

ALGEBRAIC K -THEORY VIEW ON KK -THEORY

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ABSTRACT. Let a compact group G act on real or complex C^* -algebras A and B , with A separable and B σ -unital. We express the G -equivariant Kasparov groups $KK_n(A, B)$ by algebraic K -groups of a certain additive category.

INTRODUCTION

In noncommutative topology and differential geometry one of the major interest is finding topological invariants of some class of noncommutative algebras. One of the useful and powerful tool is Kasparov KK -theory. So, its comprehensive studying, by methods of other mathematical theories, may be considered as interesting problem.

In the article [10], it is compendiously given account of the calculation of Kasparov's KK -groups by algebraic K -groups. In this paper we should like to describe it in detail.

In very first view on the problem calculation of KK -theory by algebraic K -theory seems to be improbable. There are some reasons. On the one hand, algebraic K -theory of an algebra is in general quite hard to calculate. On the other hand, the problem mentioned above seem seems impossible from the view of Kasparov KK -theory.

The key to solve the problem is to find suitable object which is sensible for both algebraic K -theory and KK -theory. In this paper we make accent on the C^* -category $\text{Rep}(A, B)$, where A is separable and B is σ -unital real or complex C^* -algebras with action of a fix compact second countable group. Stimulation factor is that, it was proved in the paper [9], for the complex algebras case, the existence of an isomorphism, up to a dimensional shift, between the topological K -theory of the C^* -category $\text{Rep}(A; B)$ and Kasparov's groups KK -groups:

$$(0.1) \quad KK_n(A, B) = K_{n+1}^t(\text{Rep}(A, B)).$$

As it was pointed out in [9], one offers, both in real and complex cases, to define algebraic, as well as topological, bivariant KK -theories by the formulas

$$(0.2) \quad KK_n^a(A, B) = K_{n+1}^a(\text{Rep}(A, B)) \quad \text{and} \quad KK_n^t(A, B) = K_{n+1}^t(\text{Rep}(A, B)),$$

where K_n^a and K_n^t are variants of algebraic and topological K -theories respectively, which will be considered below.

To solve our problem we compare three family of contravariant functors. These are

$$(0.3) \quad \{KK_{n+1}^a(-; B)\}_{n \in \mathbb{Z}}, \quad \{KK_{n+1}^t(-; B)\}_{n \in \mathbb{Z}}, \quad \{KK_n(-; B)\}_{n \in \mathbb{Z}}.$$

One of the main our result is that all these three families are so called Cuntz-Bott cohomology theories (see definition and properties in Section 3). By Corollary 3.5 the problem comparison is reduced in a fixed dimension, where the solution of problem is not difficult. So, our main result shows that algebraic and topological K -theories of $\text{Rep}(A; B)$ are essential isomorphic to the Kasparov $KK(A; B)$.

Below we shall describe the content of paper.

In the first section there are some definitions, constructions and properties of C^* -categories and C^* -categoroids. These are the objects which help us to formulate some working principles and

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interpretations, used in the next sections. In particular, here we construct category $\text{Rep}(A; B)$ that is the main subject of our research. Mainly, the material in this section is given without proof and we refer [11] for details.

In the next sections we'll need Higson's homotopy invariance theorem. Since the complexity of C^* -algebras is significant to prove Lemma 3.1.2 and Theorem 3.11 of [7], we couldn't disseminate the proof of it for real C^* -algebras. The purpose of the Section 2 is to show that it can be deduced from Kasparov's homotopy invariant theorem, both for real and complex algebras. This fact is enough for our purposes.

The next main fact, used in the solution of our problem, is Cuntz-Bott periodicity theorem. We have shown it in the weaker conditions than those are in [3]. A family of contravariant functors satisfying these conditions we'll call Cuntz-Bott cohomology. From Cuntz-Bott periodicity theorem we deduce a very simple but useful principle: Let \mathbb{E} and \mathbb{E}' be Cuntz-Bott (co)homology theories. If there exists a natural isomorphism $\mu_m : E_m \rightarrow E'_m$ for some $m \in Z$, then there exists a natural isomorphism

$$\mu_n : E_n \rightarrow E'_n$$

for all $n \in Z$ (Corollary 3.3).

Excision properties of KK-theory, which was proved in [4], tell us that KK-theory is Cuntz-Bott cohomology theory relative to the first argument, and Cuntz-Bott homology relative to the second argument.

In the next two sections we study an interpretation of algebraic and topological K -theories of C^* -categories. According to this interpretation, we define algebraic and topological K -groups for, so called, C^* -categoroids. Our definition is a modification of some arguments from [1], [5], [17]. Let us consider it more in detail.

Let a and a' be objects in an additive C^* -category A and I be a closed ideal. We say that $a \leq a'$ if there exists a morphism $s : a \rightarrow a'$ such that $s^*s = 1_a$ (such a type morphism is said to be an isometry). Denote by $\mathcal{L}(a)$ (resp. $I(a)$) the C^* -algebra $\text{hom}_A(a, a)$ (resp. $\text{hom}_I(a, a)$). We have a correctly defined inductive system of abelian groups $\{K_n^a(\mathcal{L}(a)), \sigma_{aa'}\}_a$ and $\{K_n^t(I(a)), \sigma_{aa'}\}_a$, where K_n^a (resp. K_n^t) are usual algebraic (resp. topological) K -theory groups of the algebra (resp. C^* -algebra) $\mathcal{L}(a)$. We suppose that

$$K_n^a(A) = \varinjlim_a K_n^a(\mathcal{L}(a)) \quad (\text{resp.}) \quad K_n^t(A) = \varinjlim_a K_n^t(\mathcal{L}(a)).$$

So defined algebraic K -groups are naturally isomorphic to the Quillen's K -groups $K_n^Q(A)$ (with respect to the class of all split short exact sequences), when $n \geq 0$; and the case of topological K -theory gives us an interpretation of Karoubi's topological K -theory, [13]. One can generalize this definition for an ideal I which does not depend on the choice of the enveloping additive C^* -category of I . These K -theories have an excision property generalizing the analogous property of algebraic and topological K -theories of C^* -algebras, [13], [19]. There is a natural transformation $\theta_n : K_n^a \rightarrow K_n^t$ which is a generalization of the classical natural transformation between algebraic and topological K -theories.

The section 6 of the paper is relatively difficult. In this part we'll show that

$$\{KK_n^a(-; B)\}_{n \in Z}, \quad \text{and} \quad \{KK_{n+1}^t(-; B)\}_{n \in Z}$$

have weak excision property.

In the section 7 we'll prove our main result which says that all three theories 0.3 are isomorphic Cuntz-Bott cohomology theories.

1. C^* -CATEGORIES AND C^* -CATEGORIDS

In this section we give some elementary properties of C^* -categories and C^* -categoroids, a natural categorical generalization of unital C^* -algebras and C^* -algebras. We'll mainly give basic definitions, constructions and properties without proofs here but we indicate for details the article [11].

Recall that the *diagram scheme* D consists of a class of objects $\text{Ob}D$ and a set $\text{hom}(a, b)$ for any $a, b \in \text{Ob}D$. By a *k-scheme* we mean a diagram scheme D such that $\text{hom}(a, b)$ has the structure

of a k -linear space, where $k = \mathbb{R}$ or \mathbb{C} , the fields of real and complex numbers respectively. D is called an *involutive k -scheme* if:

- an anti-linear map $*$: $\text{hom}(a, b) \rightarrow \text{hom}(b, a)$ is given for each $a, b \in \text{Ob}D$.
- the bilinear composition law

$$\text{hom}(a, b) \times \text{hom}(b, a) \rightarrow \text{hom}(a, a)$$

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is associative for any $a, b \in \text{Ob}D$.

- $(f^*)^* = f$, and $(fg)^* = g^*f^*$ if the composition fg exists.

Definition 1.1. By a C^* -scheme is meant an involutive k -scheme D such that:

1) $\text{hom}(a, b)$ is a Banach space; 2) involution is isometry; 3) $\|f\|^2 \leq \|f^*f + g^*g\|^2$ for any $f \in \text{hom}(a, b)$ and $g \in \text{hom}(a, b')$ [16] [14]; 4) the morphism f^*f is a positive element in the C^* -algebra $\text{hom}(a, a)$ for any $f \in \text{hom}(a, b)$ and $a, b \in \text{Ob}D$.

A diagram scheme \mathcal{D} is called a *categoroid* if it satisfies all the axioms of category except the existence of identities of objects. Let a and b be objects from \mathcal{D} . Then $\text{hom}(a, b)$ denotes a set of morphisms from a to b . The definition of morphisms between categoroids is analogous to that of a functor, and is called a *functoroid*. If $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{D}'$ is a functoroid of categoroids and there exists composition of morphisms f, g in \mathcal{D} , then $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$.

Definition 1.2. A categoroid A is called a C^* -categoroid if it has the structure of a C^* -scheme such that

- 1) the composition of morphisms is bilinear and $\|fg\| \leq \|f\| \cdot \|g\|$ if there exists composition of f and g ;
- 2) If A is both a category and a C^* -categoroid, then it is called a C^* -category.

Remark. With the term " C^* -categoroid" will mean a C^* -categoroid with a small underlying categoroid, while the term "large C^* -categoroid" will mean a C^* -categoroid with a usual underlying categoroid.

1) The category with all Hilbert spaces as objects and all bounded linear maps as morphisms is a large C^* -category denoted by \mathcal{H} .

2) Let A be a C^* -algebra. The category $\mathcal{H}(A)$ with all right A -modules as objects and all bounded A -linear maps with adjoint as morphisms is a large C^* -category.

3) C^* -algebra is a C^* -categoroid with one object \diamond and elements of the C^* -algebra as morphisms.

4) The category of Hilbert spaces as objects and compact linear maps as morphisms has the structure of a large C^* -categoroid.

Let A and B be C^* -categoroids. A $*$ -functoroid $\mathcal{F} : A \rightarrow B$ is given if \mathcal{F} maps the objects and morphisms of A into the objects and morphisms of B , respectively, so that:

a) $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$;

b) $\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)$;

c) $\mathcal{F}(\lambda f) = \lambda\mathcal{F}(f)$;

d) $\mathcal{F}(f^*) = \mathcal{F}(f)^*$ when the left side is defined.

If A and B are categories and $\mathcal{F}(1_a) = 1_{\mathcal{F}(1_a)}$ for any $a \in \text{Ob}A$, then \mathcal{F} is called a $*$ -functor. We say that a $*$ -functoroid between C^* -categoroids is faithful if canonical maps between objects and between morphisms are injections.

Any $*$ -functoroid is norm decreasing. Moreover, the faithful $*$ -functoroid is norm preserving.

1.1. An Ideal and the Smallest Categorization. Let A be a C^* -categoroid and $I \subset \text{Mor}A$. Let $(a, b)_I = \text{hom}(a, b) \cap I$. Then I is called a left ideal if $(a, b)_I$ is a linear subspace of $\text{hom}(a, b)$ and $f \in (a, b)_I, g \in \text{hom}(b, c)$ implies $gf \in (a, c)_I$. A right ideal is defined similarly. I is a two-sided ideal if it is both a left and a right ideal. An ideal I is closed if $(a, b)_I$ is closed in $\text{hom}(a, b)$ for each pair of objects. I determines an equivalence relation on the morphisms of A : $f \sim g$, if $f - g \in I$. If $I = I^*$ is an ideal of A , the set of equivalence classes A/I can be made into

a $*$ -categoroid in a unique way by requiring that the canonical map $f \mapsto \bar{f}$ of $A \rightarrow A/I$ be a $*$ -functoroid. A/I can be made into a normed $*$ -categoroid by defining

$$|\bar{f}| = \sup_{g \in \bar{f}} |g|.$$

Arguing as for C^* -algebras, one can show that if A be a C^* -categoroid and I a closed two-sided ideal of A . Then $I = I^*$ and A/I is a C^* -categoroid.

An ideal I in A is called *essential* if the intersection $I \cap J \neq 0$ for each nonzero ideal $J \subset A$.

Let A be a C^* -categoroid. C^* -category B is called *categorization* of A if A is contained in B as an essential ideal.

There exists *the smallest categorization* A^+ of A . This C^* -category has the same objects as A , while $\text{hom}_{A^+}(a, a) = \text{hom}_A(a, a)^+$ if the C^* -algebra $\text{hom}(a, a)$ is non-unital and remains otherwise.

1.2. Realization of a C^* -Categoroid. Any $*$ -functoroid $\mathcal{F} : A \rightarrow \mathcal{H}(B)$ is called a *representation*, where $\mathcal{H}(B)$ is a large C^* -category of right Hilbert B -modules over the C^* -algebra B and B -linear maps which have adjoint. If \mathcal{F} is faithful, then it is called a *faithful representation* or an *$*$ -embedding*.

Let A be a C^* -categoroid. There exist a C^* -algebra \mathcal{A} and a faithful representation

$$(1.1) \quad \mathcal{F} : A \rightarrow \mathcal{H}(\mathcal{A}).$$

1.3. Stable C^* -Algebra of a C^* -Categoroid and Its Representation. Let A be a C^* -categoroid and $S^0(A)$ be a $*$ -algebra of finite $I \times I$ -matrices, i.e., of matrices $(a_{ij})_{i,j \in I}$ with only finite nonzero entries, where $a_{ij} : i \rightarrow j$ is a morphism and I is the set of objects of A . As it was pointed above there exists a faithful $*$ -functoroid from A into the C^* -category of Hilbert \mathcal{A} -modules and bounded \mathcal{A} -homomorphisms, i.e., there is an injection $*$ -functoroid on objects and morphisms. Hence there exists a $*$ -monomorphism of $S^0(A)$ into $L_{\mathcal{A}}(\oplus_i E_i)$, where $L_{\mathcal{A}}(\oplus_i E_i)$ is a C^* -algebra of all bounded \mathcal{A} -homomorphisms on $\oplus_i E_i$ which have adjoint. Therefore $S^0(A)$ has a (unique) C^* -norm induced by $L_{\mathcal{A}}(\oplus_i E_i)$. A completion of $S^0(A)$ gives a C^* -algebra $S(A)$ called a *stable C^* -algebra* of the C^* -categoroid A .

Let $L = \{a_1, \dots, a_n\}$ be a finite subset of objects from A , and A_L be a full C^* -sub-categoroid of A with L as a set of objects. If R is another finite subset of $\text{Ob}A$ and $L \subset R$, then we have the canonical $*$ -homomorphism $i_{LR} : S^0(A_L) \rightarrow S^0(A_R)$ defined by $(f_{ij}) \mapsto (g_{kl})$, where $g_{kl} = f_{ij}$ for $(k, l) = (i, j)$, and is zero otherwise. Thus we obtain the direct system $\{S^0(A_L), i_{LR}\}$ of pre- C^* -algebras. One has the following statements:

- a) Each element from $S(A)$ can be represented as the $I \times I$ -matrix (a_{kl}) , where $a_{kl} \in \text{hom}(k, l)$.
- b) Let $L = \{a_1, \dots, a_n\}$ be a finite subset of objects from A , and A_L be a full C^* -sub-categoroid of A with L as a set of objects. Then $S^0(A_L)$ is a C^* -algebra.
- c) A stable C^* -algebra $S(A)$ is the direct limit of the direct system $\{S^0(A_L), i_{LR}\}$ of C^* -algebras.

1.4. Multiplier C^* -Category of a C^* -Categoroid. To construct a multiplier category, we need the following definition.

Let A be a C^* -algebra and $P = \{p_i\}_{i \in I}$ be a set of projections of $M(A)$. We say that P is a *strictly complete set of projections* if:

- (a) P is orthogonal, i.e., $p_i p_j = 0$ for every $i \neq j$ from I ;
- (b) the net $\{p_l\}$ converges strictly to 1, where l is a finite subset of I and $p_l = \sum_{a \in l} p_a$; This

means that $\sum_{a \in I} p_a = 1$ in the strict topology of $M(A)$.

Let A be a C^* -categoroid. A C^* -category $M(A)$ is called a *multiplier C^* -category* of A if A is a closed two-sided ideal in $M(A)$ and has following universal property: let D be a C^* -categoroid containing A as a closed two-sided ideal; then there exists a unique $*$ -functoroid $d : D \rightarrow M(A)$ which is the identity map on A , such that the diagram

$$(1.2) \quad \begin{array}{ccc} A & \subset & M(A) \\ \cap & \nearrow & \\ D & & \end{array}$$

is commutative.

Now, we want to construct the multiplier C^* -category for any C^* -categoroid.

Let A be a C^* -categoroid. There is an orthogonal strictly complete set of projections $\mathcal{P} = \{p_a\}_{a \in \text{Ob}A}$ in $M(S(A))$. Indeed, let A^+ be the smallest categorization of A . Since $S(A)$ is an essential ideal in $S(A^+)$ the canonical $*$ -homomorphism $S(A^+) \rightarrow M(S(A))$ is injective. For each object $a \in A^+$ there is the identity morphism 1_a which gives a set of orthogonal projections $\{p_a\}_{a \in \text{Ob}A} \subset S(A^+)$. (The image of 1_a in $M(S(A))$ is denoted by p_a). The sums $p_l = \sum_{a \in l} p_a$ are projections for any finite set l of objects in A . Thus we have a set of projections $\{p_l\}$ in $S(A^+)$ and $M(S(A))$. It can be easily shown that $\{p_l\}$ is approximate unit for $S(A^+)$ so that $\{1_l\} \rightarrow 1$ strictly in $M(S(A^+))$. On the other hand, we have the strictly continuous unital homomorphism $M(S(A^+)) \rightarrow M(S(A))$ since $S(A)$ is an essential ideal in $S(A^+)$. Thus $\{p_l\}$ converges strictly to 1 in $M(S(A))$.

The set of projections $\{p_a\}_{a \in \text{Ob}A}$ is called *the standard complete set of projections* in $M(S(A))$. Let $M(A)$ be a C^* -category with $\text{Ob}A$ as the set of objects and the set of elements of the form $p_{a'} \cdot f \cdot p_a$ from $M(S(A))$ as the set of morphisms from a to a' . The C^* -structure is induced by the corresponding structure of $M(S(A))$.

The sub-category consisting elements from $S(A)$ with property $x = p_{a'} \cdot x \cdot p_a$ form a closed two-sided ideal \mathcal{A} in $M(A)$ which is naturally isomorphic to A .

Let B be a C^* -category and A be a closed ideal in B . The category B is the multiplier C^* -category of A if and only if for any set with two objects $R = \{a, a'\}$ of $\text{Ob}B$ the canonical $*$ -homomorphism $S(B|_R) \approx M(S(A|_R))$ is an isomorphism.

1.5. Existence of Direct Limits in the Category Categoroids . Let \mathbb{C}^* be the category of C^* -categoroids and $*$ -functoroids. Bellow we'll prove existence of direct limits of C^* -categoroids over directed sets.

Let S be a directed set and $\{A_\alpha, \tau_{\alpha\beta}\}$ be an inductive system of C^* -categoroids. Let $\tau_{\alpha\beta}$ be monomorphisms on morphisms, i.e. $\tau_{\alpha\beta}(a) = 0$ iff $a = 0$, for all $\alpha \leq \beta$. Let A° be the categoroid with $\text{Ob}A^\circ = \varinjlim_\alpha \text{Ob}A_\alpha$ and $\text{hom}_{A^\circ}([A_\alpha], [A'_\alpha]) = \varinjlim_\alpha (A_\alpha, A'_\alpha)$. It is clear that A° is a $*$ -categoroid. Since $\tau_{\alpha\beta}$ are monomorphisms one can define

$$\| [f_\alpha] \| = \| f_\alpha \|$$

for all α , which gives a pre- C^* -norm structure on $\text{hom}_{A^\circ}([A_\alpha], [A'_\alpha])$. So after completion we have $\text{hom}([A_\alpha], [A'_\alpha])$. One gets in such a way a C^* -categoroid A is obtained. For each α we have the $*$ -functoroid $\tau_\alpha : A_\alpha \rightarrow A$.

