1}{p}}\big(\sigma_n f(x)\big)\le c_4\sigma_n\big(\varphi^{\frac{1}{p}}(c_4 f)\big)(x).$$

Applying once more Theorem A, we get

$$\int_{-\pi}^{\pi}\varphi\big(\sigma_n f(x)\big)w(x)\,dx=\int_{-\pi}^{\pi}\Big(\varphi^{\frac{1}{p}}\big(\sigma_n f(x)\big)\Big)^p w(x)\,dx\le$$
$$\le c_5\int_{-\pi}^{\pi}\Big(\sigma_n\Big(\varphi^{\frac{1}{p}}\big(c_4 f(x)\big)\Big)\Big)w(x)dx\le c_6\int_{-\pi}^{\pi}\Big(\varphi^{\frac{1}{p}}\big(c_4 f(x)\big)\Big)^p w(x)\,dx\le$$
$$\le c_7\int_{-\pi}^{\pi}\varphi\big(c_7 f(x)\big)w(x)\,dx. \tag{4}$$

From (3) and (4) follows (1).

(ii) $\Rightarrow$ (i). Let us note, that if $|t|\le\frac{\pi}{n+1}$, then

$$K_n(t)=\frac{2}{n+1}\Big(\frac{\sin\frac{1}{2}(n+1)t}{2\sin\frac{t}{2}}\Big)^2\ge c_1 n. \tag{5}$$

Let $I\subset(-\pi,\pi)$ be an interval, $|I|\le\frac{\pi}{4}$, $f\ge 0$ and $\operatorname{supp} f\subset I$. Suppose that n is such natural number, that $\frac{\pi}{4(n+1)}\le|I|\le\frac{\pi}{4n}$. We will estimate $\sigma_n f(x)$, when $x\in I$:

$$\sigma_n f(x)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)K_n(x-t)\,dt=\frac{1}{\pi}\int\limits_I f(t)K_n(x-t)\,dt\ge$$
$$\ge c_2 n\int\limits_I f(t)\,dt\ge\frac{c_3}{|I|}\int\limits_I f(x)\,dt.$$

From (1) then we get

$$\varphi\bigg(\frac{1}{|I|}\int\limits_I f(t)\,dt\bigg)\le\frac{c_4}{wI}\int\limits_I\varphi\big(c_4 f(x)\big)w(x)\,dx.$$

According Lemma 3, this means that φ is quasiconvex and $w\in A_{p(\varphi)}$. Theorem 1 is proved. □

Theorem 2. *Let w be a weight and $\varphi\in\Phi$. The following statements are equivalent:*

(i) *φ is quasiconvex and $w\in A_{p(\varphi)}$.*

(ii) *there is a constant c, such that for every $f\in\varphi_w(L)$*

$$\int\limits_{-\pi}^{\pi}\varphi\big(P_r f(x)\big)w(x)\,dx\le c\int\limits_{-\pi}^{\pi}\varphi\big(f(x)\big)w(x)\,dx,\quad 0\le r<1. \tag{6}$$

Proof. The implication (i)⇒(ii) can be proved the same way as for Theorem 1. Let us show that (ii) ⇒ (i). Let $I\subset(-\pi,\pi)$ be an interval. One easily can check that if $r=\max(0,1-|I|)$ and $t\in I$, then

$$P_r(t)=\frac{1}{2}\frac{1-r^2}{1-2r\cos t+r^2}\ge\frac{1}{4|I|}. \tag{7}$$

If $f\ge 0$, $\operatorname{supp} f\subset I$ and $x\in I$, then from (7) follows

$$P_r f(x)=\frac{1}{\pi}\int\limits_{\pi} f(t)p_r(x-t)\,dt\ge\frac{c_1}{|I|}\int\limits_I f(t)\,dt.$$

And the proof can be continued as in Theorem 1. □

Theorem 3. *Let $\lambda>0$, $f(\theta)\sin^{2\lambda}\theta$ is integrable on $(0,\pi)$, $\sum a_n P_n^\lambda(\cos\theta)$ is the Gegenbauer expansion of f and for $0\le r<1$, $f(r,\theta)=\sum a_n r^n P_n^\lambda(\cos\theta)$. Then the following inequalities are equivalent:*

(i) *There is a constant c, such that the inequalities*

$$\int_0^\pi \varphi\big(f(r,\theta)\big)w(\theta)\,d\theta < c\int_0^\pi \varphi\big(cf(\theta)\big)w(\theta)\,d\theta,\quad 0\le r<1,$$

hold for any $f\in\varphi_w(L)$ *on* $(0,\pi)$.

