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ONE VERSION OF GALERKIN-PETROV'S METHOD WITH ITERATIONS

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Abstract. We consider the projective-iterative method for elliptic boundary value problems in which in the capacity of the projective method we take one version of the Galerkin-Petrov method. An approximate solution is sought in the form of a linear combination of base functions of the method of finite elements, and the algebraic system is constructed by scalar multiplication of the residual by the other functions. The error estimates of the projective- iterative scheme are obtained and its stability is shown. The numerical realization of the scheme is presented.

რეზიუმე. განხილულია პროექციულ-იტერაციული მეთოდი ელიფსური სასზღვრო ამოცანებისათვის, სადაც პროექციულ მეთოდად აღებულია გალიორკინ-პეტროვის მეთოდის ერთი ვარიანტი; მიახლოებითი ამონახსნი აგებულია სასრულ ელემენტთა მეთოდის საბაზისო ფუნქციების წრფივი კომბინაციით; ალგებრული სისტემა შედგენილია გადახრის სხვა ფუნქციებზე სკალარული გამრავლების საფუძველზე. მიღებულია პროექციულ-იტერაციული სქემის ცდომილების შეფასებანი და ნაჩვენებია სქემის მდგრადობა. ჩატარებულია რიცხვითი რეალიზაცია.

1. Statement of the Problem

We consider the equation of the type ([1], p. 426)

$$Au + Ku = f, \quad u \in D(A), \quad f \in H, \tag{1.1}$$

where A is a linear self-conjugate positive definite differential operator in the Hilbert space $H \equiv L_2(\Omega)$, Ω is a bounded domain with regular boundary $\partial\Omega$, K is a linear differential operator such that $A^{-1}K$ is fully continuous in H, an energetic space $H_A \subset D(K)$, H_A is a supplement of a dense lineal $D(A) \subset H$ by the norm $\|u\|_{H_A} = [u,u]^{\frac{1}{2}} = (Au,u)^{\frac{1}{2}}$ ([1], p. 76), $D(A)$ and $D(K)$ are the domains of definition of operators A and K.

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Let the base functions of the method of finite elements $\varphi_k \equiv \varphi_k^{(h)}$, $k = 1, 2, \dots, n$, $n = n(h)$ belong to H_A, h is a lattice pitch.

Suppose that the operator A^{-1} is given explicitly by Green's functions $G(x,t)$, i.e. a solution of equation $Av = g$, $v \in D(A)$, $g \in H$ is given by the formula

$$v = A^{-1}g = \int_\Omega G(x,t)g(t)\,dt. \tag{1.2}$$

First we find the functions

$$\psi_i \equiv A^{-1}\varphi_i = \int_\Omega G(x,t)\varphi_i(t)\,dt, \quad i = 1, 2, \dots, n. \tag{1.3}$$

An approximate solution of equation (1.1) is sought in the form

$$u_h = \sum_{k=1}^{n} a_k\varphi_k, \quad n = n(h) \tag{1.4}$$

and the algebraic system is constructed by the method of Galerkin-Petrov:

$$\big(Au_h + Ku_h - f,\, \psi_i\big) = 0, \quad i = 1, 2, \dots, n,$$

or

$$\sum_{k=1}^{n} a_k\big[(\varphi_k, \varphi_i) + (K\varphi_k, A^{-1}\varphi_i)\big] = (A^{-1}f, \varphi_i), \quad i = 1, 2, \dots, n. \tag{1.5}$$

Construction of algebraic system (1.5) makes in practice difficulties in comparison with the ordinary method of finite elements, where the system is such that

$$\sum_{k=1}^{n} a_k\big\{[\varphi_k, \varphi_i] + (K\varphi_k, \varphi_i)\big\} = (f, \varphi_i), \quad i = 1, 2, \dots, n(h). \tag{1.6}$$