Let us now consider the general case. For each A_α let $I_{\alpha\beta}$ be the C^* -ideal $\ker \tau_{\beta\alpha}$ in A_α . It is clear that $I_{\alpha\beta} \subset I_{\alpha\gamma}$ if $\beta \leq \gamma$. Thus $\{I_{\alpha\beta}\}$ is a direct system of ideals and $I_\alpha = \overline{\bigcup_\beta I_{\alpha\beta}}$ is a two-sided ideal in A_α . Put $A'_\alpha = A_\alpha / I_\alpha$. Since $I_\alpha = \tau_{\beta\alpha}^{-1}(I_\beta)$, the induced $*$ -functor $\tau'_{\alpha\beta} : A'_\alpha \rightarrow A'_\beta$ is a $*$ -mono-functor. As above, $\{A'_\alpha, \tau'_{\alpha\beta}\}$ produces $\{A', \tau'_\alpha\}$, which is the direct limit. Let us show that the pair $\{A', \tau'_\alpha k_\alpha\}$, where $k_\alpha : A_\alpha \rightarrow A'_\alpha$ is the canonical $*$ -functoroid, is the direct limit of $\{A_\alpha, \tau_{\alpha\beta}\}$. We have the commutative diagram

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\tau_{\alpha\beta}} & A_\beta \\ \downarrow k_\alpha & & \downarrow k_\beta \\ A'_\alpha & \xrightarrow{\tau'_{\alpha\beta}} & A'_\beta \end{array}$$

If $\{B, \nu_\alpha\}$ is a direct system such that

- (1) $\nu_\alpha : A_\alpha \rightarrow B$ is $*$ -functoroid;
- (2) $\nu_\alpha = \nu_\beta \tau_{\alpha\beta}$, then $\ker \nu_\alpha \supset \tau_{\beta\alpha}^{-1}(\ker \nu_\beta) \supset \ker \tau_{\beta\alpha}$, so $\ker \nu_\alpha \supset I_\alpha$.

Hence there are $*$ -functoroids ν'_α such that $\nu_\alpha = k_\alpha \nu'_\alpha$; on the other hand $\nu'_\alpha = \tau'_{\beta\alpha} \nu'_\beta$. Thus there is a canonical $*$ -functoroid $\nu : A' \rightarrow B$ such that $\nu \tau_\alpha = \nu_\alpha$.

An inductive system of C^* -categories $\{A_\alpha, \tau_{\alpha\beta}\}$ and let $\{A; \tau_\alpha\}_\alpha$ be a set, where $\tau_\alpha : A_\alpha \rightarrow A$ is functor for each α . The set $\{A; \tau_\alpha\}_\alpha$ is said to be

- (1) a *direct quasi-limit* if for each $\alpha \leq \beta$, $\tau_\beta \cdot \tau_{\alpha\beta} = \tau_\alpha$;

- (2) *filled*, if for any morphism $x \in A$, the morphism $\tau(x)$ is invertible iff there exists $\beta \geq \alpha$ such that $\tau_{\alpha\beta}(x)$ is invertible in A_β ;
- (3) *full*, iff for every $x \in A$ and $\epsilon > 0$ there exist α and $x_\alpha \in A_\alpha$ such that $\|\tau_\alpha(x_\alpha) - x\| < \epsilon$.

Let $\{A, \tau_\alpha\}$ be the direct limit of the inductive system of C^* -categories $\{A_\alpha, \tau_{\alpha\beta}\}$. Then $\{A, \tau_\alpha\}$ is a filled and full direct quasi-limit. Indeed, fullness comes from the construction of the direct limit. It is also filled. Indeed, let a_α be morphisms from A_α such that $\tau_\alpha(a_\alpha)$ are invertible in A . Then by construction of the direct limit there exists β such that $k_\beta(\tau_{\alpha\beta}(a_\alpha))$ is invertible in A'_β , hence there exists an element a_β such that $k_\beta(a_\beta)$ is inverse of $k_\beta(\tau_{\alpha\beta}(a_\alpha))$. Thus $\tau_{\alpha\beta}(a_\alpha) \cdot a_\beta = 1 + i_\beta$, $a_\beta \cdot \tau_{\alpha\beta}(a_\alpha) = 1 + i'_\beta$, where $i_\beta, i'_\beta \in I_\beta$. In general $\|i_\beta\| < 1$, $\|i'_\beta\| < 1$ for enough large β since $I_\beta = \bigcup_\gamma I_{\beta\gamma}$. Therefore $\tau_{\alpha\beta}(a_\alpha) \cdot a_\beta$ and $a_\beta \cdot \tau_{\alpha\beta}(a_\alpha)$ are invertible elements, so $\tau_{\alpha\beta}(a_\alpha)$ is invertible.

1.6. Additive and Pseudo-abelian C^* -categories and C^* -categoroids. A C^* -categoroid A is said *additive C^* -categoroid*, if there exists an additive C^* -categoroid containing A as a closed two-sided ideal. Of course, in this situation the multiplier C^* -category must be additive C^* -category. A functoroid $f : A \rightarrow A'$ is said additive if it is restriction of some additive functor between additive C^* -categories containing respectively A and A' as ideals.

Let a' and a be objects in A then each element in $\text{hom}(a \oplus a')$ may be uniquely represented by matrix of form

$$\begin{pmatrix} l_{aa} & l_{a'a} \\ l_{aa'} & l_{a'a'} \end{pmatrix}$$

where $l_{aa} \in \text{hom}(a, a)$, $l_{aa'} \in \text{hom}(a, a')$, $l_{a'a} \in \text{hom}(a', a)$, $l_{a'a'} \in \text{hom}(a', a')$.

Lemma 1.3. *Let A and B be additive C^* -categoroids and $f : A \rightarrow B$ be an additive $*$ -functoroid. Then f is $*$ -isomorphism if and only if f is bijection on the objects and induced $*$ -homomorphism of C^* -algebras $f_a : \mathcal{L}(a) \rightarrow \mathcal{L}(f(a))$ is an $*$ -isomorphism for any object a in A .*

Proof. We only must show that this conditions is sufficient. It is enough to show that for any pair objects a, a' in A , the linear map

$$f_{aa'} : \text{hom}(a, a') \rightarrow \text{hom}(f(a), f(a'))$$

is an isomorphism. i. if $f_{aa'}(\alpha) = 0$ then $f_{aa'}(\alpha)f_{aa'}(\alpha^*) = f_{aa'}(\alpha^*)f_{aa'}(\alpha) = 0$. This implies that $\alpha^*\alpha = \alpha\alpha^* = 0$. Therefore $\alpha = 0$. This means that $f_{aa'}$ is injective. ii. Let $\beta \in \text{hom}(a, a')$. Consider the matrix

$$\beta' = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$$

the element of $\mathcal{L}(f(a) \oplus f(a'))$. Since $f_a : \mathcal{L}(a \oplus a') \rightarrow \mathcal{L}(f(a) \oplus f(a'))$ is $*$ -isomorphism there exists unique $\alpha \in \mathcal{L}(a \oplus a')$ such that $f_a(\alpha) = \beta'$ where $\alpha \in \mathcal{L}(a \oplus a')$. Let

$$\alpha = \begin{pmatrix} \alpha_{aa} & \alpha_{a'a} \\ \alpha_{aa'} & \alpha_{a'a'} \end{pmatrix}$$

Since $f_{a \oplus a'}$ is $*$ -isomorphism and β' has unique representation in matrix form, α must has the form

$$\alpha = \begin{pmatrix} 0 & 0 \\ \alpha_{aa'} & 0 \end{pmatrix}$$

Therefore $f_{aa'}(\alpha_{aa'}) = \beta$ □

We say that a C^* -category A has enough isometries if for any projection $p \in \mathcal{L}(a)$ and any object a in A there exists isometry $s : a' \rightarrow a$ satisfying equality $ss^* = p$. If an additive C^* -category has enough isometries then it will be said *pseudo-abelian C^* -category*.

The universal pseudo-additive C^* -category $P(A)$ of an additive C^* -category A may be constructed by the following manner [9] :

- i. objects are all pairs of form (a, p) , $a \in \text{Ob}A$, where $p \in \mathcal{L}(a)$, such that $p^2 = p$ and $p^* = p$;

ii. We say that triple (f, p', p) (or simply, f) is a morphism in $P(A)$ from (a', p') into (a, p) if $f : a' \rightarrow a$ is an morphism in A and $fp' = pf = f$.

The direct sum of objects and morphisms are defined by the formulas:

$$(a, p) \oplus (a', p') = (a \oplus a', p \oplus p') \text{ and } (f, p, q) \oplus (f', p', q') = (f \oplus f', p \oplus p', q \oplus q').$$

The category $P(A)$ has natural structure of an additive C^* -category inherited from the C^* -category A ; Besides, A can be identified with a full subcategory of $P(A)$ by the map of objects $a \mapsto (a, 1)$. Since $(q, q, p) : (a, q) \rightarrow (a, p)$ is isometry with $(q, q, p)(q, q, p)^* = (q, p, p)$ for any projection $(q, p, p) \in \mathcal{L}((a, p))$, the C^* -category $P(A)$ has enough isometries, i.e. $P(A)$ is a pseudo-abelian category.

Let A be a C^* -algebra with unit. Denote by $F(A)$ the additive C^* -category which has standard Hilbert right A -modules $A^n = A \oplus_{n\text{-times}} A$ as the objects and usual A -homomorphisms (which has an adjoint) as the morphisms. Let $P(A)$ be the standard pseudo-abelian C^* -category of $F(A)$, i.e. $P(A) = P(F(A))$.

We say that a subcategory A is a cofinal sub-category of a C^* -category A' if for any object $a' \in A'$ there exists an object $a \in A$ and an isometry $s : a' \rightarrow a$. If A is an additive full cofinal sub- C^* -category in a pseudo-abelian C^* -category A' then we say that A' is *generated* by the C^* -category A .

1.7. The C^* -categories $Rep(A; B)$ and $\text{Rep}(A; B)$. We list some examples of C^* -categories and C^* -categoroids, useful in the next sections.

1) (See [14]) Firstly we define the C^* -category $\mathcal{H}_G(B)$ over a fixed compact second countable group G . The objects of this category are countable generated right Hilbert B -modules equipped with a B -linear, norm-continuous G -action such that $g(xb) = g(x)g(b)$ and $\langle g(x), g(y) \rangle = g \langle x, y \rangle$, for all $g \in G$. A morphism $f : E \rightarrow E'$ is B -homomorphism commuting with the action of G , such that there exists $f^* : E' \rightarrow E$ satisfying the conditions: $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ where $x \in E$ and $y \in E'$. The norm of a morphism is defined as the norm of linear bounded map. $\mathcal{H}_G(B)$ is an additive C^* -category with respect to the sum of the Hilbert modules. Note that compact group acts on the morphisms by following rule: if $f : E \rightarrow E'$ then morphism $gf : E \rightarrow E'$ is defined by formula $gf(x) = g(f(g^{-1}(x)))$ (this action generally isn't norm-continuous). A morphism is called *invariant* if $gf = f$. The category $\mathcal{H}_G(B)$ contains natural class of morphism, so called compact B -homomorphisms [14]. Denote it by $\mathcal{K}_G(B)$. Properties of compact B -homomorphisms implies that $\mathcal{K}_G(B)$ is a C^* -ideal in $\mathcal{H}_G(B)$. Note that there exists a $*$ -functor $\infty : \mathcal{H}_G(B) \rightarrow \mathcal{H}_G(B)$ and a natural isomorphism of functors $id_{\mathcal{H}(B)} \oplus \infty \simeq \infty$, where $E^\infty = E \oplus E \oplus \dots$. This structure will be called an ∞ -structure.

2) Now, we define the C^* -category $\text{rep}_G(A, B)$. The objects of this category are pairs of form (E, ϕ) , where E is an object in \mathcal{H}_G and $\phi : A \rightarrow \mathcal{L}(E)$ is an equivariant $*$ -homomorphism. Objects of this type are said to be A, B -bimodules. A morphism $f : (E, \phi) \rightarrow (E', \phi')$ is an B -homomorphism $f : E \rightarrow E'$ in $\mathcal{H}_G(B)$ such that $f\phi(a) = \phi'(a)f$ for all $a \in A$. The structure of a C^* -category and action of G is inherited from the C^* -category structure of $\mathcal{H}_G(B)$. Let $\text{rep}(A, B)$ be the sub- C^* -category of $\text{rep}_G(A, B)$ which has the same object as latter but morphisms are invariant under action of G . it is easy to show that $\text{rep}(A, B)$ is an additive C^* -category (in fact, a pseudo-abelian C^* -category). The following property of $\text{rep}(A, B)$ useful for calculation of the K -groups of $\text{rep}(A, B)$. The ∞ -structure on $\mathcal{H}_G(B)$ induces a corresponding structure on $\text{rep}(A, B)$ via the formulas $(E, \phi)^\infty = (E^\infty, \phi^\infty)$, where $\phi^\infty(a) = (\phi(a))^\infty$ for all $a \in A$.

3) Consider the additive C^* -category $\mathcal{Q}_G(B)$ which is the quotient C^* -category $\mathcal{H}_G(B)/\mathcal{K}_G(B)$. It has an essential compact group action inherited from the action of a compact group on $\mathcal{H}_G(B)$. Denote by $\pi : \mathcal{H}_G(B) \rightarrow \mathcal{Q}_G(B)$ the canonical additive $*$ -functor. We need also following C^* -category denoted by $\mathcal{Q}_G(A, B)$. By definition objects of this category have the form (E, ψ) , where E is an object in $\mathcal{H}_G(B)$ and $\psi : A \rightarrow \text{hom}_{\mathcal{Q}_G(B)}(E, E)$ is an equivariantly liftable $*$ -homomorphism, i.e., there exists an A, B -bimodule (E, ϕ) such that $\pi\phi = \psi$. A morphism $f : (E, \psi) \rightarrow (E', \psi')$ is a morphism $f : E \rightarrow E'$ of the category $\mathcal{Q}_G(B)$ such that $f\psi(a) = \psi'(a)f$ for all $a \in A$. Let $\mathcal{Q}(B)$ be sub- C^* -category of $\mathcal{Q}_G(B)$ the invariant liftable morphisms of latter as the morphisms. There is a $*$ -functor $\Theta : \text{rep}(A, B) \rightarrow \mathcal{Q}(A, B)$ given by $(E, \phi) \mapsto (E', \phi)$ and $f \mapsto \pi(f)$.

4) Now, we want to define the additive C^* -category $Rep_G(A, B)$. The class of objects of this category coincides with the class of objects of $rep_G(A, B)$. But a morphism $f : (E, \phi) \rightarrow (E', \phi')$ is a morphism $f : E \rightarrow E'$ in $\mathcal{H}(B)$ such that

$$f\phi(a) - \phi'(a)f \in \mathcal{K}_G(E, E')$$

for all $a \in A$. The structure of C^* -category and action of G are inherited from $\mathcal{H}_G(B)$. It is easy to show that $Rep_G(A, B)$ is an additive C^* -category (but it isn't a pseudo-abelian C^* -category). Let $Rep(A; B)$ be sub- C^* -category of $Rep_G(A; B)$ with same class of objects as latter but morphism are those are invariant under the action of G . There is a canonical additive $*$ -functor $\Pi_{A, B} : Rep(A, B) \rightarrow \mathcal{Q}(A, B)$ defined by $(E, \phi) \mapsto (E, \pi\phi)$ and $f \mapsto \pi f$. From the definition follows that the canonical linear map

$$\text{hom}((E, \phi), (E', \phi')) \mapsto \text{hom}((E', \phi), (E', \phi'))$$

is surjective, i.e., Π is a Serre functor (see for the definition [13]).

There is an C^* -ideal $D(A, J; B)$ in $Rep(A; B)$, which is associated from any closed ideal J in A . C^* -ideal $D(A, J; B)$, which we'll use in the sequel, is defined by the following manner. Let (E, ϕ) and (E', ϕ') are objects in $Rep(A, B)$. A morphism $\alpha : (E, \phi) \rightarrow (E', \phi')$ in $Rep(A, B)$ is in $D(A, J; B)$ if $\alpha\phi(j) \in \mathcal{K}((E, \phi), (E', \phi'))$ and $\phi(j)\alpha \in \mathcal{K}((E, \phi))$. The space of all morphisms from (E, ϕ) to (E', ϕ') will be denoted by $D_{\phi, \phi'}(A, J; E, E'; B)$ (if $(E', \phi') = (E, \phi)$ then it is denoted by $D_{\phi}(A, J; E; B)$). Sometimes $Rep(A; B)$ is denoted by $D(A, 0; B)$ or $D(A; B)$.

Now we come to our main C^* -category, that is, $Rep(A, B)$.

Definition 1.4. Let $Rep(A, B)$ be the universal pseudo-abelian C^* -category of $Rep(A, B)$. Using the definition of a pseudo-abelian C^* -category, we have the following description of $Rep(A, B)$. Objects of it are triples (E, ϕ, p) , where (E, ϕ) is an object and $p : (E, \phi) \rightarrow (E, \phi)$ is a morphism in $Rep(A, B)$ such that $p^* = p$ and $p^2 = p$. A morphism $f : (E, \phi, p) \rightarrow (E', \phi', p')$ is a morphism $f : (E, \phi) \rightarrow (E', \phi')$ in $Rep(A, B)$ such that $fp = p'f = f$. In detail, f has the properties

$$(1.3) \quad f\phi(a) - \phi'(a)f \in \mathcal{K}(E, F), \quad fp = p'f = f.$$

The structure of C^* -category of $Rep(A, B)$ comes from the corresponding structure of $Rep(A, B)$.

2. HIGSON'S HOMOTOPY INVARIANCE THEOREM

In the next sections we'll need Higson's homotopy invariance theorem. The purpose of this section is to give account of Higson's homotopy invariant theorem, both for real and complex cases. Higson's theorem asserts the following. Let E be a stable and split additive functor from admissible sub-category of the category *complex* C^* -algebras into the category of abelian groups. Then it is homotopy invariance [7] (Theorem 3.2.2).

This important theorem plays major role for setting homotopy invariance of functors (for example, using this theorem, it was solved well known problem about isomorphism of algebraic and topological K -theories on stable C^* -algebras [19]). The natural question arises whether this theorem is for real C^* -algebras? The reason for it is that we couldn't dissemination the proof of Lemma 3.1.2 and Theorem 3.11 of [7] for real C^* -algebras; because the complexity of C^* -algebras is significant to prove them. We choose another point of view for Higson's theorem for both cases mentioned above, namely, an investigation of KK -theory by J. Cuntz and G. Scandalis [4]. We have chosen this interpretation first, because it is simple and only very simple arguments are used from KK -theory, and on the other hand it is right both for complex and real C^* -algebras. Equivalence relation on Kasparov bimodules is defined in this article that is called "Cobordism". It's main feature is that this equivalence coincides with the Kasparov's induced by homotopy of bimodules.

Let (E, φ, F) be a Kasparov A, B -bimodule, where E is a countable generated Hilbert right B -module, $F \in \mathcal{L}_B(E)$ is a degree one element and $\varphi : A \rightarrow \mathcal{L}_B(E)$ is a $*$ -homomorphism. We say that A, B -bimodules (E_0, φ_0, F_0) and (E_1, φ_1, F_1) are *cs*-isomorphic if there exists such a degree zero unitary $u : E_0 \rightarrow E_1$ that $u\varphi_0(a)u^* = \varphi_1(a)$ for any $a \in A$ and $uF_0u^* - F_1 \in \mathcal{K}(E_1)$. Now, one easily checks that two Kasparov A, B -bimodules (E_0, φ_0, F_0) and (E_1, φ_1, F_1) are cobordant if and only if there exists such a Kasparov A, B -bimodule (E, φ, F) that $(E_0, \varphi_0, F_0) \oplus (E, \varphi, F)$ and

$(E_1, \varphi_1, F_1) \oplus (E, \varphi, F)$ are cs -isomorphic. Thus from [4] (Theorem 3.7) one deduces as follows (cf. remark 3.8 in [4]).