(ii) φ *is quasiconvex and there exists a number* K, *independent of* I, *such that for every subinterval* I *of* $(0,\pi)$,

$$\bigg(\int_I w(\theta)\,d\theta\bigg)\bigg(\int_I \big(w(\theta)\big)^{-\frac{1}{p(\varphi)-1}}(\sin\theta)^{2\lambda p'(\varphi)}\,d\theta\bigg) < K\bigg(\int_I \sin^{2\lambda}\theta\,d\theta\bigg)^{p(\varphi)}$$

when $p(\varphi)>1$ *and*

$$\int_I w(\theta)\,d\theta < K\bigg(\int_I \sin^{2\lambda}\theta\,d\theta\bigg)\operatorname*{essinf}_{y\in I}\big(w(y)\sin^{-2\lambda}y\big)$$

when $p(\varphi)=1$.

To prove this theorem we will need a lemma, concerning the A_p classes generated by continuous Borel measures.

Let μ be a continuous Borel measure on the real line (for every point $\alpha\in\mathbf{R}$, $\mu\{a\}=0$), $\varphi\in\Phi$ and w is a weight function (a.e., positive, locally integrable function). By definition $w\in A_p(\mu)$ $(0\le r<1)$ if

$$\sup_{I\subset\mathbf{R}}\bigg(\frac{1}{\mu I}\int_I w(x)\,d\mu(x)\bigg)\bigg(\frac{1}{\mu I}\int_I w(x)^{-\frac{1}{p-1}}\,d\mu(x)\bigg)^{p-1}<\infty,\quad\text{when } 1< p<\infty$$

and

$$\frac{1}{\mu I}\int_I w(x)\,d\mu(x) <\le c\operatorname*{essinf}_{y\in I}w(y),\quad\text{when}\quad p=1,$$

where c is independent of I. Here and everywhere the ratio is supposed to be zero when $\mu I=0$.

Let $f\in\mathbf{L}^1_{\mathrm{loc}}(\mu)$ and define Maximal function

$$M_\mu f(x)=Mf(x)=\sup_{x\in I}\frac{1}{\mu I}\int_I |f(x)|\,d\mu(x). \tag{8}$$

Lemma 4. *Let* $\varphi\in\Phi$ *and* w *is a weight on* $\mathbf{R}$. *The following conditions are equivalent:*

(i) *There exists* $c_1>0$, *such that for every* $f\in\mathbf{L}^1_{\mathrm{loc}}(\mu)$

$$\varphi(\lambda)w\big\{x: Mf(x)>\lambda\big\}\le\int_{-\infty}^{\infty}\varphi\big(c_1f(x)\big)w(x)\,d\mu(x). \tag{9}$$

(ii) *There exists* $c_2 > 0$, *such that for every interval* I *and* $f \in \mathbf{L}^1_{\rm loc}(\mu)$ *with* $\operatorname{supp} f \subset I$

$$\varphi\biggl(\frac{1}{\mu I}\int\limits_I f(x)\,d\mu(x)\biggr) \le \frac{c_2}{wI}\int\limits_I \varphi\bigl(c_2 f(x)\bigr)w(x)\,d\mu(x). \tag{10}$$

(iii) φ *is quasiconvex and* $w \in A_{p(\varphi)}(\mu)$.

Lemma 4 was proved by A. Gogatisahvili and V. Kokilashvili for homogenous-type spaces ([5], [6]). The proof is same in our case and we will not repeat it.

Proof of Theorem 3. (ii) $\Rightarrow$ (i). As it is known

$$f(r,\theta) = \int\limits_0^\pi P(r,\theta,t)f(t)\,dm_\lambda(t), \tag{11}$$

where $dm_\lambda = \sin^{2\lambda} t\,dt$ and

$$P(r,\theta,t) = \frac{\lambda}{\pi}(1-r^2)\int\limits_0^\pi \frac{\sin^{2\lambda-1}\tau}{(1-2r(\cos\theta\cos t+\sin\theta\sin t)+r^2)^{\lambda+1}}\,d\tau.$$

The kernel $P(r,\theta,t)$ has approximation unit properties, and because of this, the rest of the implication (ii) $\Rightarrow$ (i) is the same as in Theorem 1.