But system (1.5) possesses the following property: a number of conditionality $\varkappa_n \equiv \lambda_{\max}\lambda_{\min}^{-1}$ of symmetric matrices in (1.5), generated by the operator A in case of a uniform lattice with pitch h, is uniformly bounded as $h \to 0$ ([2], p. 104; [3], p. 240), which guarantees the stability. Numbers of conditionality of corresponding matrices in scheme (1.6) is $\varkappa_n \sim h^{-2m}$ ([3], p. 243), where $2m$ is an order of the differential operator A. On the basis of the Galerkin-Petrov method, order of convergence of the projective-iterative scheme increases for each cycle even for $\varphi_k \overline{\in} D(A)$.

Note that in the iteration we shall need the functions $A^{-1}\varphi_k$, $A^{-1}K\varphi_k$, $k = 1, 2, \dots, n$.

The aim of the present paper is to get an error estimate for scheme (1.5) by iterations in the spaces $L_2(\Omega)$, $W_2^{2m}(\Omega)$, $C(\overline{\Omega})$ and to show the stability of the projective-iterative scheme. The numerical realization is given.

2. Projective-Iterative Scheme

Introduce the operator

$$P_h v = \sum_{k=1}^{n} C_k \varphi_k, \quad n = n(h),$$

where coefficients $C_1, C_2, \dots, C_n$ are defined from the condition

$$\Big\| v - \sum_{k=1}^{n} C_k \varphi_k \Big\|_H^2 = \min .$$

Then we can write (1.5) in the form

$$u_h + P_h A^{-1} K u_h = P_h A^{-1} f, \quad u_h \in S_h, \tag{2.1}$$

where S_h is the linear shell of functions $\varphi_1, \varphi_2, \dots, \varphi_n$, $(H_n \equiv S_h)$.

From (1.1) we obtain the second kind equation

$$u + A^{-1} K u = A^{-1} f \tag{2.2}$$

and its Galerkin's approximation (2.1) ([4], p. 199). If the operator $I + A^{-1}K$ is invertible in H and $\|P^h A^{-1} K\| \to 0$ as $h \to 0$, $(P_h \equiv I - P^h)$ then for sufficiently small h equation (2.1) has a unique solution u_h. Moreover, if $\|A^{-1} K P^h\|_H \to 0$ as $h \to 0$, then we can apply the following projective-iterative method ([5]):

of the solution u_h we take the iteration

$$\widetilde{u}_h = -A^{-1} K u_h + A^{-1} f, \tag{2.3}$$

(1) calculate the residual

$$r_0 \equiv A^{-1} f - \widetilde{u}_h - A^{-1} K \widetilde{u}_h$$

and the scalar product (r_0, φ_i), $i = 1, 2, \dots, n$;

(2) solve the algebraic system

$$\sum_{k=1}^{n} a_k^{(1)} \big[(\varphi_k, \varphi_i) + (K\varphi_k, A^{-1}\varphi_i) \big] = (r_0, \varphi_i), \quad i = 1, 2, \dots, n, \tag{2.4}$$

the left-hand sides of algebraic systems (1.5) and (2.4) are the same;

(3) of the solution $u_h^{(1)} = \sum_{k=1}^{n} a_k^{(1)} \varphi_k$ we take the iteration

$$\widetilde{u}_h^{(1)} = -A^{-1} K u_h^{(1)} + r_0,$$

(4) summarize the results of iterations

$$\widetilde{u}_{h,1} \equiv \widetilde{u}_h + \widetilde{u}_h^{(1)}.$$

Cycle (1)–(4) can be repeated several times. After l cycles we obtain an approximate solution by means of the projective-iterative method

$$\widetilde{u}_{h,l} \equiv \widetilde{u}_h + \widetilde{u}_h^{(1)} + \cdots + \widetilde{u}_h^{(l)}. \tag{2.5}$$