Let A be a separable Z_2 -graded C^* -algebra and B be a Z_2 -graded σ -unital C^* -algebra. Let $\widehat{KK}(A, B)$ be the cancellation monoid of the abelian monoid of classes A, B -bimodules identified by cs -isomorphism and $KK(A, B)$ be Kasparov group. Then the natural homomorphism

$$\tau : \overline{KK}(A, B) \rightarrow KK(A, B)$$

is an isomorphism. In particular, $\overline{KK}(A, B)$ is an abelian group and homotopy invariance for both arguments.

Bellow, we give a slightly different variant of this theorem, which will be our fulcrum to approach Higson's theorem.

Consider Kasparov A, B -bimodules satisfying conditions $F^* = F$ and $F^2 = 1$ (we call them *special*). Let $\widehat{KK}(A, B)$ be the cancellation monoid of the abelian monoid of classes special A, B -bimodules identified by cs -isomorphism. Then one formulate the above mentioned result in the following form.

Theorem A. *Let A be a separable Z_2 -graded C^* -algebra and B it be a Z_2 -graded σ -unital C^* -algebra. Let $\widehat{KK}(A, B)$ be the cancellation monoid of the abelian monoid of classes special A, B -bimodules identified by cs -isomorphism and $KK(A, B)$ be Kasparov group. Then the natural homomorphism $\tau : \widehat{KK}(A, B) \rightarrow KK(A, B)$ is an isomorphism. In particular, $\widehat{KK}(A, B)$ is an abelian group and homotopy invariance for both arguments.*

The proof of this theorem is based on the fact that cobordism and homotopy of Kasparov bimodules coincides, and from Lemmas 17.4.2-17.4.3 in [2]. We shortly remind the content of them:

Kasparov KK -groups won't be changed if in their definition will be taken only

- (1) Kasparov bimodules with property $F^* = F$, and one takes only homotopy or only cobordism;
- (2) Kasparov bimodules with property $F^* = F$ and $F^2 = 1$, and one takes homotopy or only cobordism;
- (3) special Kasparov bimodules, and one takes only cs -isomorphism.

We remark that the functional calculus of a self-adjoint element is used to show (1)-(3) in [2]. The functional calculus for a self-adjoint element x in complex C^* -algebra A is the $*$ -monomorphism $\Phi : C(\text{sp}x) \rightarrow A$, defined by the rule $id_{\text{sp}x} \mapsto x$. For the case of a real C^* -algebra under functional calculus we mean the following. Let A be a real C^* -algebra, and consider the complex involutive algebra $A \otimes_R C$ with involution $(a \otimes c)^* = a^* \otimes \bar{c}$. Then there exists a C^* -norm, with respect to which $A \otimes_R C$ is a complex C^* -algebra, and canonical $*$ -embedding $i : A \rightarrow A \otimes_R C$ defined by $a \mapsto a \otimes 1$ is an isometry (cf. Theorem 2 and Corollary 2 in [16], also [14]). For real C^* -algebras there exists functional calculus for self-adjoint elements. To be more precise, let $r \in A$ is a self-adjoint element and $A(r)$ be the real closed sub-algebra in A generated by r . It is real part of the sub- C^* -algebra $C^*(r \otimes 1)$, generated by the element $r \otimes 1$ in $A \otimes_R C$. According to functional calculus of complex C^* -algebras one has the $*$ -monomorphism $A(r) \rightarrow A \otimes_R C$ defined by rule $r \mapsto r \otimes 1$. Thus under functional calculus we mean the functional calculus of then element $r \otimes 1$ in $A \otimes_R C$ (For example, if f is continuous real function on the $\text{sp}(r \otimes 1)$ then there exists the unique element $f(r) \in A(r)$ that $i(f(r)) = f(r \otimes 1)$).

Now theorem A may be interpreted for trivially graded A and B in the following way.

Let (φ, ψ, U) be a triple, where H_φ and H_ψ are countable generated Hilbert right B -modules with trivial grading; $\varphi : A \rightarrow \mathcal{L}_B(H_\varphi)$ and $\psi : A \rightarrow \mathcal{L}_B(H_\psi)$ are $*$ -homomorphisms; $U : H_\varphi \rightarrow H_\psi$ is a B -homomorphism which has an adjoint. A triple (φ, ψ, U) is said to be *unitary Fredholm A, B -bimodule* if U is a unitary and satisfies the following condition:

$$U\varphi(a) - \psi(a)U \in \mathcal{K}(H_\varphi, H_\psi), \text{ for all } a \in A.$$

A, B -bimodules (φ_0, ψ_0, U_0) and (φ_1, ψ_1, U_1) are said to be cs -isomorphic if there exists degree zero unitaries $u : H_{\varphi_0} \rightarrow H_{\varphi_1}$ and $v : H_{\psi_0} \rightarrow H_{\psi_1}$ that $u\varphi_0(a)u^* = \varphi_1(a)$, $v\psi_0(a)v^* = \psi_1(a)$ and

$$vU_0u^* - U_1 \in \mathcal{K}(H_{\varphi_1}, H_{\psi_1})$$

for any $a \in A$.

The sum of A, B -bimodules are defined in usual manner

$$(\varphi_0, \psi_0, U_0) \oplus (\varphi_1, \psi_1, U_1) = (\varphi_0 \oplus \varphi_1, \psi_0 \oplus \psi_1, U_0 \oplus U_1).$$

In the definition of group $\widetilde{KK}(A, B)$ Kasparov special A, B -bimodules can be replaced by Fredholm unitary A, B -bimodules. This statement is proved by the following way. First of all, if B is trivially graded C^* -algebra then any graded Hilbert B -module E can be represented in this form $E = E_0 \oplus \bar{E}_1$ where E_0 and E_1 are trivially graded Hilbert B -modules and \bar{E}_1 is the opposite to E_1 graded Hilbert B -module, i.e. $\bar{E}_1^{(0)} = 0$ and $\bar{E}_1^{(1)} = E_1$. Let $i_d : E_1 \rightarrow \bar{E}_1$ be the degree one B -linear map defined by the identity map. Now, if (E, ϕ, F) is a Kasparov special A, B -bimodule then one has

$$(2.1) \quad E = E_0 \oplus \bar{E}_1, \quad \phi = \begin{pmatrix} \varphi & 0 \\ 0 & \bar{\psi} \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} 0 & \bar{U}^* \\ \bar{U} & 0 \end{pmatrix}.$$

Here $\varphi : A \rightarrow \mathcal{L}_B(E_0)$ and $\psi : A \rightarrow \mathcal{L}_B(E_1)$ are $*$ -homomorphisms, $U : E_0 \rightarrow E_1$ is unitary B -homomorphism, $\bar{U} = i_d \cdot U \cdot i_d^*$ and $\bar{\psi}(a) = i_d \cdot \psi(a) \cdot i_d^*$. It is easily check that (φ, ψ, U) is Fredholm unitary A, B -bimodule. Conversely, let (φ, ψ, U) be a Fredholm unitary A, B -bimodule, where $\varphi : A \rightarrow \mathcal{L}_B(E_0)$ and $\psi : A \rightarrow \mathcal{L}_B(E_1)$ are $*$ -homomorphisms such that E_0 and E_1 are trivially graded Hilbert B -modules. Then the correspondence Kasparov special A, B -bimodule (E, ϕ, F) is given by the formulas 2.1. Therefore Theorem A may be formulated as follows.

Theorem 2.1. *Let A be a separable trivially graded C^* -algebra and B be a trivially graded σ -unital C^* -algebra. Let $\widetilde{KK}(A, B)$ be the cancellation monoid of the abelian monoid of classes Fredholm A, B -bimodules identified by cs -isomorphism, and $KK(A, B)$ be Kasparov group. Then the natural homomorphism*

$$\tilde{\tau} : \widetilde{KK}(A, B) \rightarrow KK(A, B)$$

is an isomorphism. In particular, $\widetilde{KK}(A, B)$ is an abelian group and homotopy invariance for both arguments.

Let \mathcal{S}_0 be a sub-category of C^* -algebras containing two C^* -algebras, \mathbf{F} and $\mathbf{F}[0; 1]$, as the objects; and two non-trivial morphisms, these are evolution maps $ev_0 : \mathbf{F}[0; 1] \rightarrow \mathbf{F}$ and $ev_1 : \mathbf{F}[0; 1] \rightarrow \mathbf{F}$ given by the formulas $ev_0(f(t)) = f(0)$ and $ev_1(f(t)) = f(1)$, where $f(t) \in \mathbf{F}[0; 1]$. The following definition points out an important class of functors on \mathcal{S}_0 .

Definition 2.2. A functor E from \mathcal{S}_0 into a category is said to be

- (1) *weak homotopy* (or shortly *w-homotopy*) if $E(ev_0) = E(ev_1)$.
- (2) *weak K -hereditary* if there is a natural transformation of functors

$$\chi : \widetilde{KK}(-; \mathbf{F}) \rightarrow \text{hom}(E(-); E(\mathbf{F}))$$

so that $\chi(e, \theta, 1)$ is the identity morphism from $E(\mathbf{F})$ in itself, where $(e, \theta, 1)$ is Fredholm unitary \mathbf{F}, \mathbf{F} -module; $e : \mathbf{F} \rightarrow \mathcal{K}(\mathcal{H})$ is a $*$ -homomorphism which maps $1 \in \mathbf{F}$ in a rank one projection, $\theta : \mathbf{F} \rightarrow \mathcal{K}(\mathcal{H})$ is the zero homomorphism and 1 is unit of the algebra $\mathcal{L}(\mathcal{H})$, where \mathcal{H} is a countable generated Hilbert space.

For example, let E be a homotopy invariant covariant functor on the category S of separable C^* -algebras and $*$ -homomorphisms. Then functor $E(A \otimes -)$ is a *w-homotopy* functor on \mathcal{S}_0 , for any $A \in S$. Conversely, if $E(A \otimes -)$ is a *w-homotopy* functor on \mathcal{S}_0 , for any $A \in S$ then, of course, E is homotopy invariant too.

Next we'll only need the following very particular case of Theorem 2.1.

Corollary 2.3. *The contravariant functor $\widetilde{KK}(-; \mathbf{F})$ on the category \mathcal{S}_0 is w-homotopy invariant.*

Now, from Corollary 2.3 we deduce the following lemma. We'll use this lemma to obtain Higson's theorem.

Lemma 2.4. *Let E be a weak K -hereditary functor from \mathcal{S}_0 into a category. Then E is w-homotopy functor.*

Proof. Since K -homology, by Corollary 2.3, is w -homotopy and E is a weak K -hereditary on S_0 , the diagram

$$\begin{array}{ccc} \widetilde{KK}(\mathbf{F}, \mathbf{F}) & \xrightarrow{\chi_{\mathbf{F}}} & \text{hom}(E(\mathbf{F}), E(\mathbf{F})) \\ \downarrow e\bar{v}_0=e\bar{v}_1 & & \downarrow e\bar{v}_0 \downarrow e\bar{v}_1 \\ \widetilde{KK}(\mathbf{F}[0;1], \mathbf{F}) & \xrightarrow{\chi_{\mathbf{F}[0;1]}} & \text{hom}(E(\mathbf{F}[0;1]), E(\mathbf{F})) \end{array}$$

commutes. Thus $e\bar{v}_0 \cdot \chi_{\mathbf{F}} = e\bar{v}_1 \cdot \chi_{\mathbf{F}}$. Again, since E is a weak K -hereditary, one has $E(ev_0) = e\bar{v}_0(\chi_{\mathbf{F}}(\iota)) = e\bar{v}_1(\chi_{\mathbf{F}}(\iota)) = E(ev_1)$ where ι is the class of Fredholm \mathbf{F}, \mathbf{F} -bimodule $(e, \theta, 1)$. Therefore E is a w -homotopy functor. \square

2.1. Pairings with Fredholm Pairs. Let $E : S \rightarrow Ab$ be a functor, where Ab is the category abelian groups and homomorphisms. A pairing of E with the set of Fredholm pairs is defined in [7]. We'll recall it.

A Fredholm B -pair is such a pair (φ, ψ) where φ and ψ are $*$ -homomorphisms from B into $\mathcal{L}_{\mathbf{F}}(\mathcal{H})$ that $\varphi(b) - \psi(b) \in \mathcal{K}(\mathcal{H})$ for any $b \in B$, where \mathcal{H} is a countable generated Hilbert space over \mathbf{F} , Here $\mathcal{K}(\mathcal{H})$ is the C^* -algebra of compact operators. A pairing of E with the set of Fredholm B -pairs is a rule. This assigns to each Fredholm B -pair (φ, ψ) a morphism $\times(\varphi, \psi) : E(A \otimes B) \rightarrow E(A \otimes \mathbf{F})$ in the category C , for any $A \in S$ and $B \in S_0$, with the following properties:

- (1) Functoriality. If (φ, ψ) is a Fredholm B' -module, and if $f : B \rightarrow B'$ is a $*$ -homomorphism from S , then the diagram

$$\begin{array}{ccc} E(A \otimes B) & \xrightarrow{\times(\varphi f, \psi f)} & E(A \otimes \mathbf{F}) \\ \downarrow E(\text{id}_A \otimes f) & & \parallel \\ E(A \otimes B') & \xrightarrow{\times(\varphi, \psi)} & E(A \otimes \mathbf{F}) \end{array}$$

commutes.

- (2) Additivity. If (φ, χ) and (χ, ψ) are Fredholm B -pairs, then

$$\times(\varphi, \chi) + \times(\chi, \psi) = \times(\varphi, \psi).$$

- (3) Stability. If (φ, ψ) is a Fredholm B -pair and $\eta : B \rightarrow \mathcal{L}(\mathcal{H})$ is any $*$ -homomorphism then

$$\times(\varphi, \psi) = \times \left(\left(\begin{array}{cc} \varphi & 0 \\ 0 & \eta \end{array} \right), \left(\begin{array}{cc} \psi & 0 \\ 0 & \eta \end{array} \right) \right).$$

- (4) Non-degeneracy. If (e, θ) is a Fredholm \mathbf{F} -module for which $e : \mathbf{F} \rightarrow \mathcal{K}(\mathcal{H})$ maps $1 \in \mathbf{F}$ to p , where p is a rank one projection in $\mathcal{K}(\mathcal{H})$ and θ is the zero homomorphism. Then

$$\times(e, 0) : E(A \otimes \mathbf{F}) \rightarrow E(A \otimes \mathbf{F})$$

is the identity morphism.

- (5) Unitary equivalence. If $U \in \mathcal{L}(\mathcal{H})$ is a unitary then

$$\times(\varphi, \psi) = \times(U\varphi U^*, U\psi U^*)$$

- (6) Compact perturbations. If $U \in \mathcal{L}(\mathcal{H})$ is a unitary which is equal to the identity modulo compacts, then

$$\times(\varphi, U\varphi U^*) = 0.$$

The following theorem is exactly theorem 3.1.4 of [7] (not only on the category of complex C^* -algebras as in [7], but on the category of real C^* -algebras too.)

Theorem 2.5. *Let be E a functor from the category \mathcal{S} into the category Ab of abelian groups and their homomorphisms and the functor admits a pairing with the set of Fredholm B -pairs, $B \in S_0$. Then E is a homotopy functor.*

Proof. As it was pointed out after definition 2.2, it is enough to show that $E(A \otimes -)$ is a weak K -hereditary functor. So, we'll have to construct a natural transformation functors

$$\chi : \widetilde{KK}(-; \mathbf{F}) \rightarrow \text{hom}(E(A \otimes -); E(\mathbf{A} \otimes \mathbf{F})).$$

Let (φ, ψ, U) be a Fredholm unitary B, \mathbf{F} -bimodule, then by definition

$$\chi(\varphi, \psi, U) = \times(\varphi, U^*\psi U).$$

The correspondence χ is correctly defined, since $\widetilde{KK}(-; \mathbf{F})$ is the cancellation monoid of the monoid of the classes of *cs*-isomorphic unitary A, \mathbf{F} -bimodules and the following hold.

- (1). If (φ_0, ψ_0, U_0) and (φ_1, ψ_1, U_1) are *cs*-isomorphic then $\chi(\varphi_0, \psi_0, U_0) = \chi(\varphi_1, \psi_1, U_1)$.
Indeed, let (u, v) be a *cs*-isomorphism from (φ_0, ψ_0, U_0) in (φ_1, ψ_1, U_1) , then

$$\begin{aligned} \chi(\varphi_0, \psi_0, U_0) &= \times(\varphi_0, U_0^* \psi_0 U_0) = (\text{by unitary equivalence}) \\ &= \times(\varphi_1, u^*(U_0^* \psi_0 U_0)u) = \times(\varphi_1, (u^* U_0^* v)(v^* \psi_0 v)(v^* U_0 u)) = (\text{by additivity}) \\ &= \times(\varphi_1, U_1^* \psi_1 U_1) + \times(U_1^* \psi_1 U_1, (u^* U_0^* v) \psi_1 (v^* U_0 u)) = (\text{by compact perturbation}) \\ &= \times(\varphi_1, U_1^* \psi_1 U_1) = \chi(\varphi_1, \psi_1, U_1). \end{aligned}$$

- (2). $\chi(\varphi_0 \oplus \varphi_1, \psi_0 \oplus \psi_1, U_0 \oplus U_1) = \chi(\varphi_0, \psi_0, U_0) + \chi(\varphi_1, \psi_1, U_1)$.
Indeed,

$$\begin{aligned} \chi(\varphi_0 \oplus \varphi_1, \psi_0 \oplus \psi_1, U_0 \oplus U_1) &= (\text{by additivity}) \\ &= \chi(\varphi_0 \oplus \varphi_1, \psi_0 \oplus \varphi_1, U_0 \oplus 1) + \chi(U_0^* \psi_0 U_0 \oplus \varphi_1, U_0^* \psi_0 U_0 \oplus \psi_1, 1 \oplus U_1) = (\text{by stability}) \\ &= \chi(\varphi_0, \psi_0, U_0) + \chi(\varphi_1, \psi_1, U_1) \end{aligned}$$

□

Now, Higson's theorem may be formulated in the following way.

Theorem 2.6. *Let E be stable and split exact covariant or contravariant functor on the category of separable complex or real C^* -algebras. Then E is homotopy invariant.*

Proof. The complex case, using theorem 3.1.4, was proved in [7]. The latter may be applied, mutatis mutandis, to the real case, according to Theorem 2.5. The contravariant case can be deduced from the covariant case in following manner. Let Φ be a contravariant functor with above mentioned properties and A be a separable C^* -algebra, then consider a covariant functor $\text{hom}(\Phi(-), \Phi(A))$ which, of course, is split exact and stable and thus homotopy invariant. Therefore,

$$\Phi(ev(0)) = \Phi(ev(0))(id_{\Phi(A)}) = \Phi(ev(1))(id_{\Phi(A)}) = \Phi(ev(1)),$$

where $ev(0), ev(1) : A([0; 1]) \rightarrow A$ are evolutions at 0 and 1. □

Let S_G be the category of separable C^* -algebras with action of a fixed compact second countable group as the objects and the equivariant $*$ -homomorphisms as the morphisms. For a fixed $B \in S_G$, one has natural functor $B \otimes - : S \rightarrow S_G$ sending C^* -algebra A in the $B \otimes A$ with diagonal action of G . Trivial checking shows that latter functor sends a split exact sequence from S in the split exact sequence of the category S_G .

Further, consider algebra \mathcal{K} as the object in S_G , assuming that action of G on \mathcal{K} is trivial. We say that functor E is stable if $E(e_A)$ is an isomorphism, where e_A is a natural equivariant inclusion $e_A : A \rightarrow A \otimes \mathcal{K}$ given by the map $a \mapsto a \otimes p$, p is rank one projection.