(i) $\Rightarrow$ (ii). Let $I \subset (0,\pi)$, $f \ge 0$, $\operatorname{supp} f \subset I$ and $r = 1 - \frac{|I|}{6}$. As is shown in [7], there exists a constant c, independent of I, θ and t, such that

$$P(r,\theta,t) \ge c\biggl(\int\limits_I \sin^{2\lambda}\tau\,d\tau\biggr)^{-1},$$

when θ, $t \in I$. Then from (11) follows that

$$f(r,\theta) \ge \frac{c}{m_\lambda I}\int\limits_I f(t)\,dm_\lambda(t). \tag{12}$$

If we apply (12) in (9), will get

$$\varphi\biggl(\frac{1}{m_\lambda I}\int\limits_I f(x)\,dm_\lambda(x)\biggr)\int\limits_I w(\theta)\,d\theta \le c\int\limits_I \varphi\bigl(cf(\theta)\bigr)w(\theta)\,d\theta.$$

Now, using Lemma 4 for the weight $w(\theta)\sin^{-2\lambda}\theta$, we obtain (i). Theorem 3 is proved.

As we touched the operator M_μ let us formulate and prove one theorem, concerning the boundedness of this operator in the classes $\varphi_w(L)$. The same problem for homogenous-type spaces was investigated by A. Gogatishvili and V. Kokilashvili [5].

Theorem 4. *Let μ be nonnegative, continuous Borel measure on the real axis, $\varphi \in \Phi$, w is a weight function and the operator M be defined by the equality* (9)*. Then the following statements are equivalent:*
(i) *φ^α is quatsiconvex for some α, $0 < \alpha < 1$ and $w \in A_{p(\varphi)}(\mu)$.*
(ii) *There exists a conctant c, such that for every $f \in \mathbf{L}^1_{\text{loc}}(\mu)$*

$$\int_{-\infty}^{\infty} \varphi\big(Mf(x)\big) w(x)\, d\mu(x) \le c \int_{-\infty}^{\infty} \varphi\big(f(x)\big) w(x)\, d\mu(x).$$

Lemma 5. *If μ is continuous Borel measure on the real axis and $\mu(a,b) < \infty$, then there exists a point $c \in (a,b)$, such that*

$$\mu(a,c) = \frac{1}{2}\mu(a,b). \tag{13}$$

Proof. Define on $[a,b]$ a function m by the equality: $m(x) = \mu(a,x)$. We will show that m is continuous on $[a,b]$. Let $x \in [a,b]$ and $h > 0$. Then

$$m(x+h) - m(x) = \mu[x, x+h) \to \mu\{x\}$$

when $h \to 0$. But $\mu\{x\} = 0$ and therefore $m(x+h) \to m(x)$. So, $\lim\limits_{t\to x+} m(t) = m(x)$. In the same manner we can show that $\lim\limits_{t\to x-} m(t) = m(x)$. Thus m is continuous. As $m(a) = 0$ and $m(b) = \mu(a,b)$, by the Cauchy Theorem we can find a point $c \in (a,b)$, for which the (13) holds. □

Lemma 6. *Let μ be nonnegative, continuous Borel measure on the real axe, $\varphi \in \Phi$ and $\mu E > 0$. If there exists a number c, such that the inequality*

$$\int_E \varphi\big(Mf(x)\big)\, d\mu(x) \le c \int_E \varphi\big(f(x)\big)\, d\mu(x) \tag{14}$$

holds for every measurable function f with $\operatorname{supp} f \subset E$*, then φ^α is quasiconvex for some α, $0 < \alpha < 1$.*

Proof. Define a Borel measure ν by the following way:

$$\nu e = \mu\big(e \cap E\big) = \int_e \chi_E(x)\, d\mu(x)$$

where e is any Borel measurable set. The measure ν is absolutely continuous with respect of the measure μ and $\frac{d\nu}{d\mu}(x) = \chi_E(x)$ for a.e., x by the sense of the measure μ. Without restricting generality we can assume that $0 < \mu(0,1) < \infty$, $0 \in E$ and $\frac{d\nu}{d\mu}(0) = 1$. We can also assume that $\frac{\nu(0,x)}{\mu(0,x)} \ge \frac{3}{4}$ when $x \le 1$. By the definition of ν it means that

$$\mu(0,x) \cap E \ge \frac{3}{4}\mu(0,x) \tag{15}$$

when $x \leq 1$. By Lemma 5, there exists a decreasing sequence (r_j) of real numbers, such that $r_0 = 1$ and

$$\mu(0, r_j) = 2^{-j}\mu(0, 1). \tag{16}$$

Let us estimate $\mu E \cap (r_{j+1}, r_j)$, using (15) and (16):

$$\mu E \cap (r_{j+1}, r_j) = \mu E \cap (0, r_j) - \mu E \cap (0, r_{j+1}) \geq \frac{3}{4}\mu(0, r_j) - \mu(0, r_{j+1}) =$$
$$= \frac{3}{2}\mu(r_{j+1}, r_j) - \mu(r_{j+1}, r_j) = \frac{1}{2}\mu(r_{j+1}, r_j) = \frac{\mu(0,1)}{2^{j+2}}. \tag{17}$$