The error of an approximate solution for sufficiently small h (similarly to [5]) is expressed by the formula

$$u-\widetilde{u}_{h,\,l}=$$
$$=(I+A^{-1}KP_h)^{-1}(-A^{-1}KP_h)\cdots(I+A^{-1}KP_h)^{-1}(-A^{-1}KP_hu). \quad (2.6)$$

3. Error Estimates in H

Lemma 1. *Let $H_A\subset D(K^*)$ and for $\forall\, v\in H_A$ the inequality*

$$\|K^*v\|\le C\|v\|_{H_A}, \quad (3.1)$$

is fulfilled. Then the operator $A^{-\frac{1}{2}}K$ is bounded in H.

Indeed, the condition $v\in H_A$ implies that $g\equiv A^{\frac{1}{2}}v\in H$, $v=A^{-\frac{1}{2}}g$. From (3.1) we have

$$\|K^*A^{-\frac{1}{2}}g\|\le C\|g\|, \quad \forall\, g\in H,$$

i.e., the operator $K^*A^{-\frac{1}{2}}$ is bounded in H. The operator K^* is conjugate in H, and the norm $\|A^{-\frac{1}{2}}K\|=\|K^*A^{-\frac{1}{2}}\|$.

It is well-known ([3], p. 172) that if power of the linear shell S_h of base functions is equal to $(k-1)$, the base is homogeneous of order q, an order of all derivatives which are connected with nodal parameters is less than $k-\frac{p}{2}$, p is dimension of the domain Ω and the function $u\in W_2^k(\Omega)$, then for the interpolation u_I the estimate

$$\|u-u_I\|_s\le C_sh^{k-s}\|u\|_k, \quad s=0,1,2\dots,q. \quad (3.2)$$

is valid. Here we suppose that $s=0$, $k=2m$, where $2m$ is order of the differential operator A. Then

$$\|P^hu\|=\|u-P_hu\|_0\le\|u-u_s\|_0\le C_0h^{2m}\|u\|_{2m}. \quad (3.3)$$

Lemma 2. *In condition* (3.3) *the norm*

$$\|A^{-\frac{1}{2}}P^h\|\le Ch^m. \quad (3.4)$$

Indeed, we have

$$\big(A^{-\frac{1}{2}}P^hg,\,A^{-\frac{1}{2}}P^hg\big)=\big(A^{-1}P^hg,\,P^hg\big)\le\big\|A^{-1}P^h\big\|\|g\|^2,$$
$$\big\|A^{-\frac{1}{2}}P^h\big\|\le\big\|A^{-1}P^h\big\|^{\frac{1}{2}}, \quad \big(A^{-1}P^h\big)^*=P^hA^{-1}.$$

Moreover, it follows from (3.3) that

$$\big\|P^hA^{-1}g\big\|\le C_0h^{2m}\big\|A^{-1}g\big\|_{2m}\le\overline{C}h^{2m}\big\|g\big\|_H,$$

i.e.,

$$\big\|P^hA^{-1}\big\|\le\overline{C}h^{2m}, \quad \overline{C}\equiv C_0\widetilde{C}, \quad \big\|A^{-1}g\big\|_{2m}\le\widetilde{C}\big\|g\big\|$$

and finally,

$$\big\|A^{-\frac{1}{2}}P^h\big\|\le(\overline{C_0})^{\frac{1}{2}}h^m, \quad C=(\overline{C})^{-\frac{1}{2}}.$$

Lemma 3. *If all the conditions of Lemmas* 1 *and* 2 *are fulfilled and the operator* $A^{-1}KA^{\frac{1}{2}}$ *is bounded in* H*, then*

$$\left\|A^{-1}KP^h\right\| \leq \overline{C}_0 h^m. \tag{3.5}$$

Indeed,

$$\left\|A^{-1}KP^h\right\| \leq \left\|A^{-1}KA^{\frac{1}{2}}\right\|\left\|A^{-\frac{1}{2}}P^h\right\| \leq \overline{C}_0 h^m,$$

where

$$\overline{C}_0 \equiv \left\|A^{-1}KA^{\frac{1}{2}}\right\|(\overline{C})^{\frac{1}{2}}.$$