Now, we'll make some remarks on the notation of homotopy. Let A be an algebra in S_G . Consider algebra $A[0; 1]$ with action of G defined by equality $(g \cdot f)(t) = g(f(t)), \forall t \in [0; 1]$. It is well known that the latter algebra is isomorphic to the tensor product $A \otimes \mathbf{F}[0; 1]$ where the action of G on $\mathbf{F}[0; 1]$ is trivial. Let E be a functor on S_G . It is easily to show that E is homotopy invariant if only if $E(e_A(0)) = E(e_A(1))$ for any $A \in S_G$, where $e_t : A[0; 1] \rightarrow A$ the evolution at $t \in [0; 1]$. This fact may be trivially reformulated in the following way. A functor E on S_G is homotopy invariant if and only if the functor $E(A \otimes -)$ is homotopy invariant on the category S for all $A \in S_G$.

A functor E will be said *split exact* if a functor $E(A \otimes -)$ is split exact on S , for any algebra A in S_G .

The following corollary is a consequence of Theorem 2.6.

Corollary 2.7. *Let E be stable and split exact functor on the category S_G of real or complex C^* -algebras with the action of a fixed compact second countable group G and equivariant $*$ -homomorphisms. Then E is homotopy invariant.*

Proof. For any object A in S_G , consider the functor $E(A \otimes -)$ which, of course is stable and split exact on the category S , thus homotopy invariant by Theorem 2.6. Therefore E is homotopy invariant on the category S_G , by the principle pointed after Theorem 2.6. □

3. ON THE CUNTZ-BOTT PERIODICITY

As before, let S_G denotes category of separable complex or real C^* -algebras with action of compact second countable group G and equivariant $*$ -homomorphisms. let $\mathbb{E} = \{E_n\}_n \in \mathbf{mathbf{Z}}$ be a set of covariant functors from S_G into the category of abelian groups and homomorphisms, indexed by the integer numbers. One says that \mathbb{E} is Cuntz-Bott homology theory on the category \mathbb{E} if

- (1) \mathbb{E} has weak excision property. Namely, for any exact sequence $0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ of algebras from S_G , where epimorphism admits equivariant completely positive and contractive section. Then

i. there exist homomorphism $\delta_n : E_n(A) \rightarrow E_{n-1}(I)$, for any $n \in \mathbf{Z}$, non-depending on a completely positive and contractive section of p , and natural in the following sense. let

$$\begin{array}{ccccccccc} 0 & \rightarrow & I & \rightarrow & B & \rightarrow & A & \rightarrow & 0 \\ & & \downarrow f_I & & \downarrow f_B & & \downarrow f_A & & \\ 0 & \rightarrow & I' & \rightarrow & B' & \rightarrow & A' & \rightarrow & 0 \end{array}$$

be a commutative diagram such that in the horizontal short exact sequences epimorphisms have completely positive and contractive sections. Then diagram

$$\begin{array}{ccc} E_n(A) & \xrightarrow{\delta} & E_{n-1}(I) \\ \downarrow E_n(f_A) & & \downarrow E_n(f_I) \\ E_n(A') & \xrightarrow{\delta'} & E_{n-1}(I') \end{array}$$

commutes. ii. The natural two-sided sequence of abelian groups

$$\cdots \rightarrow E_n(I) \rightarrow E_n(B) \rightarrow E_n(A) \xrightarrow{\delta_n} E_{n-1}(I) \rightarrow \cdots$$

is exact.

- (2) \mathbb{E} is stable. This means that if $e_A : A \rightarrow A \otimes \mathcal{K}$ is a homomorphism defined by a map $a \mapsto a \otimes p$, here p is a rank one projection in \mathcal{K} , then $E_n(e_A)$ is an isomorphism.

Now, we make some remarks about Cuntz's results on Bott periodicity. Let $\mathbf{T}_{\mathbb{C}}$ Toeplitz complex C^* -algebra generated by an isometry. Denote by $C_{\mathbb{C}}(S^1)$ the C^* -algebra of continuous complex functions on the standard unite cycle S^1 of the module one complex numbers. There is a short exact sequence

$$0 \rightarrow \mathcal{K}_{\mathbb{C}} \rightarrow \mathbf{T}_{\mathbb{C}} \xrightarrow{t} C_{\mathbb{C}}(S^1) \rightarrow 0.$$

The real case may be considered in the following way. There is a "Real" structure on $\mathbf{T}_{\mathbb{C}}$ defined by equality $\bar{v} = v$. Similarly, on the $C_{\mathbb{C}}(S^1)$ a "Real" structure is defined by the map $f(z) \mapsto \overline{f(\bar{z})}$. Denote by $C_{\mathbb{R}}(S^1)$ the real sub-algebra of fixed elements in $C_{\mathbb{C}}(S^1)$ relative the latter conjugation. Let $t : \mathbf{T}_{\mathbb{R}} \rightarrow C_{\mathbb{R}}(S^1)$ is given by a map $v \mapsto \text{id}_{S^1}$. One has a short exact sequence

$$0 \rightarrow \mathcal{K}_{\mathbb{R}} \rightarrow \mathbf{T}_{\mathbb{R}} \xrightarrow{t} C_{\mathbb{R}}(S^1) \rightarrow 0.$$

Note that natural projection $p : \mathbf{T}_{\mathbb{F}} \rightarrow \mathbb{F}$ defined by the map $v \mapsto 1$ splits by the map $j : \mathbb{F} \rightarrow \mathbf{T}_{\mathbb{F}}$ defined by $1 \mapsto 1$. Let $\mathbf{T}'_{\mathbb{F}}$ be the kernel of p . The following proposition is completely analogue of the Proposition 4.3. [3].

Proposition 3.1. *Let \mathbb{E} be a Cuntz-Bott homology. Then the homomorphisms $E_n(\text{id}_A \otimes j)$ and $E_n(\text{id}_A \otimes p)$ are isomorphisms between $E_n(A)$ and $E_n(A \otimes \mathbf{T}_{\mathbb{F}})$ for arbitrary $A \in S_G$ and $n \in \mathbf{Z}$. In particular, $E_n(A \otimes \mathbf{T}'_{\mathbb{F}}) = 0$.*

Proof. The sequence

$$0 \rightarrow A \otimes \mathbf{T}'_{\mathbb{F}} \rightarrow A \otimes \mathbf{T}_{\mathbb{F}} \rightarrow A \otimes \mathbb{F} \rightarrow 0$$

is split exact. Since functors E_n are split exact and stable, they are homotopy invariant by Higson's theorem. Then the proof literally coincides with the proof of Proposition 4.3. in [3]. \square

Now, let $\mathcal{U}_{\mathbb{F}}$ be a sub-algebra in $C_{\mathbb{F}(S^1)}$ making sequence

$$0 \rightarrow \mathcal{U}_{\mathbb{F}} \rightarrow C_{\mathbb{F}(S^1)} \rightarrow \mathbb{F} \rightarrow 0$$

exact. Then one has natural exact sequence

$$0 \rightarrow \mathcal{K}_{\mathbb{F}} \rightarrow \mathbf{T}'_{\mathbb{F}} \rightarrow \mathcal{U}_{\mathbb{F}} \rightarrow 0.$$

From the definition of $\mathcal{U}_{\mathbb{F}}$ it follows that latter algebra is nuclear C^* -algebra. This implies that epimorphism in the latter exact sequence has a completely positive and contractive section. Thus the epimorphism has completely positive and contractive section in the exact sequence

$$0 \rightarrow A \otimes \mathcal{K}_{\mathbb{F}} \rightarrow A \otimes \mathbf{T}'_{\mathbb{F}} \rightarrow A \otimes \mathcal{U}_{\mathbb{F}} \rightarrow 0,$$

by Lemma 1.3.4. in [7]. It implies, According to the Proposition 3.1, that $E_n(A)$ is naturally isomorphic to $E_{n+1}(A \otimes \mathcal{U}_{\mathbb{F}})$. Summarize this fact we have the following.

Theorem 3.2. (cf. Cuntz, [3]) *Let \mathbb{E} be a Cuntz-Bott homology theory. Then there is a natural isomorphism*

$$(3.1) \quad E_n(A) = E_{n+1}(A \otimes \mathcal{U}_{\mathbb{F}}).$$

From this useful theorem we deduce an elementary but applicable principle which we'll use in the sequel.

Corollary 3.3. *Let \mathbb{E} and \mathbb{E}' are Cuntz-Bott homology theories. If there exists natural isomorphism $\mu_m : E_m \rightarrow E'_m$ for some $m \in \mathbb{Z}$, then there exists natural isomorphism $\mu_n : E_n \rightarrow E'_n$ for all $n \in \mathbb{Z}$.*

Proof. Consider the following short exact sequence

$$0 \rightarrow \mathbf{F}(0; 1) \rightarrow \mathbf{F}(0; 1] \rightarrow \mathbf{F} \rightarrow 0$$

where \mathbf{F} is \mathbb{R} or \mathbb{C} the fields of real or complex numbers respectively. Since each algebra in the short exact sequence are nuclear C^* -algebras, according to Lemma 1.3.4. [7], one concludes that in the exact sequence

$$0 \rightarrow A \otimes \mathbf{F}(0; 1) \rightarrow A \otimes \mathbf{F}(0; 1] \rightarrow A \otimes \mathbf{F} \rightarrow 0$$

the epimorphism has a completely positive and contractive section, for any separable C^* -algebra A . From definition of Cuntz-Bott homology theory and Higson's homotopy invariant theorem immediately follows that any functor E_n, E'_n are homotopy invariant. From the latter short exact sequence it follows that there are natural isomorphisms

$$(3.2) \quad E_{n+1}(A) \simeq E_n(A \otimes \Omega_{\mathbb{F}}) \quad \text{and} \quad E'_{n+1}(A) \simeq E'_n(A \otimes \Omega_{\mathbb{F}}),$$

where $\Omega_{\mathbb{F}} = \mathbf{F}(0; 1)$. On the one hand, the formulas 3.2 guarantees natural isomorphism $E_n(A) \simeq E'_n(A)$ for $n \geq m$. On the other hand, Cuntz isomorphisms 3.1 guarantees natural isomorphism $E_n(A) \simeq E'_n(A)$ for $n \leq m$. \square

The definition of Cuntz-Bott cohomology theory is dual to homology theory case. Let $\mathbb{E} = \{E^n\}_{n \in \mathbb{Z}}$ be a Cuntz-Bott cohomology theory, We use the following identification $E^n = E_{-n}$ in sequel. Then Definition of Cuntz-Bott cohomology theory has the following form.

Let, as before, S_G denotes category of separable complex or real C^* -algebras with action of compact second countable group G and equivariant $*$ -homomorphisms. let $\mathbb{E} = \{E_n\}_n \in \mathbb{Z}$ be a set of contravariant functors from S_G into the category of abelian groups and homomorphisms, indexed by the integer numbers. One says that \mathbb{E} is Cuntz-Bott cohomology theory on the category \mathbb{E} if

- (1) \mathbb{E} has weak excision property. Namely, for any exact sequence $0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ of algebras from S_G , where epimorphism admits equivariant completely positive and contractive section. Then

i. there exist homomorphism $\delta_n : E_n(I) \rightarrow E_{n-1}(A)$, for any $n \in Z$, non-depending on a completely positive and contractive section of p , and natural by the following sense. let

$$\begin{array}{ccccccccc} 0 & \rightarrow & I & \rightarrow & B & \rightarrow & A & \rightarrow & 0 \\ & & \downarrow f_I & & \downarrow f_B & & \downarrow f_A & & \\ 0 & \rightarrow & I' & \rightarrow & B' & \rightarrow & A' & \rightarrow & 0 \end{array}$$

be a commutative diagram such that in the horizontal short exact sequences epimorphisms have completely positive and contractive sections. Then diagram

$$\begin{array}{ccc} E_n(I) & \xrightarrow{\delta} & E_{n-1}(A) \\ \downarrow E_n(f_I) & & \downarrow E_n(f_A) \\ E_n(I') & \xrightarrow{\delta'} & E_{n-1}(A') \end{array}$$

commutes.

ii. The natural two-sided sequence of abelian groups

$$\cdots \rightarrow E_n(A) \rightarrow E_n(B) \rightarrow E_n(I) \xrightarrow{\delta_n} E_{n-1}(A) \rightarrow \cdots$$

is exact.

(2) \mathbb{E} is stable. This means that if $e_A : A \rightarrow A \otimes \mathcal{K}$ is a homomorphism defined by a map $a \mapsto a \otimes p$, here p is a rank one projection in \mathcal{K} , then $E_n(e_A)$ is an isomorphism.

In this case one has the properties of cohomology theory like to the above properties of homology theory. We'll list them.

Theorem 3.4. *Let \mathbb{E} be a Cuntz-Bott cohomology theory. Then there are natural isomorphisms*

$$(3.3) \quad E_{n+1}(A) = E_n(A \otimes \mathcal{U}_{\mathbb{F}}) \quad \text{and} \quad E_{n-1}(A) = E_n(A \otimes \Omega_{\mathbb{F}}).$$

Corollary 3.5. *Let \mathbb{E} and \mathbb{E}' be Cuntz-Bott cohomology theories. If there exists natural isomorphism $\mu_m : E_m \rightarrow E'_m$ for some $m \in Z$, then there exists natural isomorphism $\mu_n : E_n \rightarrow E'_n$ for all $n \in Z$.*

4. ON THE ALGEBRAIC K -THEORY OF C^* -CATEGORIES

Before introducing our view on algebraic K -theory of C^* -categories, let us make some more comment on the results of A. Suslin and M. Wodzicki in algebraic K -theory. It is well known fact that any C^* -algebra has a right (left) bounded approximate unit. They have, by Proposition 10.2 of [19], the factorization property $(\mathbf{TF})_{\text{right}}$. Thus any C^* -algebra possesses property \mathbf{AH}_Z and according on Proposition 1.21 of [19], one concludes that C^* -algebras satisfy excision in algebraic K -theory. The results mentioned above leads to the anew definition of algebraic K -theory of C^* -algebras, which is flexible in the comparison of it with the topological K -theory of C^* -algebras.

Let A be a C^* -algebra and denote by A^+ the C^* -algebra obtained by adjoining a unit to A . If A is unital denote by $GL_n(A)$ the group of invertible elements in the C^* -algebra $M_n(A)$, and in the non-unital case define $GL_n(A)$ as the group $\ker(GL_n(A^+) \rightarrow GL_n(A))$. Since any C^* -algebra has a right (left) bounded approximate unit, it implies well known fact $A = A^2$. Thus, by Corollary 1.13 of [19] the group of elementary matrices $E(A)$ is a perfect normal subgroup of $GL(A)$ with an abelian quotient $GL(A)/E(A)$. So one can apply Quillen plus construction to the classifying space $BGL(A)$. The resulting space denote by $BGL(A)^+$.

The algebraic K -theory groups are defined by the following manner:

$$K_n^a(A) = \begin{cases} K_0(A) & \text{if } n = 0 \\ \pi_{n-1}(B^+(GL(A))) & \text{if } n \in \mathbb{N}. \end{cases}$$

and for negative n the group $K_n^a(A)$ (so called Bass K -groups, which sometimes will be denoted by $K_n^B(A)$) is defined so that the following sequence

$$K_{1-n}^a(A) \rightarrow K_{1-n}^a(A[x, x^{-1}]) \rightarrow K_n^a(A) \rightarrow 0$$

is exact.

Now, according to the results of [19], one has the following properties of algebraic K -theory of C^* -algebras.

- (1) K_i is a covariant functor from the category of C^* -algebras and their $*$ -homomorphisms into the category of abelian groups for any $i \geq 1$;
- (2) For every unital C^* -algebra R , which contains a C^* -algebra A as a two-sided ideal, the canonical map $K_*(A) \rightarrow K_*(R, A)$ an isomorphism;
- (3) The natural embedding in the left upper corner $A \hookrightarrow M_n(A)$ induces, for every natural n , an isomorphism $K_*(A) \simeq K_*M_n(A)$;
- (4) Any extension of C^* -algebras

$$0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$$

induces functorial and infinite two-sided long exact sequence of algebraic groups

$$(4.1) \quad \cdots \rightarrow K_{i+1}(A) \rightarrow K_i(I) \rightarrow K_i(B) \rightarrow K_A(I) \rightarrow \cdots \quad (i \in \mathbb{Z}).$$

Using the property (3) and Lemma 2.6.12 in [7], one gets the following property

- (*Invariance under inner automorphism*) Let A be a C^* -algebra and u be an unitary element in an unital C^* -algebra containing A as a closed two-sided ideal. Then the inner automorphism $ad(u) : A \rightarrow A$ induces identity map of K -groups.

4.1. Algebraic K -functors of C^* -categoroids . In this subsection we define algebraic K -theory of C^* -categoroids in the form which suites our purposes in the sequel.

Let J be a C^* -categoroid and A be an additive C^* -category containing J as a closed C^* -ideal. Let $\mathcal{L}(a) = \text{hom}_A(a, a)$ and $\mathcal{L}(a, J) = \text{hom}_J(a, a)$. The latter is a closed ideal in the C^* -algebra $\mathcal{L}(a)$. A morphism $v : a \rightarrow a'$ in A will be called *isometry* if $v^*v = 1_a$. Let us write $a \leq a'$ if there is an isometry $v : a \rightarrow a'$. The relation " $a \leq a'$ " makes the set of objects into direct system. Any isometry $v : a \rightarrow a'$ defines $*$ -homomorphisms of C^* -algebras

$$\text{Ad}(v) : \mathcal{L}(a) \rightarrow \mathcal{L}(a')$$

by the rule $x \mapsto vxv^*$. It maps $\mathcal{L}(a, J)$ into $\mathcal{L}(a', J)$.

Let $v_1 : a \rightarrow a'$ and $v_2 : a \rightarrow a'$ are two isometries. Then $\text{Ad}v_1$ and $\text{Ad}v_2$ induce the same homomorphisms

$$\text{Ad}_*v_1 = \text{Ad}_*v_2 : K_n^a(\mathcal{L}(a)) \rightarrow K_n^a(\mathcal{L}(a'))$$

and

$$\text{Ad}_*v_1 = \text{Ad}_*v_2 : K_n^a(\mathcal{L}(a, J)) \rightarrow K_n^a(\mathcal{L}(a', J)).$$

Indeed, let u be an unitary element in an unital C^* -algebra containing A as an ideal, then for the inner automorphism $ad(u) : A \rightarrow A$ the homomorphism $K_n^a(ad(u))$ is the identity map. Therefore, the maps

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$$

taking $\mathcal{L}(a')$ into $M_2(\mathcal{L}(a'))$, induces the same isomorphisms after using the functor K_n^a . So, it is enough to show that the maps

$$x \mapsto \begin{pmatrix} v_1xv_1^* & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & v_2xv_2^* \end{pmatrix},$$

which take $\mathcal{L}(a)$ into $M_2(\mathcal{L}(a'))$, induce by K_n^a the same map. Indeed, the second is obtained from the first conjugating by the unitary

$$\begin{pmatrix} 1 - v_1v_1^* & v_1v_2^* \\ v_2v_1^* & 1 - v_2v_2^* \end{pmatrix},$$

which is an element of $M_2(\mathcal{L}(a'))$.

This discussion shows that the homomorphism $\nu_*^{aa'} = K_n^a(\nu^{aa'})$ is not depended on choosing an isometry $\nu^{aa'} : a \rightarrow a'$. Therefore one has direct system $\{K_n^a(\mathcal{L}(a)), \nu_*^{aa'}\}_{a, a' \in \text{ob}A}$.

Definition 4.1. Let J be an additive C^* -categoroid and A be an additive C^* -category containing J as a closed C^* -ideal. Define

$$(4.2) \quad \mathbf{K}_n^a(A, J) = \varinjlim K_n^a(\mathcal{L}(a, J)).$$

Lemma 4.2. *Let J be a C^* -categoroid considered as a closed ideal in an additive C^* -category A . Then $\mathbf{K}_n^a(A, J) = \mathbf{K}_n^a(M(J), J)$. where $M(J)$ is the multiplier (additive) C^* -category of J .*

Proof. Since there exists natural $*$ -functor $G : A \rightarrow M(J)$ identity on J , the relation " $a \leq a'$ " in A implies " $a \leq a'$ " in $M(J)$. This means that there is natural morphism of direct systems

$$\{K_n^a(\mathcal{L}(a)), \nu_*^{aa'}\}_{a, a' \in \text{ob} A} \rightarrow \{K_n^a(\mathcal{L}(a)), \nu_*^{aa'}\}_{a, a' \in \text{ob} M(J)},$$

which is given by identity homomorphism on each $K_n^a(\mathcal{L}(a))$. This morphism is cofinal, since if " $a \leq a'$ " in $M(J)$ then " $a \leq a \oplus a'$ " in A and " $a' \leq a \oplus a'$ " in $M(J)$. \square

According to Lemma 4.2, one has the following.