Let $t > 0$, $k \in N$ and $f(t) = \chi_{E \cap (0, r_k)}(t)$. Suppose that $r_{j+1} < x \leq r_j$. If $j \geq k$, then

$$Mf(x) \geq \frac{1}{\mu(0, r_k)} \int\limits_{(0, r_k)} f\, d\mu = t\frac{\mu E \cap (0, r_k)}{\mu(0, r_k)} > \frac{t}{2} \tag{18}$$

and when $j < k$,

$$Mf(x) \geq \frac{1}{\mu(0, r_j)} \int\limits_{(0, r_j)} f\, d\mu = t\frac{\mu E \cap (0, r_k)}{\mu(0, r_k)} > \frac{t}{2}\frac{\mu(0, r_k)}{\mu(0, r_j)} = t2^{j-k-1}. \tag{19}$$

From (18), (19) and (17) follows:

$$\int\limits_E \varphi\big(Mf(x)\big)\, d\mu(x) \geq \sum_{j=0}^{k-1} \int\limits_{E \cap (r_{j+1}, r_j)} \varphi\big(Mf(x)\big)\, d\mu(x) >$$
$$> \sum_{j=0}^{k-1} \varphi\Big(\frac{t}{2^{k-j+1}}\Big)\mu E \cap (r_{j+1}, r_j) + \varphi\Big(\frac{t}{2}\Big)\mu E \cap (0, r_k) >$$
$$> \frac{1}{2}\sum_{j=0}^{k-1} \varphi\Big(\frac{t}{2^{k-j+1}}\Big)\frac{\mu(0,1)}{2^{j+1}}.$$

On the other hand,

$$\int\limits_E \varphi\big(cf(x)\big)\, d\mu(x) = \varphi(t)\mu E \cap (0, r_k) \leq \varphi(ct)2^{-k}\mu(0, 1),$$

or, which is the same,

$$\sum_{i=1}^{k+1} 2^i \varphi\Big(\frac{t}{2^i}\Big) \leq c_1\varphi(ct).$$

Tending k to infinity we get

$$\sum_{i=1}^{\infty} 2^i \varphi\Big(\frac{t}{2^i}\Big) \leq c_1\varphi(ct). \tag{20}$$

It can be easily shown that from (20) follows the inequality:

$$\int_0^t \frac{\varphi(s)}{s^2}\,ds \le c_2 \frac{\varphi(ct)}{t}.$$

Then, by Lemma 2, φ^α is quasiconvex for some α, $0<\alpha<1$. □

Proof of Theorem 4. (i) ⇒ (ii). Let $p<p(\varphi)$ be such that $w\in A_p$. By the definition of $p(\varphi)$, $\varphi^{\frac{1}{p}}$ is quasiconvex. Then, by Lemma 1,

$$\varphi^{\frac{1}{p}}\big(Mf(x)\big) \le c_1 M\Big(\varphi^{\frac{1}{p}}(c_1 f)\Big)(x),$$

and, as the operator is bounded in $L^p_w(\mu)$ space ([8], Theorem 7),

$$\int_{-\infty}^{\infty} \varphi\big(Mf(x)\big)w(x)\,d\mu(x) = \int_{-\infty}^{\infty} \Big(\varphi^{\frac{1}{p}}\big(Mf(x)\big)\Big)^p w(x)\,d\mu(x) \le$$

$$\le c_2 \int_{-\infty}^{\infty} \Big(M\Big(\varphi^{\frac{1}{p}}(c_1 f)\Big)(x)\Big)^p w(x)\,d\mu(x) \le c_3 \int_{-\infty}^{\infty} \varphi\big(c_3 f(x)\big)w(x)\,d\mu(x).$$

(i) ⇒ (ii). Let k be such that the set $E=\{x:\frac{1}{k}\le w(x)\le k\}$ is of positive measure, $f\in \mathbf{L}^1_{\rm loc}(\mu)$ and $\operatorname{supp} f\subset E$. Then

$$\int_E \varphi\big(Mf(x)\big)\,d\mu(x) \le c\int_E \varphi\big(f(x)\big)\,d\mu(x).$$

By Lemma 6 φ^α is quasiconvex and by Lemma 4 $w\in A_{p(\varphi)}(\mu)$. The proof is complete.

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