Note that

$$\left\|P^hA^{-1}K\right\| \leq \left\|P^hA^{-\frac{1}{2}}\right\|\left\|A^{-\frac{1}{2}}K\right\|.$$

Theorem 1. *If the operator* $I + A^{-1}K$ *is invertible in* H*, inequalities* (3.1) *and* (3.3) *are fulfilled, the operator* $A^{-1}KA^{\frac{1}{2}}$ *is bounded in* H*, the exact solution* $u \in W_2^k(\Omega)$*,* $k \geq 2m$*, then for sufficiently small* h *the error estimate*

$$\begin{gathered}\left\|u - \widetilde{u}_{h,l}\right\| \leq \\ \leq \left\|(I + A^{-1}KP_h)^{-1}\right\|^{l+1}\left\|A^{-1}KA^{\frac{1}{2}}\right\|^{l+1}\left\|P^hA^{-\frac{1}{2}}\right\|^{l+1}\left\|P^hu\right\|, \\ l = -1, 0, 1, \dots\end{gathered} \tag{3.6}$$

is valid. When l *is fixed and* $h \to 0$*, the above estimate has the order*

$$\left\|u - \widetilde{u}_{h,l}\right\| = O\big(h^{k+m(l+1)}\big). \tag{3.7}$$

This theorem follows from (2.6), (3.2), from the above lemmas and the fact that the operator $A^{-1}KA^{\frac{1}{2}}$ is bounded.

The case $l = -1$ corresponds to an approximate solution u_h of equation (2.1), whereas in [4] (p. 200) we have

$$\left\|u - u_h\right\| = O(h^k).$$

In particular problems we have to prove the boundedness of the operators $A^{-\frac{1}{2}}K$ and $A^{-1}KA^{\frac{1}{2}}$. The boundedness of $A^{-\frac{1}{2}}K$ follows from (3.1) and the latter have to be proved. We can prove the boundedness of $A^{-1}KA^{\frac{1}{2}}$ as follows: introduce the operator $L \equiv A^{\frac{1}{2}}K - KA^{\frac{1}{2}}$. If we prove that the operator $A^{-1}L$ is bounded, then the operator $A^{-1}KA^{\frac{1}{2}} = A^{-\frac{1}{2}}K - A^{-1}L$ will be bounded as well.

4. Stability

In [6] we can find definitions of stability of the projective-iterative scheme for the second kind equation $(I + T)u = f$, u, $f \in H$. For the j-th cycle, a non-perturbed approximate equation has the form

$$u_n^j + P_nTu_n^j = P_nr_{j-1}, \;\; u_n^j \in H_n, \;\; j = 0, 1, \dots, l, \tag{4.1}$$

$$r_{-1} = f, \;\; u_n^0 = u_n, \;\; r_{j-1} \equiv f - \widetilde{u}_{n,j-1} - T\widetilde{u}_{n,j-1},$$

and a perturbed approximate equation has the form

$$\big(I+P_nT+\Delta_n\big)v_n^j=P_nr_{j-1}+P_n\big(\overline{r}_{j-1}-r_{j-1}\big)+\Delta\big(P_nr_{j-1}\big), \qquad (4.2)$$
$$v_n\in H_n,\quad j=0,1,\dots,l,\quad v_n^0=v_n,$$
$$\overline{r}_{j-1}\equiv f-\widetilde{v}_{n,j-1}-T\widetilde{v}_{n,j-1},\quad \overline{r}_{-1}=r_{-1}=f.$$

To the operator $(I+P_nT):H_n\to H_n$ there corresponds the matrix $B_n\equiv((I+T)\varphi_k,\varphi_i)_{i,k=1}^{(n)}$ and to the operator $\Delta_n:H_n\to H_n$ there corresponds the error matrix $\Gamma_n\equiv(\gamma_{ki})_{i,k=1}^{(n)}$. They are independent of the norm of cycle j.