Definition 4.3. Let J be an additive C^* -categoroid. Then by definition

$$\mathbf{K}_n^a(J) = \mathbf{K}_n^a(M(J), J).$$

Definition 4.4. Let A and B are C^* -categories. A $*$ -functor $G : A \rightarrow B$ is said to be cofinal if for of a for any object $b \in B$ there exists an object $a \in A$ and an isometry $s : b \rightarrow G(a)$. A C^* -category A is said to be a *cofinal* sub-category in B , if natural inclusion functor is cofinal.

The following lemma is trivial but useful in the next part of paper.

Lemma 4.5. *Let A' be a cofinal full sub- C^* -category of an additive C^* -category. Then $\mathbf{K}_n^a(A') = \mathbf{K}_n^a(A)$.*

Now, we prove excision property of algebraic K -theory on the category of C^* -categoroids, which plays major role in this paper.

Proposition 4.6. *Let A and B be additive C^* -categories and J be an ideal in B such that the sequence*

$$0 \rightarrow J \rightarrow B \rightarrow A \rightarrow 0$$

is exact. Then two-side sequence of algebraic K -groups

$$(4.3) \quad \begin{array}{ccccccc} \dots & \rightarrow & \mathbf{K}_n^a(A) & \rightarrow & \mathbf{K}_{n-1}^a(J) & \rightarrow & \mathbf{K}_{n-1}^a(B) & \rightarrow & \dots \\ & & \dots & \rightarrow & \mathbf{K}_0^a(A) & \rightarrow & \dots & & \dots & \rightarrow & \mathbf{K}_{-m}^a(A) & \rightarrow & \mathbf{K}_{-m-1}^a(J) & \rightarrow & \dots \end{array}$$

is exact $n, m \in \mathbb{N}$.

Proof. Consider exact sequence of C^* -algebras

$$0 \rightarrow \mathcal{L}(a, J) \rightarrow \mathcal{L}(a) \rightarrow \mathcal{L}(a)/\mathcal{L}(a, J) \rightarrow 0.$$

By excision property of algebraic K -theory of C^* -algebras [19], one has two sided long exact sequence

$$\dots \rightarrow K_n^a(\mathcal{L}(a)/\mathcal{L}(a, J)) \rightarrow K_{n-1}^a(\mathcal{L}(a, J)) \rightarrow K_{n-1}^a(\mathcal{L}(a)) \rightarrow K_{n-1}^a(\mathcal{L}(a)/\mathcal{L}(a, J)) \rightarrow \dots$$

Using lemma 4.2 and definitions 4.3 and 4.1, one can have exactness of (4.3), because direct limit preserves exactness. \square

4.2. Comparison of "+" and "Q" Variants of Algebraic K -theory of C^* -categories .

Let A be a C^* -algebra with unit. Denote by $F(A)$ the additive C^* -category which has standard Hilbert right A -modules $A^n = A \oplus_{n\text{-times}} \oplus A$ as the objects and usual A -homomorphisms (which has an adjoint) as the morphisms. Let $P(A)$ be the standard pseudo-abelian C^* -category of $F(A)$, i.e. $P(A) = P(F(A))$.

If an additive full sub-category A' in a pseudo-abelian category A' is cofinal subcategory then we'll say that A generates A' .

Let a be an object in a pseudo-abelian C^* -category A . Put $a^{\oplus n} = a \oplus \dots \oplus_{n\text{-times}} \dots \oplus a$, $n \in \mathbb{N}$. Let F_a be full additive subcategory of A which has $\{a^{\oplus n} | n = 1, \dots\}$ as the set of objects. Consider a full sub- C^* -category A_a consisting all such objects a' in A for which there exists an isometry $s : a' \rightarrow a^{\oplus n}$, where $a^{\oplus n} \in F_a$. It is clear, that A_a is a pseudo-abelian C^* -category, which is said to be *maximal pseudo-abelian sub- C^* -category* generated by an object $a \in A$.

Lemma 4.7. *Let A be an additive C^* -category and $J : A \rightarrow B$ be a $*$ -additive functor, where B is a pseudo-additive C^* -category. Then there is a functor $\mathbb{J} : P(A) \rightarrow B$ extending J . Moreover, if $J' : A \rightarrow B$ is an other functor isomorphic to J then \mathbb{J} is isomorphic to \mathbb{J}' , where latter functor is an extension of J' .*

Proof. Let a be an object in A and $p_a \in \mathcal{L}(a)$ a projection. Since B is a pseudo-abelian category, one can choose an object $[p_a]$ in B such that $[1_a] = J(a)$ and an isometry $s_{p_a} : [p_a] \rightarrow J(a)$ such that $ss^* = J(p_a)$. Define a functor $\mathbb{J} : P(A) \rightarrow B$ by the maps $(a, p) \mapsto [p_a]$ and $f \mapsto s_{p'_a}^* \cdot J(f) \cdot s_{p_a}$, where $f : (a, p_a) \rightarrow (a', p'_a)$ is a morphism in $P(A)$. Simple checking shows that \mathbb{J} satisfies the requirement of the lemma. To show the second part of the lemma, let $p \in \mathcal{L}(a)$ be a projection. Then there is an isometry $\{p\} : (a, p) \rightarrow (a, 1)$ induced by projection p . So, one has the isometries $s_p : [p] \rightarrow J(a)$ and $s'_p : [p] \rightarrow J'(a)$ such that $\mathbb{J}(a, p) = [p]$, $\mathbb{J}'(a, p) = [p]'$, and $\mathbb{J}(\{p\}) = s_p$ and $\mathbb{J}'(\{p\}) = s'_p$. It is easily to check that the collection $\{s'_p \cdot \tau_a \cdot s_p\}$ is the natural isomorphism of functors from \mathbb{J} to \mathbb{J}' , where $\{\tau_a\}$ is a natural isomorphism from J to J' . \square

Remark 4.8. Let A be an unital C^* -algebra. Then Lemma 4.7 implies that $P(F(A))$ (further it is also denoted by $P(A)$) is equivalent to the category $\mathcal{P}(A)$ of finitely generated projective right A -modules, where $F(A)$ is the additive C^* -category of standard finitely generated free right Hilbert A -modules. From now on, we consider the pseudo-abelian C^* -category $P(A)$ as the substitute of the category $\mathcal{P}(A)$.

Now, we give an interpretation of Quillens K -groups [17], [8] on which is based our calculations in the next part of paper.

Definition 4.9. Let A be a pseudo-abelian C^* -category. Under $K_n^Q(A)$, $n \geq 0$ we mean Quillen's K -groups relative to the family of split short exact sequences in A ; when A is unital C^* -algebra, by definition $K_n^Q(A) = K_n^Q(P(A))$, $n \geq 0$.

Let A and B be unital C^* -algebras and $\varphi : A \rightarrow B$ be a $*$ -homomorphisms (not unital in general). Then one has a $*$ -functor $P(\varphi) : P(A) \rightarrow P(B)$ defined by the maps $(A^n, p) \mapsto (B^n, \varphi^n(p))$ and $(f_{ij}) \mapsto (\varphi(f_{ij}))$ where (f_{ij}) is $n \times m$ -matrix which defines a A -homomorphism from A^n into A^m . Therefore we get a functor P from the category of unital C^* -algebras and $*$ -homomorphisms (non-unital in general) into category of pseudo-abelian C^* -categories and $*$ -functors.

4.2.1. *Construction.* Let A be a pseudo-abelian C^* -category and let $s : a' \rightarrow a$ be an isometry in A . There are a $*$ -homomorphism $f_s : \mathcal{L}(a') \rightarrow \mathcal{L}(a)$ defined by the map $\alpha \mapsto s\alpha s^*$ and the induced $*$ -functor

$$P(f_s) : P(\mathcal{L}(a')) \rightarrow P(\mathcal{L}(a)).$$

If $s_1 : a' \rightarrow a$ is an other isometry there exists natural isomorphism $v : P(f_s) \rightarrow P(f_{s_1})$ defined by the following way. For an object of form (a', p) in $P(\mathcal{L}(a'))$ let's define $v_{(a', p)} : (a, sps^*) \rightarrow (a, s_1ps_1^*)$ by equality $v_{(a', p)} = s_1ps^*$. Since

$$(sps_1^*)(s_1ps^*) = sps^* \quad \text{and} \quad (s_1ps^*)(sps_1^*) = s_1ps_1^*,$$

$v_{(a', p)}$ is an isomorphism in $P(\mathcal{L}(a))$. In general, for the objects of form $(a^{n'}, p)$ we define $v_{(a^{n'}, p)} = s_1^n ps^{n*}$. Since isomorphic additive functors induce the same homomorphism after using algebraic K -functor, one has

$$(4.4) \quad K_n^a(P(f_s)) = K_n^a(P(f_{s_1})).$$

Let a and a' be objects in A . we write $a' \leq a$ if there is an isometry $s : a' \rightarrow a$. The relation " $a' \leq a$ " makes the set of objects of a pseudo-abelian category A into a direct system $\{obA, \leq\}$. Therefore one has correctly defined direct system of abelian groups.

$$(4.5) \quad \Omega_1(A) = \{K_n^a(\mathcal{L}(a)), \kappa_{a'a}^n\}_{\{obA, \leq\}}$$

where $\kappa_{a'a}^n$ is the homomorphism $K_n^a(P(f_s))$ and by (4.4) it is not depended from the choosing of an isometry $s : a' \rightarrow a$.

4.2.2. *Construction* . Consider, now, a second direct system of abelian groups for A . Let, as above, A_a be the maximal pseudo-abelian sub- C^* -category generated by an object $a \in A$. It is evident that if there exists isometry $s : a' \rightarrow a$ then one has natural inclusion additive $*$ -functor (not depended on s) $i_{a'a} : A_{a'} \rightarrow A_a$ and thus we have the direct system $\{A_a, i_{a'a}\}_{(obA, \leq)}$ of the pseudo-abelian C^* -categories. Let $\mu_{a'a}^n = K_n(i_{a'a})$. Therefore we have the following direct system of abelian groups

$$(4.6) \quad \Omega_2(A) = \{K_n(A_a), \mu_{a'a}^n\}_{(obA, \leq)},$$

which is connected to the direct system $\Omega_1(A)$.

There is a natural isomorphism from the direct system $\Omega_1(A)$ into the direct system $\Omega_2(A)$. Indeed, consider natural $*$ -functor $\omega_a : F(\mathcal{L}(a)) \rightarrow A$ which is given by the maps $\mathcal{L}^n(a) \mapsto a^n$ and $(f_{ij}) \mapsto (f_{ij})$. Then by Lemma 4.7 one can choose $\omega'_a : P(\mathcal{L}(a)) \rightarrow A$, an extension of ω_a for every $a \in A$. Elementary checking shows that ω'_a is equivalence from $P(\mathcal{L}(a))$ onto A_a .

Proposition 4.10. *On the category of pseud-abelian C^* -categories and additive $*$ -functors the functors \mathbf{K}^a and \mathbf{K}^Q are naturally isomorphic.*

Proof. Let $s : a \rightarrow a'$ be an isometry. Then one has the isometries $s^n : a^n \rightarrow a'^n$ for any natural n , where $s^n = s \oplus \dots \oplus_{n\text{-times}} \dots \oplus s$. Define a functor $f_s : P(a) \rightarrow P(a')$ by the maps $(a^n, p) \mapsto (a'^n, s^n p s^{*n})$ and $l \mapsto s^m l s^{*m}$, where $l : (a^n, p) \rightarrow (a^m, q)$ is a morphism in $P(a)$. We assert that the following diagram

$$(4.7) \quad \begin{array}{ccc} P(\mathcal{L}(a')) & \xrightarrow{\omega_{a'}} & A_{a'} \\ f_s \uparrow & & \cup^{i_{a'a}} \\ P(\mathcal{L}(a)) & \xrightarrow{\omega_a} & A_a \end{array}$$

is commutative up to isomorphism of functors, i.e. $\omega_{a'} \cdot f_s \approx i_{a'a} \cdot \omega_a$. According to Lemma 4.7, it is enough to construct an isomorphism

$$g : \omega_{a'}(f_s(\mathcal{L}(a))) \rightarrow i_{a'a}(\omega_a(\mathcal{L}(a))).$$

But $\omega'_{a'}(f_s(\mathcal{L}(a))) = \omega'_{a'}(a', ss^*)$ and $i_{a'a}(\omega_a(\mathcal{L}(a))) = a$. Note that in the definition of ω' the object $\omega'((a', ss^*) = [ss^*])$ is taken such that there exists isometry $s_1 : [ss^*] \rightarrow a'$ that $s_1^* s_1 = 1_{ss^*}$ and $s_1 s_1^* = ss^*$. Now, by definition $g = s^* s_1$ which, of course, is an isomorphism from $[ss^*]$ into a . Therefore the latter diagram is commutative up to isomorphism of functors. Now, apply K^Q -functor. Since isomorphic functors have the same K -value, the diagram

$$(4.8) \quad \begin{array}{ccc} K_n^Q(\mathcal{L}(a)) & \xrightarrow{\zeta_n^a} & K_n^Q(A_a) \\ v_{a'a}^n \downarrow & & \downarrow \mu_{a'a}^n \\ K_n^Q(\mathcal{L}(a')) & \xrightarrow{\zeta_n^{a'}} & K_n^Q(A_{a'}) \end{array}$$

is commutative, where $\zeta_n^a = K_n^Q(\omega_a)$. Thus we give the natural isomorphism of direct systems $\{\zeta_n^a\} : \Omega_1 \rightarrow \Omega_2$. It is clear, that algebraic (C^* -algebraic) direct limit of the direct system Ω_2 is the category A . Since algebraic K -functors commute with direct limits one has

$$(4.9) \quad \varinjlim_a K_n^Q(P(\mathcal{L}(a))) \approx K_n^Q(A).$$

But $K_n^Q(P(\mathcal{L}(a)))$ is naturally isomorphic to $K^a(\mathcal{L}(a))$. Therefore \mathbf{K}^a is isomorphic to \mathbf{K}^Q . \square

Remark 4.11. Let A be a pseudo-abelian C^* category. Then, for $n \geq 0$, the groups $\mathbf{K}_n^a(A)$, by Proposition 4.10, is exactly $K_n^Q(A)$. For $n < 0$, the groups $\mathbf{K}_n^a(A)$ can be considered as the generalization of Bass groups, and in this case this K -groups sometimes will be denoted by $\mathbf{K}_n^B(A)$.

5. ON THE TOPOLOGICAL K -THEORY OF C^* -CATEGOROIDS

Let A be a C^* -algebra and A^+ be the C^* -algebra obtained by adjoining a unit to A . If A is unital denote by $GL_n(A)$ the topological group of invertible elements in the C^* -algebra $M_n(A)$, and in the non-unital case define $GL_n(A)$ as the topological subgroup

$$\ker(GL_n(A^+) \rightarrow GL_n(A)).$$

The topological K -theory groups are defined in the following way:

$$K_n^t(A) = \begin{cases} K_0(A) & \text{if } n = 0 \\ \pi_{n-1}(GL(A)) & \text{if } n \in \mathbb{N}. \end{cases}$$

The following properties of topological K -groups.

1. If homomorphisms $f, g : A \rightarrow B$ are homotopic then induced homomorphisms $K_0^t(f)$ and $K_0^t(g)$ are equal.

2. if $0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ is an exact sequence of C^* -algebras then the following sequence abelian groups

$$\cdots \rightarrow K_n^t(I) \rightarrow K_n^t(B) \rightarrow K_n^t(A) \xrightarrow{\delta} K_n^t(I) \rightarrow \cdots \rightarrow K_0^t(I) \rightarrow K_0^t(B) \rightarrow K_0^t(A)$$

is exact.

3. Let $A \rightarrow A \otimes \mathcal{K}$ be a homomorphism, defined, for any C^* -algebra A , by the map $a \mapsto a \otimes p$, where $p \in \mathcal{K}$ is rank one projection. Then induced homomorphism is an isomorphism $K_n^t(A) \simeq K_n^t(A \otimes \mathcal{K})$.

4. Let $\{A_i; f_{ij}\}_I$ be a direct system of C^* -algebras and $\{A; f_i\}$ is the direct limit. Then natural homomorphism

$$\varinjlim K_n^t(f_i) : \varinjlim K_n^t(A_i) \rightarrow K_n^t(A)$$

is an isomorphism for any C^* -algebra A and natural n .

5. By Cuntz-Bott periodicity theorem [3], there are the natural isomorphisms

$$K_n^t(A) = \begin{cases} K_n^t(A \otimes C_0(\mathbb{R}) \otimes C_0(\mathbb{R})) & \text{in complex case,} \\ K_n^t(A \otimes C_0(\mathbb{R}) \otimes C_0^R(i\mathbb{R})) & \text{in real case.} \end{cases}$$

Note that so defined functors has period 2 in the complex C^* -algebras case, and period 8 in the real C^* -algebra case.

From the latter property follows that for negative integers K -groups may be defined by the formulas

- $K_{-n}^t(A) = K_0^t(A \otimes C_0(\mathbb{R})^{\otimes n})$ for the complex case;
- $K_{-n}^t(A) = K_0^t(A \otimes C_0^R(i\mathbb{R})^{\otimes n})$ for the real case

where $C_0^R(i\mathbb{R})$ is Cuntz's algebra defined in [3].

From the property 3 and Lemma 2.6.12 in [7] immediately follows that topological (as well as algebraic) K -theory has the following property.

- (*Invariance under inner automorphism*) Let A be a C^* -algebra and u be an unitary element in a unital C^* -algebra containing A as a closed two-sided ideal. Then the inner automorphism $ad(u) : A \rightarrow A$ induces identity map of topological K -groups.

Now, we remark that in the subsection 4.1 one can replaces algebraic K -groups by topological K -groups then all the results are true. This is possible since invariance under inner automorphism of algebraic K -theory was used and the same property has topological K -theory too. So we have the following definitions and properties of topological K -theory of C^* -categories.

Let A be an additive C^* -categoroid and $M(A)$ be the additive C^* -category centralizers. Let $\mathcal{L}(a) = \text{hom}_{M(A)}(a, a)$ and $A(a) = \text{hom}_A(a, a)$. The latter is a closed ideal in the C^* -algebra $\mathcal{L}(a)$. Any isometry $v : a \rightarrow a'$ in $M(A)$ defines $*$ -homomorphisms of C^* -algebras

$$\text{Ad}(v) : A(a) \rightarrow A(a')$$

by the rule $x \mapsto vxv^*$. Thus one has an induced homomorphism

$$\text{Ad}_n(v) : K_n^t(A(a)) \rightarrow K_n^t(A(a'))$$

which isn't depended on choosing of isometry $v : a \rightarrow a'$. Denote this homomorphism by $\tau_{aa'}$. We have the direct system of abelian groups $\{K_n^t(A(a)); \tau_{aa'}\}_{a \in A}$

Definition 5.1. By definition

$$(5.1) \quad \mathbf{K}_n^t(A) = \varinjlim K_n^t(A(a)).$$

One can easily prove the following.

Lemma 5.2. *Let A be an additive C^* -category and A' be a cofinal additive C^* -subcategory. Then the canonical additive functor $A' \subset A$ induces an isomorphism*

$$\mathbf{K}_n^t(A') \approx \mathbf{K}_n^t(A).$$

In particular, $\mathbf{K}_n^t(A) = \mathbf{K}_n^t(\mathbf{P}(A))$.

The given bellow is excision property of topological K -theory on the category of C^* -categoroids.

Proposition 5.3. *Let A and B be additive C^* -categoroids and J be an ideal in B such that the sequence*

$$0 \rightarrow J \rightarrow B \rightarrow A \rightarrow 0$$

is exact. Then following two-side sequence of topological K -groups

$$(5.2) \quad \begin{array}{ccccccc} \dots & \rightarrow & \mathbf{K}_n^t(A) & \rightarrow & \mathbf{K}_{n-1}^t(J) & \rightarrow & \mathbf{K}_{n-1}^t(B) & \rightarrow & \dots \\ & & & & \dots & \rightarrow & \mathbf{K}_0^t(A) & \rightarrow & \dots & & \dots & \rightarrow & \mathbf{K}_{-m}^t(A) & \rightarrow & \mathbf{K}_{-m-1}^t(J) & \rightarrow & \dots \end{array}$$

is exact $n, m \in \mathbb{N}$.

Now, we'll give an interpretation of Karoubi's K -groups [12], [13] by the functors K^t . The methods, used in the subsection 4.2, may be applied.

Since C^* -category A is algebraic limit of direct system of C^* -categories $\{A_a, i_{aa'}\}$, $a \in A$, it is C^* -algebraic direct limit too. From the construction of C^* -algebraic direct limit and its property (see subsection 1.5) implies that the natural homomorphism

$$\varinjlim(i_{aa'}) : \varinjlim K^{-n}(A_a) \rightarrow K^{-n}(A)$$

is an isomorphism Karoubi's K -groups.

By analogy with Subsection 4.2, one can easily proof that K_n^t naturally isomorphic to Karoubi's K^{-n} , where $n = 0, 1, \dots$

5.1. Karoubi's Topological K -theory of C^* -categories. The purpose of this subsection is to transform some main results of K -theory of Banach categories, introduced by M. Karoubi in [12], [13], to C^* -categories.

The group $K^0(A)$ of an additive C^* -category is the Grothendieck group of the abelian monoid of unitary isomorphism classes of objects of A . Note that this definition coincides with usual definition because in a C^* -category, objects are isomorphic if and only if they are unitarily isomorphic. Indeed, if $u : E \rightarrow F$ is isomorphism then $u_0 = u\sqrt{(u^*u)^{-1}}$ is a unitary isomorphism.

Let A be an additive C^* -category. The canonical functor induces an isomorphism $i_* : K^0(\tilde{A}) \rightarrow \mathbb{K}^0(\xi A)$, where the left-hand K -group is the same as in the definition above, and the right one as in [12], [13].

Now we'll give discussion analogous questions for the K^{-1} group (cf. [12], [13]). Let A be an additive C^* -category. Consider the set of pairs (E, α) , where $E \in A$ and $\alpha \in \text{hom}(E, E)$ is a unitary automorphism.

a). The pairs (E, α) and (E', α') are said to be *unitarily isomorphic* if there exists a unitary isomorphism $u : E \rightarrow E'$ such that diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow \alpha & & \downarrow \alpha' \\ E & \xrightarrow{u} & E' \end{array}$$

is commutative.

b). The pairs (E, α) and (E, α') are said to be *homotopic* if α and α' are homotopic in $\text{Aut } E$.

c). A pair (E, α) is said to be *elementary* if it is homotopic to $(E, 1_E)$.

d). The sum is defined by the formula

$$(E, \alpha) \oplus (E', \alpha') = (E \oplus E', \alpha \oplus \alpha').$$

e). The pairs (E, α) and (E', α') are said to be *stably isomorphic* if there exist elementary pairs $(\bar{E}, \bar{\alpha})$ and $(\hat{E}, \hat{\alpha})$, and a unitary isomorphism

$$(E, \alpha) \oplus (\bar{E}, \bar{\alpha}) \simeq (E', \alpha') \oplus (\hat{E}, \hat{\alpha}).$$

f). The abelian monoid $K^{-1}(A)$ is defined as the monoid of classes of stably isomorphic pairs. Denote by $d(E, \alpha)$ the class of (E, α) in $K^{-1}(A)$.

There are the following relations in $K^{-1}(A)$:

- a) $d(E, \alpha) + d(E, \alpha^*) = 0$;
- b) If α and α' are homotopic unitary isomorphisms, then $d(E, \alpha) = d(E, \alpha')$.
- c) $d(E, \alpha) + d(E, \beta) = d(E, \beta\alpha)$;

In particular, $K^{-1}(A)$ is an abelian group.

The next proposition is analogous to the corresponding property of $K^0(A)$.

Proposition 5.4. *Let A be an additive C^* -category. The canonical homomorphism*

$$i_* : K^{-1}(A) \rightarrow \mathbb{K}^{-1}(A),$$

defined by $d(E, \alpha) \mapsto d(E, \alpha)$ is an isomorphism. Here $\mathbb{K}^{-1}(A)$ is Karoubi's group.

Proof. i_* is an epimorphism: Let (E, α) be a pair with α an isomorphism. Consider the unitary isomorphism $\bar{\alpha} = \alpha\sqrt{\alpha^*\alpha}^{-1}$. It is homotopic to α , because $\alpha^*\alpha$ is homotopic to 1_E . We get that $d(E, \alpha) = d(E, \bar{\alpha})$. i is a monomorphism: If $i(d(E, \alpha)) = 0$, then there exists elementary (E', e') such that $(E \oplus E', \alpha \oplus e')$ is elementary. Then $(E \oplus E', \overline{\alpha \oplus e'})$ is also elementary, that is $(E \oplus E', \alpha \oplus \bar{e}')$ elementary. This means $d(E, \alpha) = 0$. \square

Thus the properties of $K^{-1}(A)$ are inherited from the corresponding properties of $\mathbb{K}^{-1}(A)$. In particular, we get the following:

Theorem 5.5. *Let A be an additive C^* -category, \tilde{A} be the associated pseudoabelian C^* -category and $i : A \rightarrow \tilde{A}$ the canonical additive $*$ -functor. Then the induced homomorphism*

$$(5.3) \quad i_* : K^{-1}(A) \rightarrow K^{-1}(\tilde{A})$$

is isomorphism.

Let A and B be additive C^* -categories and $\mathcal{F} : A \rightarrow B$ be an additive $*$ -functor. Denote by $\Gamma(\mathcal{F})$ the set of triples (E, F, α) , where E and F are objects in A , and $\alpha : \mathcal{F}(E) \rightarrow \mathcal{F}(F)$ is a unitary isomorphism in B .

a) Two triples (E, F, α) and (E', F', α') are *unitarily isomorphic* if there exist unitary isomorphisms $f : E \rightarrow E'$ and $g : F \rightarrow F'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ \downarrow f & & \downarrow g \\ E' & \xrightarrow{\alpha'} & F' \end{array}$$

is commutative.

b). Two triples (E, F, α) and (E, F, α') are called *homotopic* if α and α' are homotopic in the subspace of unitary isomorphisms in $\text{hom}(E, F)$.

c). The triple $(E, E, 1_E)$ is called trivial. A triple (E, F, α) is said to be *elementary* if this triple is homotopic to the trivial triple.

e). The sum of triples is defined by the formula $(E, F, \alpha) \oplus (E', F', \alpha') = (E \oplus E', F \oplus F', \alpha \oplus \alpha')$.

f). Two triples $\sigma = (E, F, \alpha)$ and $\sigma' = (E', F', \alpha')$ are *stably unitarily isomorphic* if there exist elementary pairs $\tau = (\bar{E}, \bar{E}, \bar{\alpha})$ and $\tau' = (\bar{E}', \bar{E}', \bar{\alpha}')$ such that $\sigma \oplus \tau$ and $\sigma' \oplus \tau'$ are unitarily isomorphic.

The set $K(\mathcal{F})$ of stably isomorphic triples is an abelian monoid with respect to the sum of triples. Denote by $d(E, F, \alpha)$ the class of (E, F, α) in $K(\mathcal{F})$. The monoid $K(\mathcal{F})$ is an abelian group. Moreover $d(E, F, \alpha) + d(F, E, \alpha^*) = 0$. Note that $d(E, F, \alpha) + d(F, E, \alpha^*) = d(E \oplus F, F \oplus E, \alpha \oplus \alpha^*)$ The last triple is isomorphic to $(E \oplus F, \beta)$, where

$$\beta = \begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}$$

which is homotopic to $1_{\mathcal{F}(E) \oplus \mathcal{F}(F)}$ by $u(t) = \sigma(t)\sqrt{\sigma^*(t)\sigma(t)}$, where

$$\sigma(t) = \begin{pmatrix} 1 & -t\alpha^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^* \\ 0 & 1 \end{pmatrix}$$

The following theorem compares our definition of $K(\mathcal{F})$ with the corresponding one of Karoubi.

Theorem 5.6. *The canonical homomorphism $i : K(\mathcal{F}) \rightarrow \mathbb{K}(\mathcal{F})$ defined by*

$$d(E, F, \alpha) \mapsto d(E, F, \alpha)$$

is an isomorphism.

Proof. Let (E, F, α) be a triple which defines an element in $\mathbb{K}(\mathcal{F})$, where α is an isomorphism (but not unitary isomorphism). Let $\bar{\alpha} = \alpha\sqrt{\alpha^*\alpha}$. $\bar{\alpha}$ is unitary and is homotopic to α because $\alpha^*\alpha$ is homotopic to $1_{\mathcal{F}(E)}$. This proves that i is an epimorphism. Now, let $d(E, F, \alpha) \in K(\mathcal{F})$ defines 0 in $\mathbb{K}(\mathcal{F})$. This means, by [13], that there exist objects G and H and isomorphisms (but after polar decomposition we may suppose they are unitary isomorphisms) $u : E \oplus G \rightarrow H$ and $v : F \oplus G \rightarrow H$ that $\mathcal{F}(v)(\alpha \oplus 1_{\mathcal{F}(G)})\mathcal{F}(u^*)$ is homotopic to $1_{\mathcal{F}(H)}$ (see [13]) by a homotopy $h(t)$. Then $\bar{h}(t) = h(t)\sqrt{(h^*(t)h(t))^{-1}}$ gives homotopy between $(E, F, \alpha) \oplus (G, G, 1_G)$ and $(H, H, 1_H)$. This means $d(E, F, \alpha) = 0$ in $K(\mathcal{F})$. \square

This theorem shows that all properties of $K(\mathcal{F})$ inherited from the corresponding properties of $\mathbb{K}(\mathcal{F})$. In particular, we'll get the following results. (Cf. [12], [13].) There are the following relations in $K(\mathcal{F})$:

- a) If α and α' are homotopic, then $d(E, F, \alpha) = d(E, F, \alpha')$;
- b) $d(E, F, \alpha) + d(F, G, \beta) = d(E, G, \beta\alpha)$.

Let $\mathcal{F} : A \rightarrow B$ be a Serre quasi-surjective additive $*$ -functor. Then

- a) if in the definition of $K(\mathcal{F})$ we replace elementary triples by trivial triples we get the same group.
- b) $d(E, F, \alpha) = 0$ iff there exist an object G from A and unitary isomorphism $\beta : E \oplus G \rightarrow F \oplus G$ such that $\mathcal{F}(\beta) = \alpha \oplus 1_{\mathcal{F}(G)}$.

Proposition 5.7. *Let $\mathcal{F} : A \rightarrow B$ be a quasi-surjective additive $*$ -functor. Then the sequence of abelian groups*

$$(5.4) \quad K^{-1}(A) \xrightarrow{f_1} K^{-1}(B) \xrightarrow{\partial} K^0(\mathcal{F}) \xrightarrow{i} K^0(A) \xrightarrow{\partial} K^0(B)$$

is exact, where $i(d(E, F, \alpha)) = d(E) - d(F)$ (for the definition of ∂ see [13]). In addition, if there exists a functor $\Psi : B \rightarrow A$ such that $\mathcal{F} \cdot \Psi \simeq Id_B$, then there exists a split exact sequence

$$(5.5) \quad 0 \rightarrow K^0(\mathcal{F}) \xrightarrow{i} K^0(A) \xrightarrow{j} K^0(B) \rightarrow 0.$$

Now, we'll discuss two examples, which we'll need in the sequel.

1) Recall that an object of $\text{rep}(A, B)$ has the form (E, ϕ) , where E is a right Hilbert B -module with action of compact group and $\phi : A \rightarrow \mathcal{L}(E)$ is supposed equivariant. A morphism from (E, ϕ) to (E', ϕ') is by definition an invariant B -homomorphism $f : E \rightarrow E'$ such that $f\phi(a) = \phi'(a)f$. Note that $\text{rep}(A, B)$ is a pseudoabelian C^* -category. To show that $K^i(\text{rep}(A, B)) = 0$ for all $i \in \mathbb{Z}_2$, consider the ∞ -structure of $\text{rep}(A, B)$ $E^\infty = E \oplus E \oplus \dots$, $\alpha^\infty = \alpha \oplus \alpha \oplus \dots$, and $\phi^\infty(a) = (\phi(a))^\infty$. Let

$$\infty : \text{rep}(A, B) \rightarrow \text{rep}(A, B)$$

be the $*$ -functor defined by the formula $\infty(E) = E^\infty$, $\infty(\phi) = \phi^\infty$, and if α is a morphism in $\text{rep}(A, B)$, then $\infty(\alpha) = \alpha^\infty$. There exists a natural isomorphism $id_{\text{rep}(A, B)} \oplus \infty \simeq \infty$. From this it follows that the groups $K^i(\text{rep}(A, B))$ of classes of isomorphic objects of $\text{rep}(A, B)$ have an automorphism I with property that

$$id_{K^i(\text{rep}(A, B))} + I = I.$$

From this fact it follows that $K^i(\text{rep}(A, B)) = 0$.

2) Consider the canonical quasi-surjective functor

$$\Theta_{A, B} : \text{rep}(A, B) \rightarrow \text{Cal}(A, B).$$

Applying the exact sequence (5.4) of K -groups and result of example 1, one gets that the canonical homomorphism

$$(5.6) \quad \partial : K^{-1}(\text{Cal}(A, B)) \rightarrow K^0(\Theta_{A, B})$$

is an isomorphism.

6. WEAK EXCISION

In this section we'll show that the contravariant functors

$$\mathbf{K}_n^a(\text{Rep}(-; B)) \quad \text{and} \quad \mathbf{K}_n^t(\text{Rep}(-; B))$$

have weak excision property for all $n \in \mathbb{Z}$.

At first one needs following proposition, which easily comes from the proposition 4.6.

Proposition 6.1. *Let J be a G -invariant C^* -ideal in a separable $G - C^*$ -algebra A , and let*

$$0 \rightarrow D(A, J; B) \rightarrow \text{Rep}(A, B) \rightarrow \text{Rep}(A, B)/D(A, J; B) \rightarrow 0$$

be induced exact sequence of C^ -categoroids. Then for the algebraic K -theory one has the following exact sequences*

$$(6.1) \quad \dots \rightarrow K_n^a(D(A, J; B)) \rightarrow K_n^a(\text{Rep}(A, B)) \rightarrow K_n^a(\text{Rep}(A, B)/D(A, J; B)) \rightarrow K_{n-1}^a(D(A, J; B)) \rightarrow \dots,$$

respectively for topological K -groups

$$(6.2) \quad \dots \rightarrow K_n^t(D(A, J; B)) \rightarrow K_n^t(\text{Rep}(A, B)) \rightarrow K_n^t(\text{Rep}(A, B)/D(A, J; B)) \rightarrow K_{n+1}^t(D(A, J; B)) \rightarrow \dots$$

Let $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ be exact sequence of C^* -algebras such that epimorphism has an equivariant completely positive and contractive section. The solution of the problem comes from the showing of

- (1) $\mathbf{K}_n^a(\text{Rep}(A; B)/D(A, J; B)) \simeq \mathbf{K}_n^a(\text{Rep}(J; B))$
- (2) $\mathbf{K}_n^a(\text{Rep}(A/J; B)) \simeq \mathbf{K}_n^a(D(A, J; B))$

6.1. The Isomorphism $\mathbf{K}_n^a(\text{Rep}(A; B)/D(A, J; B)) \approx \mathbf{K}_n^a(\text{Rep}(J; B))$. Let (E, ϕ) be an object in $\text{Rep}(A, B)$ and $j : J \rightarrow A$ natural equivariant inclusion. There is the canonical $*$ -functor, induced by the natural inclusion j

$$(6.3) \quad j : \text{Rep}(A; B) \rightarrow \text{Rep}(J; B)$$

defined by maps $(E, \phi) \mapsto (E, \phi j)$ and $x \mapsto x$.

Proposition 6.2. *The canonical $*$ -functor 6.3 maps $D(A, J; B)$ into $D(J, J; B)$ and the induced $*$ -functor*

$$(6.4) \quad \xi : \text{Rep}(A; B)/D(A, J; B) \rightarrow \text{Rep}(J; B)/D(J, J; B)$$

is an isomorphism of C^ -categories.*

Proof. (cf. [5]) By lemma 1.3 it is enough to show that for any object (E, ϕ) the $*$ -homomorphism C^* -algebras

$$\xi_{J, \phi} : D_\phi(A; E; B)/D_\phi(A, J, E; B) \rightarrow D_{\phi \cdot j}(J, E; B)/D_{\phi \cdot j}(J, J, E; B)$$

is an $*$ -isomorphism. It is easy to show that $\xi_{J, \phi}$ is a monomorphism. To show that $\xi_{J, \phi}$ is an epimorphism, let $x \in D_{\phi \cdot j}(J, E; B)$ and let E_1 be a $G - C^*$ -algebra in $\mathcal{L}(E)$ generated by $\phi(J) \cup \mathcal{K}(E)$; E_2 be the separable $G - C^*$ -algebra generated by all elements of form $[x, \phi(y)]$, $y \in J$; and \mathcal{F} be the G -invariant separable linear space generated by x and $\phi(A)$. One has

- $E_1 \cdot E_2 \subset \mathcal{K}(E)$, because $\phi(b)[\phi(a), x] \sim [\phi(ba), x] \in \mathcal{K}(E)$, $a \in A$, $b \in J$,
- $[\mathcal{F}, E_1] \subset E_1$, because $[x, \phi(J)] \subset \mathcal{K}(E)$ and $[\phi(A), \phi(J)] \subset \phi(J)$.

From the Kasparov technical theorem follows that there exists positive G -invariant operator X such that

- (1) $X \cdot \phi(J) \subset \mathcal{K}(E)$;
- (2) $(1 - X) \cdot [\phi(A), x] \subset \mathcal{K}(E)$;
- (3) $[x, X] \in \mathcal{K}(E)$.

Since $[(1 - X)x, \phi(a)] = (1 - X)[x, \phi(a)] - [X, \phi(a)]x$, it follows from (2) and (3) that $(1 - X)x \in D_\phi(A, E; B)$. In addition, it follows from (2) that $Xx \in D_{\phi \cdot j}(J, J, E; B)$, and so that the image of $(1 - X)x$ in $D_{\phi \cdot j}(J, E; B)/D_{\phi \cdot j}(J, J, E; B)$ coincides with the image of x . \square

Now, we prove the following.

Theorem 6.3. *Let A be a separable $G - C^*$ -algebra and B be a σ -unital $G - C^*$ -algebra. Let J be a closed ideal in A . There exists the essential isomorphism*

$$(6.5) \quad \mathbf{K}_n^a(\text{Rep}(A, B)/D(A, J; B)) \approx \mathbf{K}_n^a(\text{Rep}(J, B))$$

Proof. It follows from the Proposition 6.2 that

$$\mathbf{K}_n^a(\text{Rep}(A; B)/D(A, J; B)) \approx \mathbf{K}_n^a(\text{Rep}(J; B)/D(J, J; B)).$$

Thus it is enough to show that the natural homomorphism

$$\mathbf{K}_*^a(\text{Rep}(J; B)) \rightarrow \mathbf{K}_*^a(\text{Rep}(J; B)/D(J, J; B))$$

is an isomorphism. From the exact sequence 6.1 it follows that it is enough to show that $K_*(D(J, J; B)) = 0$. By Lemma 4.5, we can use the cofinal subcategory $\text{Rep}_{H_B^G}(J; B)$, with the objects of form (H_B^G, φ) , where H_B^G is Kasparov's universal Hilbert B -module [14]. Note that the canonical isometry

$$i_1^{H_B} : H_B^G \rightarrow H_B^G \oplus H_B^G$$

in the first summand is in $D_{\phi, \phi \oplus 0}(J; H_B^G, H_B^G \oplus H_B^G; B)$ and induces inner homomorphism

$$\text{ad}(i_1^{H_B^G}) : D_{\phi}(J, J; H_B^G; B) \rightarrow D_{\phi \oplus 0}(J, J; H_B^G \oplus H_B^G; B).$$

Consider a sequence of $*$ -homomorphisms

$$(6.6) \quad D_{\phi}(J, J; H_B^G; B) \rightarrow D_{\phi \oplus \phi}(J, J; H_B^G \oplus H_B^G; B) \subset D_{\phi \oplus 0}(J, J; H_B^G \oplus H_B^G; B)$$

where the inclusion is given by the map $x \mapsto x$. If the first arrow is induced by the inclusion $\iota_1 : H_B^G \rightarrow H_B^G \oplus H_B^G$ in the first summand, then the composition is $\text{ad}(i_1^{H_B^G})$. If the first arrow is induced by the inclusion $\iota_2 : H_B \rightarrow H_B \oplus H_B$ in the first summand, one gives a homomorphism $\lambda^{H_B^G}$. On the other hand, the homomorphism $\lambda^{H_B^G}$ is the composition of the natural $*$ -homomorphisms of C^* -algebras

$$D_{\phi}(J, J; H_B^G; B) \rightarrow D_0(J, J; H_B^G; B) \rightarrow D_{\phi \oplus 0}(J, J; H_B^G \oplus H_B^G; B),$$

given by the maps

$$x \mapsto x \text{ and } x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}.$$

Remark that $D_0(J, J; H_B^G; B) = M(J \otimes \mathcal{K}_G)$. It is well known that the latter algebra has trivial algebraic (as well as topological) K -theory groups. If we apply K -functors then corresponding homomorphism of $\lambda^{H_B^G}$, is zero homomorphisms. Now, let $x \in K_n^a(D_{\phi}(J, J; H_B; B))$ represents an element in $\mathbf{K}_n^a(D(J, J; B))$. Then x and $\mathbf{K}_n^a(\text{ad}(i_1^{H_B^G})(x))$ represent the same element. Since

$$\mathbf{K}_n^a(\text{ad}(i_1^{H_B^G})) = \mathbf{K}_n^a(\lambda^{H_B^G}),$$

the element represented by x must be zero. Therefore $\mathbf{K}_*(D(J, J; B)) = 0$. \square

6.2. The Isomorphism $\Gamma_n : \mathbf{K}_n^a(\text{Rep}(A/J; B)) \rightarrow \mathbf{K}_n^a(D(A, J; B))$. Let A and J be as in the last subsection. Let $p : A \rightarrow A/J$ be the canonical homomorphism which admits an equivariant completely positive and contractive section $s : A/J \rightarrow A$. Let (E, ϕ) be an object in $\text{Rep}(A; B)$ i.e. there is given an equivariant $*$ -homomorphism $\phi : A \rightarrow \mathcal{L}(E)$. A $*$ -homomorphism

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} : A/J \rightarrow \mathcal{L}(E \oplus E')$$

will be called $G - s$ -dilation for ϕ if $\psi_{11}(a) = \phi(s(a))$, where E' is a right Hilbert B -module.

According to generalized Stinespring's theorem there exists a right Hilbert B -module E' and a $G - s$ -dilation

$$\psi = \begin{pmatrix} \phi \cdot s & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} : A/J \rightarrow \mathcal{L}(E \oplus E')$$

for any completely positive and contractive section $s : A/J \rightarrow A$ [14].

Lemma 6.4. *Let ψ be a s -dilation for ϕ . Then*

- (1) $\psi_{12}(a^*) = \psi_{21}(a)^*$;
- (2) for any $a, b \in A$ there exists $j \in J$ such that $\psi_{12}(a)\psi_{21}(b) = \phi(j)$.
- (3) $\psi_{12}(a)x$ and $x\psi_{21}(a)$ are compact morphisms for any $a \in A$ and $x \in D_\phi(A, J; B)$.

Proof. The case (1) is trivial, because ψ is a $*$ -homomorphism. The case (2). Since ψ is a $*$ -homomorphism, $\phi \cdot (s(ab) - s(a) \cdot s(b)) = \psi_{12}(a) \cdot \psi_{21}(b)$. But $j = s(ab) - s(a) \cdot s(b) \in J$. Therefore $\psi_{12}(a) \cdot \psi_{21}(b) = \phi(j)$. The case (3). If $x \in D_\phi(A, J; B)$ then $x\phi(j)$ and $\phi(j)x$ are compact morphisms for any $j \in J$. Then $x\psi_{12}(a) \cdot \psi_{21}(a)x^* = x\phi(j')x^*$ for some $j' \in J$. This fact implies that $x\psi_{12}(a) \cdot \psi_{21}(a)x^*$ is compact morphism. Therefore $x\psi_{12}(a)$ and $\psi_{21}(a)x$ ($= (x^*\psi_{12}(a^*))^*$) are compact morphisms. \square

Lemma 6.5. *Let A be separable C^* -algebra and J be a closed ideal such that the projection $p : A \rightarrow A/J$ has a completely and contractive section s . Let $\phi : A/J \rightarrow \mathcal{L}(E)$ is a $*$ -homomorphism and $\psi : A/J \rightarrow \mathcal{L}(E \oplus E')$ is s -dilation for ϕ . There exists a $*$ -homomorphism $\varphi : A/J \rightarrow \mathcal{L}(E')$ such that*

$$\psi = \begin{pmatrix} \phi & 0 \\ 0 & \varphi \end{pmatrix}$$

Proof. A s -dilation for ϕ has the form

$$\begin{pmatrix} \phi & \psi_{12} \\ \psi_{21} & \varphi \end{pmatrix}.$$

Since ψ and ϕ are $*$ -homomorphisms, $\psi_{12}(a)\psi_{21}(b) = 0$. According to (1) of Lemma 6.4, we have $\psi_{12}(a)\psi_{12}(a)^* = 0$. Therefore $\psi_{12}(a) = 0$ (similarly, $\psi_{21}(a) = 0$). These facts easily imply that φ is a $*$ -homomorphism. \square

Lemma 6.6. *The map $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x' = \begin{pmatrix} x_{11} & 0 & x_{12} \\ 0 & 0 & 0 \\ x_{21} & 0 & x_{22} \end{pmatrix}$ defines a $*$ -monomorphism*

$$(6.7) \quad \xi : M_2(D_\phi(A, J, E \oplus E; B)) \rightarrow D_{\psi \cdot p \oplus \phi}(A, J, E \oplus E' \oplus E; B).$$

Proof. By assumption one has $(\phi(a) \oplus \phi(a))x - x(\phi(a) \oplus \phi(a)) \in \mathcal{K}(E \oplus E)$, for any $a \in A$, and $(\phi(b) \oplus \phi(b))x \in \mathcal{K}(E \oplus E)$, $x(\phi(b) \oplus \phi(b)) \in \mathcal{K}(E \oplus E)$ for any $b \in J$. It implies that

$$\begin{aligned} \phi(a)x_{mn} - x_{mn}\phi(a) &\in \mathcal{K}(E), \quad a \in A, \quad \text{and} \\ \phi(b)x_{mn} &\in \mathcal{K}(E), \quad x_{mn}\phi(b) \in \mathcal{K}(E), \quad b \in J. \end{aligned}$$

Then $(\psi(p(a)) \oplus \phi(a)) \cdot x' - x' \cdot (\psi(p(a)) \oplus \phi(a)) =$

$$= \begin{pmatrix} x_{11}\psi_{11}(p(a)) - \psi_{11}(p(a))x_{11} & x_{11}\psi_{12}(p(a)) & x_{12}\phi(a) - \psi_{11}(p(a))x_{12} \\ \psi_{21}(p(a))x_{11} & 0 & \psi_{21}(p(a))x_{12} \\ x_{21}\psi_{11}(p(a)) - \phi(a)x_{21} & x_{21}\psi_{12}(p(a)) & x_{22}\phi(a) - \phi(a)x_{22} \end{pmatrix}.$$

As in Lemma 6.4, one has $\psi_{21}(p(a))x_{11} \in \mathcal{K}(E, E')$, $x_{11}\psi_{12}(p(a)) \in \mathcal{K}(E', E)$, $x_{21}\psi_{12}(p(a)) \in \mathcal{K}(E', E)$ and $\psi_{21}(p(a))x_{12} \in \mathcal{K}(E, E')$. Using the fact $\phi(a) - \psi_{11}(p(a)) \in \phi(J)$, one has

$$(\psi(p(a)) \oplus \phi(a)) \cdot x' - x' \cdot (\psi(p(a)) \oplus \phi(a)) \in \mathcal{K}(E \oplus E' \oplus E), \quad a \in A.$$

To show that $(\psi(p(b)) \oplus \phi(b)) \cdot x'$ and $x' \cdot (\psi(p(b)) \oplus \phi(b))$ are in $\mathcal{K}(E \oplus E' \oplus E)$ when $b \in J$, note that $(\psi(p(b)) \oplus \phi(b)) \cdot x'$ and $x' \cdot (\psi(p(b)) \oplus \phi(b))$ are equal to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi(b)x_{21} & 0 & \phi(b)x_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & x_{12}\phi(b) \\ 0 & 0 & 0 \\ 0 & 0 & x_{22}\phi(b) \end{pmatrix}$$

respectively. They are compact morphisms because each entries of matrices are compact. \square

Let A be a separable C^* -algebra, J be a closed ideal in A and $p : A \rightarrow A/J$ be the canonical $*$ -homomorphism. Let $D^{(p)}(A, J; B)$ be a full C^* -sub-categoroid in $D(A, J; B)$ which has all pair of the form $(E, \phi \cdot p)$ as objects, where a pair (E, ϕ) is an object in $D(A/J; B)$.

Consider a $*$ -functoroid $\Gamma' : \text{Rep}(A/J; B) \rightarrow D^{(p)}(A, J; B)$ defined by the following rules:

- (1) if (E, ϕ) is an object in $\text{Rep}(A/J; B)$, then the corresponding object in $D^{(p)}(A, J; B)$ is the object $(E, \phi \cdot p)$;
- (2) if $x : (E, \phi) \rightarrow (E', \phi')$ is a morphism in $\text{Rep}(A/J; B)$, the corresponding morphism is $x : (E, \phi \cdot p) \rightarrow (E', \phi' \cdot p)$.

Let a functoroid $\Gamma : \text{Rep}(A/J; B) \rightarrow D(A, J; B)$ be the composition of Γ' with the natural $*$ -inclusion $\varepsilon : D^{(p)}(A, J; B) \subset D(A, J; B)$.

Lemma 6.7. *The $*$ -functoroid Γ' is an $*$ -isomorphism of C^* -categoroids.*

Proof. Since $\text{Rep}(A/J; B)$ and $D^{(p)}(A, J; B)$ are additive C^* -categoroids, it is enough, by Lemma 1.3, to show that for any object (E, ϕ) the induced $*$ -homomorphism

$$\Gamma'_{(E, \phi)} : D_\phi(A/J; E; B) \rightarrow D_{\phi \cdot p}(A, J; E; B)$$

is an $*$ -isomorphism. Indeed, if $x\phi(a') - \phi(a')x \in \mathcal{K}(E)$, $\forall a' \in A/J$, then $x\phi(p(a)) - \phi(p(a))x \in \mathcal{K}(E)$, $\forall a \in A$ and $x\phi(p(j)) - \phi(p(j))x = 0 \in \mathcal{K}(E)$, $\forall j \in J$. Conversely is more trivial. \square

In the theorem bellow we'll need a $*$ -functoroid

$$\varepsilon : D(A, J; B) \rightarrow D^{(p)}(A, J; B)$$

defined in the following way. For any object (E, ϕ) choose an object $(E \oplus E', \psi^\phi \cdot p)$ such that ψ^ϕ must be a s -dilation for ϕ . This is possible, since p has completely positive and contractive section s , and by Stinespring's theorem there exists a s -dilation for any completely positive and contractive section. If $x : (E, \phi) \rightarrow (E', \phi')$ is a morphism in $D(A, J; B)$ then

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : (E, \psi^\phi \cdot p) \rightarrow (E', \psi^{\phi'} \cdot p)$$

is a morphism in $D^{(p)}(A, J; B)$. Indeed,

$$(6.8) \quad \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{11}^\phi(p(a)) & \psi_{12}^\phi(p(a)) \\ \psi_{21}^\phi(p(a)) & \psi_{22}^\phi(p(a)) \end{pmatrix} - \begin{pmatrix} \psi_{11}^{\phi'}(p(a)) & \psi_{12}^{\phi'}(p(a)) \\ \psi_{21}^{\phi'}(p(a)) & \psi_{22}^{\phi'}(p(a)) \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \\ = \begin{pmatrix} x\psi_{11}^\phi(p(a)) - \psi_{11}^{\phi'}(p(a))x & x\psi_{12}^\phi(p(a)) \\ -\psi_{21}^{\phi'}(p(a))x & 0 \end{pmatrix}.$$

Since

$$x\psi_{11}^\phi(p(a)) - \psi_{11}^{\phi'}(p(a))x = x\phi(s \cdot p(a)) - \phi(s \cdot p(a))x = (x\phi(a) - \phi(a)x) + x\phi(j) - \phi(j)x,$$

$\forall a \in A$, where $j = a - s \cdot p(a) \in J$, one has that $x\psi_{11}^\phi(p(a)) - \psi_{11}^{\phi'}(p(a))x$ is a compact morphism. The morphisms $x\psi_{12}^\phi(p(a))$ and $\psi_{21}^{\phi'}(p(a))x$ are, by Lemma 6.4, compact morphism too. Therefore (6.8) is a compact morphism. Besides,

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{11}^\phi(p(j)) & \psi_{12}^\phi(p(j)) \\ \psi_{21}^\phi(p(j)) & \psi_{22}^\phi(p(j)) \end{pmatrix} = \begin{pmatrix} \psi_{11}^\phi(p(j)) & \psi_{12}^\phi(p(j)) \\ \psi_{21}^\phi(p(j)) & \psi_{22}^\phi(p(j)) \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Thus $*$ -functoroid ε is correctly defined.

Theorem 6.8. *Let A be a separable C^* -algebra, J be a closed ideal in A such that canonical $*$ -homomorphism $p : A \rightarrow A/J$ has a completely and contractive section. Then the functoroid Γ induces an isomorphism*

$$\Gamma_n : \mathbf{K}_n^a(\text{Rep}(A/J; B)) \rightarrow \mathbf{K}_n^a(D(A, J; B))$$

Proof. According to Lemma 6.7, it is enough to show that $\varepsilon : D^{(p)}(A, J; B) \subset D(A, J; B)$ induces the isomorphism

$$\varepsilon_n : \mathbf{K}_n^a(D^{(p)}(A, J; B)) \rightarrow \mathbf{K}_n^a(D(A, J; B)).$$

Consider homomorphism

$$\varepsilon_n : \mathbf{K}_n^a(D(A, J; B)) \rightarrow \mathbf{K}_n^a(D^{(p)}(A, J; B)).$$

We assert, that $\varepsilon_n \varepsilon_n$ and $\varepsilon_n \varepsilon_n$ are the identity homomorphisms. Let consider the first case. Let $(E, \phi \cdot p)$ be an object in $D^{(p)}(A, J; B)$. Then functor ε sends it in the object of form $(E \oplus E', \psi \cdot p)$

where ψ is s -dilation of $\phi \cdot p$. According to Lemma 6.5 $\psi = \begin{pmatrix} \phi & 0 \\ 0 & \varphi \end{pmatrix}$ where φ is $*$ -homomorphism from A into $\mathcal{L}(E')$. If $x \in D_{\phi \cdot p}(A, J; E; B)$ then $\varepsilon \varepsilon(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in D_{\psi p}(A, J; E \oplus E'; B)$. Let $i_1 : E \rightarrow E \oplus E'$ be the inclusion in the first summand. We assert, that $i_1 \phi(p(a)) = \psi(p(a))i_1$ and $\varepsilon \varepsilon(x) = i_1 x i_1^*$ for $\forall a \in A$. Indeed, $i_1(\phi(p(a))(\xi)) = \phi(p(a))(\xi) \oplus 0$ and

$$\psi(p(a))i_1(\xi) = \psi(p(a))(\xi \oplus 0) = \phi(p(a))(\xi) \oplus 0, \quad \xi \in E.$$

Now, let $k \in K_n^a(D_{\phi \cdot p}(A, J; E; B))$ represents an element $\{k\}$ in $\mathbf{K}_n^a(D^{(p)}(A, J; E; B))$. Then $\varepsilon_n \varepsilon_n(\{k\}) = \{\varepsilon_n \varepsilon_n(k)\}$. Since $\varepsilon \varepsilon = ad(i_1)$ one has $\{\varepsilon_n \varepsilon_n(k)\} = \{k\}$. Therefore $\varepsilon_n \varepsilon_n$ is identity homomorphism. Now, we show that homomorphism $\varepsilon_n \varepsilon_n$ is the identity homomorphism. Recall that restriction of $\varepsilon \cdot \varepsilon$ on the object (E, ϕ) induces the $*$ -homomorphism

$$D_\phi(A, J; E; B) \rightarrow D_{\psi p}(A, J; E \oplus E' B)$$

given by the map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. Consider a homomorphism ϑ which is the composition

$$D_\phi(A, J; E; B) \rightarrow D_{\psi p}(A, J; E \oplus E' B) \rightarrow D_{\psi p \oplus \phi}(A, J; E \oplus E' \oplus E; B),$$

where the first is induced by $\varepsilon \varepsilon$ and the second arrow is induced by the isometry in the first two summands. The $*$ -homomorphism $\vartheta : D_\phi(A, J; B) \rightarrow D_{\psi \cdot p \oplus \phi}(A, J; B)$ is defined by

$$x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that this homomorphism has the decomposition:

$$D_\phi(A, J; B) \xrightarrow{i_1} M_2(D_\phi(A, J; B)) \xrightarrow{\xi} D_{\psi \cdot p \oplus \phi}(A, J; B)$$

where i_1 is given by the map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and second arrow is (6.7). Consider η the composition of the sequence of $*$ -homomorphisms

$$D_\phi(A, J; B) \xrightarrow{i_2} M_2(D_\phi(A, J; B)) \xrightarrow{\xi} D_{\psi \cdot p \oplus \phi}(A, J; B)$$

where i_2 is given by the correspondence $x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$. The homomorphisms ϑ and η induces the same homomorphisms of K -groups, since i_1 and i_2 induce the same homomorphisms of K -groups. Let $k \in K_n^a(D_\phi(A, J; E; B))$ defines an element $\{k\} \in \mathbf{K}_n^a(D(A, J; B))$. Let k' be the image of k relative to the homomorphism induced by the $*$ -homomorphism ϑ . Of course, $\{\varepsilon_n \varepsilon_n(k)\} = \{k'\}$. On the other hand, k' coincides with the image of k relative to the homomorphism induced by the $*$ -homomorphism η . Since η is defined by the isometry of E in the third summand of $E \oplus E' \oplus E$, one has $\{k\} = \{k'\} = \{\varepsilon_n \varepsilon_n(k)\}$. This means that $\varepsilon_n \varepsilon_n$ is the identity homomorphism. \square

Now, we are ready to prove the following excision property, which plays one of the main role in this article.

Theorem 6.9. *Let B be a σ -unital C^* -algebra. If in an exact sequence of separable C^* -algebras $0 \rightarrow I \rightarrow A \xrightarrow{p} A/I \rightarrow 0$ p admits a completely positive and contractive section. Then there exist two-sided long exact sequence*

$$(6.9) \quad \dots \rightarrow \mathbf{K}_n^a(\text{Rep}(A, B)) \rightarrow \mathbf{K}_n^a(\text{Rep}(J, B)) \rightarrow \mathbf{K}_{n-1}^a(\text{Rep}(A/J, B)) \rightarrow \mathbf{K}_{n-1}^a(\text{Rep}(A, B)) \rightarrow \dots$$

Proof. Consider short exact sequence of C^* -categoroids

$$0 \rightarrow D(A, J; B) \rightarrow \text{Rep}(A, B) \rightarrow \text{Rep}(A, B)/D(A, J; B) \rightarrow 0.$$

We have, by Proposition 4.6, the two-sided long exact sequence

$$\dots \rightarrow \mathbf{K}_n^a(\text{Rep}(A, B)) \rightarrow \mathbf{K}_n^a(\text{Rep}(A, B)/D(A, J; B)) \xrightarrow{\cong} \mathbf{K}_{n-1}^a(D(A, J; B)) \rightarrow \mathbf{K}_{n-1}^a(\text{Rep}(A, B)) \rightarrow \dots$$

According to Theorem 6.3 and Theorem 6.8, one has

$$(6.10) \quad \mathbf{K}_n^a(\text{Rep}(A; B)/D(A, J; B)) \approx \mathbf{K}_n^a(\text{Rep}(J; B))$$

and

$$(6.11) \quad \mathbf{K}_n^a(\text{Rep}(A/J; B)) \approx \mathbf{K}_n^a(D(A, J; B)).$$

It gives us the two-sided long exact sequence (6.9). \square

One has also similar sequence for the case of topological K -theory:

$$(6.12) \quad \dots \rightarrow \mathbf{K}_n^t(\text{Rep}(A, B)) \rightarrow \mathbf{K}_n^t(\text{Rep}(J, B)) \rightarrow \mathbf{K}_{n-1}^t(\text{Rep}(A/J, B)) \rightarrow \mathbf{K}_{n-1}^t(\text{Rep}(A, B)) \rightarrow \dots$$

7. THE ISOMORPHISMS OF ALGEBRAIC, TOPOLOGICAL AND KASPAROV KK -GROUPS

In this section we'll turn to the our main problem mentioned in the introduction.

Define algebraic bivariant KK -groups by

$$(7.1) \quad KK_n^a(-; B) = \begin{cases} \mathbf{K}_{n+1}^Q(\text{Rep}(-; B)) & \text{if } n \geq -1 \\ \mathbf{K}_{n+1}n^B(\text{Rep}(-; B)) & \text{if } n < -1, \end{cases}$$

and topological KK -groups by

$$(7.2) \quad KK_n^t(-; B) = \mathbf{K}_{Kar}^{-n-1}(\text{Rep}(-; B))$$

where \mathbf{K}_n^Q , \mathbf{K}_n^B and \mathbf{K}_{kar}^n are Quillen, Bass and Karoubi K -groups respectively. We'll write by $KK_n(A, B)$ Kasparov's group $KK^{-n}(A, B)$. Our main goal is proof of isomorphism of families mentioned above.

In the first place we proof isomorphism in the fix dimension.

7.1. On the Isomorphism $\mathbf{K}_0^a(\text{Rep}(-; B)) \simeq KK_{-1}(-; B)$. Consider a triple $(\varphi, E; p)$, where E is trivially graded countable generated right B -module, $\varphi : A \rightarrow \mathcal{L}_B(E)$ is a $*$ -homomorphism and $p \in \mathcal{L}_B(E)$, so that

$$(7.3) \quad \begin{aligned} p\varphi(a) - \varphi(a)p &\in \mathcal{K}_B(E), \\ (p^* - p)\varphi(a) &\in \mathcal{K}_B(E), \quad (p^2 - p)\varphi(a) \in \mathcal{K}_B(E), \quad \forall a \in A. \end{aligned}$$

Such a triple is called *Kasparov-Fredholm A, B -module*. If all left parts in 7.3 are zero, then such a triple is said to be degenerate.

Define sum Kasparov-Fredholm A, B -modules by the formula

$$(\varphi, E; p) \oplus (\varphi', E'; p') = (\varphi \oplus \varphi', E \oplus E'; p \oplus p').$$

Consider the equivalence relations:

- (*Unitary isomorphism*) A, B -modules $(\varphi, E; p)$ and $(\varphi', E'; p')$ will be said to be unitary isomorphic if there exists unitary isomorphism $u : E \rightarrow E'$ such that

$$u\varphi(a)u^* = \varphi'(a), \quad upu^* = p', \quad \forall a \in A.$$

- (*Homology*) A, B -modules $(\varphi, E; p)$ and $(\varphi', E'; p')$ will be said to be homological if

$$p'\varphi'(a) - p\varphi(a) \in \mathcal{K}_B(E), \quad \forall a \in A.$$

Simple checking shows that the equivalence relations, defined above, are well behaved relative to the sum.

Let $\mathcal{E}^1(A, B)$ be a abelian monoid of classes of equivalence A, B -modules generated by the unitary isomorphism and homology. Denote by $\mathcal{D}^1(A, B)$ a sub-monoid of $\mathcal{E}^1(A, B)$ consisting of only those classes which are classes of all degenerate triples. By definition

$$E^1(A, B) = \mathcal{E}^1(A, B)/\mathcal{D}^1(A, B)$$

Using the Kasparov stabilization theorem, one can to show that definition of $E^1(A, B)$ coincides with Kasparov's original definition of $E^1(A, B)$. Therefore one sees that the last monoid (group) may be considered as a model of $KK^1(A, B)$ by lemma 2 of section 7 of [15].

Recall, that the objects in $\text{Rep}(A, B)$, by definition, has the form $(\varphi, E; p)$, where $p : (\varphi, E) \rightarrow (\varphi, E)$ is projection in the category $\text{Rep}(A; B)$. More precisely,

$$\varphi(a)p - p\varphi(a) \in \mathcal{K}_B(E), \quad p^* = p, \quad p^2 = p.$$

An unitary isomorphism $s : (\varphi, E; p) \rightarrow (\psi, E'; q)$ is partial isomorphism $s : E \rightarrow E'$ such that

$$s\varphi(a) - \psi(a)s \in \mathcal{K}_B(E, E'), \quad s^*s = p, \quad ss^* = q.$$

Let $\mathcal{E}_\pi^1(A, B)$ be abelian monoid of unitary isomorphic objects in $\text{Rep}(A; B)$. Thus Grothendieck group of $\mathcal{E}_\pi^1(A, B)$ is exactly $K_0(\text{Rep}(A, B))$.

Kasparov-Fredholm A, B -modules $(\varphi, E; p)$ and $(\varphi', E'; p')$ will be said to be strong unitary isomorphic if there exists unitary isomorphism $u : E \rightarrow E'$ such that

$$u\varphi(a)u^* - \varphi'(a) \in \mathcal{K}_B(E'), \quad p' = upu^*, \quad \forall a \in A.$$

Let $s : (\varphi, E; p) \rightarrow (\varphi', E'; p')$ be an unitary isomorphism in $\text{Rep}(A; B)$. Then

$$\bar{s} : \left(\begin{array}{ccc} \varphi \oplus \psi & E \oplus E & \bar{p} \end{array} \right) \rightarrow \left(\begin{array}{ccc} \psi \oplus \varphi & E \oplus E & \bar{p}' \end{array} \right)$$

is strong isomorphism, where

$$\bar{s} = \left(\begin{array}{cc} s & 1 - ss^* \\ 1 - s^*s & s \end{array} \right), \quad \bar{p} = \left(\begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) \quad \text{and} \quad \bar{p}' = \left(\begin{array}{cc} 1 & 0 \\ 0 & p' \end{array} \right).$$

Other hand, simple checking shows that strong unitary isomorphic Kasparov-Fredholm A, B -modules is contained in the equivalence generated by unitary isomorphism and homology. This means that one has correctly defined homomorphism

$$\lambda_1 : K^0(\text{Rep}(A, B)) \rightarrow E^1(A, B),$$

given by the map $[(\varphi, E; p)] \mapsto \{(\varphi, E; p)\}$.

Let $\text{Rep}(A, B)/\text{D}(A, A; B)$ be pseudo-abelian category of the category $\text{Rep}(A, B)/\text{D}(A, A; B)$. Let $(\varphi; E; p)$ be a Kasparov-Fredholm A, B -module. The p defines projector \dot{p} in the category $\text{Rep}(A, B)/\text{D}(A, A; B)$. Thus the triple $(\varphi; E; \dot{p})$ is an object in $\text{Rep}(A, B)/\text{D}(A, A; B)$. We want definition of a homomorphism

$$\mu : E^1(A, B) \rightarrow K_0(\text{Rep}(A, B)/\text{D}(A, A; B))$$

by the map $(\varphi; E; p) \mapsto (\varphi; E; \dot{p})$. We'll show that this is correct.

We recall definition of operatorial homotopy:

- (*Operatorial homotopy*) A, B -module $(\varphi, E; p)$ is operatorial homotopic to a triple $(\varphi, E; p')$ if there exist a continuous map $p_t : [0; 1] \rightarrow \mathcal{L}_B(E)$ such that $(\varphi, E; p_t)$ is A, B -module for any $t \in [0; 1]$.

If $(\varphi, E; p)$ is homological to $(\psi, E; q)$, then $(\varphi, E; p) \oplus (\psi, E; 0)$ is operatorial homotopic to $(\varphi, E; 0) \oplus (\psi, E; q)$. Indeed, Desired homotopy is defined by the formula

$$\left(\left(\begin{array}{cc} \varphi & 0 \\ 0 & \psi \end{array} \right), E \oplus E, \frac{1}{1+t^2} \left(\begin{array}{cc} p & tpq \\ tqp & t^2q \end{array} \right) \right), \quad t \in [0; \infty].$$

(cf. section 7 in [15]). Thus the projections $p \dot{\oplus} 0$ and $0 \dot{\oplus} q$ are homotopic. Then, using Lemma 4 section 6 in [15], one concludes that the objects $(\varphi, E; \dot{p}) \dot{\oplus} (\psi, E; \dot{0})$ and $(\varphi, E; \dot{0}) \dot{\oplus} (\psi, E; \dot{q})$ are unitary isomorphic objects in $\text{Rep}(A, B)/\text{D}(A, A; B)$. Let $(\varphi, E; p)$ is unitary isomorphic to $(\psi, E; q)$. Then $(\varphi, E; \dot{p})$ is isomorphic to $(\psi, E; \dot{q})$ in the category $\text{Rep}(A, B)/\text{D}(A, A; B)$. Therefore μ is correctly defined.

We are just ready to prove the following theorem.

Theorem 7.1. *The natural homomorphism*

$$\lambda_1 : K^0(\text{Rep}(A, B)) \rightarrow E^1(A, B)$$

is an isomorphism.

Proof. Of course, λ_1 is epimorphism. Indeed, let $(\varphi, E; p)$ be Kasparov-Fredholm A, B -module. Applying analogue to Lemmas 17.4.2-17.4.3 in [2], one can suppose that $p^* = p$ and $\|p\| \leq 1$. Then it is equivalent to $(\varphi \oplus 0, E \oplus E; p')$, where

$$p' = \begin{pmatrix} p & \sqrt{p-p^2} \\ \sqrt{p-p^2} & 1-p \end{pmatrix}.$$

Simple checking shows that p' is a projection and $(\varphi \oplus 0, E \oplus E; p')$ is an object in $\text{Rep}(A, B)$. To show that λ_1 is monomorphism, consider commutative diagram

$$\begin{array}{ccc} K^0(\text{Rep}(A, B)) & \xrightarrow{\lambda_1} & E^1(A, B) \\ \parallel & & \downarrow \mu \\ K^0(\text{Rep}(A, B)) & \xrightarrow{\xi} & K^0(\text{Rep}(A, B)/D(A, A; B)). \end{array}$$

By Theorem 6.3, ξ is an isomorphism. Therefore λ_1 is monomorphism. \square

7.2. The Main Theorem. Now, we present our main result in the following theorem.

Theorem 7.2. *Let B be a σ -unital C^* -algebra. Then the families of functors*

$$(7.4) \quad \{KK_n^a(-; B)\}_{n \in \mathbb{Z}}, \quad \{KK_n^t(-; B)\}_{n \in \mathbb{Z}}, \quad \{KK_n(-; B)\}_{n \in \mathbb{Z}}.$$

are naturally isomorphic Cuntz-Bott cohomology theories on the category of separable C^ -algebras and $*$ -homomorphisms.*

Proof. By Proposition 4.10 the functor $KK_n^a(-; B)$ is naturally isomorphic to $\mathbf{K}_{n+1}^a(\text{Rep}(-; B))$ (respectively for $KK_n^t(-; B)$. See 5). But by Theorem 6.9 family

$$\{\mathbf{K}_n^a(\text{Rep}(-; B))\}_{n \in \mathbb{Z}}.$$

has weak excision property. The same property has also

$$\{\mathbf{K}_n^t(\text{Rep}(-; B))\}_{n \in \mathbb{Z}}.$$

Further, thanks to the result of [4], the family

$$\{KK_n(-; B)\}_{n \in \mathbb{Z}}$$

has weak excision property. Now, we show:

- functors $\mathbf{K}_n^a(\text{Rep}(-; B))$, $\mathbf{K}_n^t(\text{Rep}(-; B))$ and $KK_n(-; B)$ have stable property, for all $n \in \mathbb{Z}$.

This fact with the Theorem 6.9 implies that all three families are Cuntz-Bott cohomology theories. Let us go on to show it.

Let $p \in \mathcal{K}$ be a rank one projection and A be a separable C^* -algebra. The $*$ -homomorphism $e_A : A \rightarrow A \otimes \mathcal{K}$ defined by the map $a \mapsto a \otimes p$, $\forall a \in A$, induces a $*$ -functor

$$e_A^* : \text{Rep}(A \otimes \mathcal{K}; B) \rightarrow \text{Rep}(A; B).$$

Now, we construct a $*$ -functor

$$\varepsilon : \text{Rep}(A; B) \rightarrow \text{Rep}(A \otimes \mathcal{K}; B)$$

which is somehow a right inverse to e_A^* . Let $\phi : A \rightarrow \mathcal{L}(E)$ be an object in $\text{Rep}(A; B)$. One has the induced $*$ -homomorphism

$$\phi \otimes id_{\mathcal{K}} : A \otimes \mathcal{K} \rightarrow \mathcal{L}(E \otimes_C \mathcal{H})$$

defined as the composition

$$A \otimes \mathcal{K} \rightarrow \mathcal{L}(E) \otimes \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(E \otimes_C \mathcal{H}),$$

of natural maps. Of course, this is an object in $\text{Rep}(A \otimes \mathcal{K}; B)$. Let $f : (E, \phi) \rightarrow (E', \phi')$ be a morphism in $\text{Rep}(A; B)$, i.e. $f\phi(a) - \phi(a)f \in \mathcal{K}(E)$, $\forall a \in A$. Then

$$\varepsilon(f) = f \otimes p : (E \otimes_C \mathcal{H}, \phi \otimes id_{\mathcal{K}}) \rightarrow (E' \otimes_C \mathcal{H}, \phi' \otimes id_{\mathcal{K}})$$

is a morphism in $\text{Rep}(A \otimes \mathcal{K}; B)$.

Now we construct an useful isometry $\sigma_E : E \rightarrow E \otimes_k \mathcal{H}$, for any countable generated B -module E . Choose $y \in \mathcal{H}$ such that $p(y) = y$ and $\|y\| = 1$ and consider a B -homomorphism σ_E given by the formula $x \mapsto x \otimes y$. For a $z \in \mathcal{H}$, there exists $\lambda_z \in k$ determined uniquely by the equation $p(z) = \lambda_z y$ (note, that p is rank one projection). Define σ_E^* by the correspondence $x \otimes z \mapsto \lambda_z x$. The B -homomorphism σ_E^* is the adjoint to σ_E . Indeed, since $p(y) = y$ and $\|y\| = 1$,

$$\langle \sigma_E(x); x' \otimes z \rangle = \langle x \otimes y; x' \otimes z \rangle = \langle x; x' \rangle \cdot \langle y, pz \rangle = \langle x; \lambda_z x' \rangle = \langle x; \sigma_E^*(x' \otimes z) \rangle \quad \forall x, x', z \in E.$$

Since $\sigma_E^* \sigma_E(x) = \sigma_E^*(x \otimes y) = x$, one concludes σ_E is an isometry.

Since

$$\sigma_E \phi(a)(z) - ((\phi \otimes id_{\mathcal{K}})e_A(a))\sigma_E(z) = \phi(a)(z) \otimes y - (\phi(a) \otimes p)(z \otimes y) = 0, \quad \forall z \in E,$$

the isometry σ_E is a morphism from (E, ϕ) into $(E \otimes_C \mathcal{H}, (\phi \otimes id_{\mathcal{K}})e_A)$.

Consider restriction of $e_A^* \varepsilon$ on the $D_\phi(A; E; B)$. Thus we have the $*$ -homomorphism

$$(7.5) \quad (e_A^* \varepsilon)_E : D_\phi(A; E; B) \rightarrow D_{(\phi \otimes id)e_A}(A; E \otimes_k \mathcal{H}; B)$$

which maps x to $x \otimes p$. But

$$(\sigma_E x)(z) = (\sigma_E)(x(z)) = x(z) \otimes y = ((x \otimes p)\sigma_E)(z), \quad \forall x \in D_\phi(A; E; B), \quad \forall z \in E.$$

This means

$$(7.6) \quad (e_A^* \varepsilon)_E(x) = \sigma_E x \sigma_E^*.$$

Now, we show the functor $e_A^* \varepsilon$ induces the identity homomorphism of group $\mathbf{K}_n^a(\text{Rep}(A, B))$ onto itself. Indeed, choose an element $r \in \mathbf{K}_n^a(\text{Rep}(A, B))$. By definition of \mathbf{K}_n^a -groups the element r is represented by an element $r_\phi \in K_n^a(D_\phi(A; E; B))$. Then the element $\mathbf{K}_n^a(e_A^* \varepsilon)(r)$ is represented by the element

$$(7.7) \quad K_n^a((e_A^* \varepsilon)_E)(r_\phi).$$

The equation 7.6 implies that the element 7.7 represents the element r . This means that $e_A^* \varepsilon$ induces identity homomorphism of $\mathbf{K}_n^a(\text{Rep}(A, B))$ onto itself. Thus $\varepsilon_n = K_n^a(\varepsilon)$ is a right inverse of $K_n^a(e_A^*)$. This means that $K_n^a(e_A^*)$ is epimorphism. Thus it is enough to show that $K_n^a(e_A^*)$ is a monomorphism. Consider commutative diagram

$$\begin{array}{ccc} \mathbf{K}_n^a(\text{Rep}(A, B)) & \xrightarrow{K_n^a(e_A^*)} & \mathbf{K}_n^a(\text{Rep}(A^I \otimes \mathcal{K}, B)) \\ \downarrow \mathbf{K}_n^a(e_A^*) & & \downarrow \mathbf{K}_n^a(e_0^A) \\ \mathbf{K}_n^a(\text{Rep}(A^I, B)) & \xrightarrow{\varepsilon_n} & \mathbf{K}_n^a(\text{Rep}(A^I \otimes \mathcal{K}, B)), \end{array}$$

where $e_t^A : A^I \rightarrow A$ is the evolution at $t \in I = [0; 1]$. Since $\mathbf{K}_n^a(\text{Rep}(- \otimes \mathcal{K}, B))$ is split and stable functor, it is homotopy invariant functor. This fact implies that right vertical arrow is an isomorphism. Therefore $K_n^a(e_A^*)$ is a monomorphism. Therefore the family $\{\mathbf{K}_n^a(\text{Rep}(-; B))\}_{n \in \mathbb{Z}}$ is Cuntz-Bott cohomology theory. According to Corollary 3.5 and Theorem 7.1, one concludes that all three theories are isomorphic. \square

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