Here we cite Definition 2 from [6] which concerns the stability.

The projective-iterative method is said to be stable from the space $l_2^{(n)}$ to the space H, if there exist independent of n constants $r>0$, $C_1^{(l)}>0$ and $C_2^{(l)}>0$ such that perturbed equations (4.2) for $\|\Gamma_n\|\le r$ have unique solutions v_n^j, $j=0,1,\dots,l$ and the estimate

$$\big\|\widetilde{v}_{n,l}-\widetilde{u}_{n,l}\big\|_H\le C_1\max_{-1\le k\le l-1}\big\|\overline{\delta}^{(n,k)}\big\|_{l_2^n}+C_2\big\|\Gamma_n\big\|_{l_2^n} \qquad (4.3)$$

is valid; the vector $r^{(n)}\in l_2^{(n)}$, the norm $\|\tau^{(n)}\|=(\sum\limits_{k=1}^{n}\tau_k^2)^{1/2}$, the norm of the matrix $\|\Gamma_n\|\le(\sum\limits_{i,k=1}^{n}\gamma_{ki}^2)^{1/2}$ and $\overline{\delta}^{(n,k)}$ is the error of the scalar products $(\overline{r}_k,\varphi_i),(\overline{r}_k,\varphi_2),\dots,(\overline{r}_k,\varphi_n)$, $k=-1,0,1,\dots,l-1$.

The case $l=0$, $k=-1$ is the stability of the initial projective method without iteration.

Here we quote Theorem 2 from [6] (p. 1040). Uniform linear independence (almost orthonormalization) of the base system $\varphi_1,\varphi_2,\dots$ in H is sufficient for the projective-iterative method to be stable in a sense of Definition 2.

If $\varphi_1,\varphi_2,\dots,\varphi_n$, $n=n(h)$, $\forall\,n\in N$ are uniformly linearly independent, i.e., eigen numbers of symmetric matrices $(\varphi_k,\varphi_i)_{i,k=1}^{(n)}$, $\forall\,n(h)$ satisfy the conditions

$$0<\lambda_0\le\lambda_1^{(n)}\le\cdots\le\lambda_n^{(n)}\le\Lambda_0,$$

then the numbers of conditionality of these matrices is $\varkappa_n=\lambda_n^{(n)}(\lambda_1^{(n)})^{-1}\le\Lambda_0\lambda_0^{-1}$, and vice versa, if numbers of conditionality of matrices $(\varphi_k,\varphi_i)_{i,k=1}^{(n)}$, are $\varkappa_n\le\varkappa$, $\forall\,n(h)$, then the functions $\widetilde{\varphi}_k\equiv(\lambda_1^{(n)})^{-\frac12}\varphi_k$, $k=1,2,\dots,n(h)$, $n(h)\in\mathcal{N}$ are uniformly linearly independent.

Indeed, eigen numbers of the matrix $(\widetilde{\varphi}_k,\widetilde{\varphi}_i)_{i,k=1}$ are bounded below $\overline{\lambda}_k^{(n)}$, $k=1,2,\dots,n(h)$, $\overline{\lambda}_1^{(n)}=\lambda_1^{(n)}(\lambda_1^{(n)})^{-1}=1$ and bounded above $\overline{\lambda}_n^{(n)}=\lambda_n^{(n)}(\lambda_1^{(n)})^{-1}=\varkappa_n\le\varkappa$, $\forall\,n(h)\in\mathcal{N}$.

Normalization of base functions does not change approximate solutions u_n^j, v_n^j, $j=0,1,\dots,l$, it changes only solutions of algebraic systems.

As is mentioned in §1, numbers of conditionality of matrices generated by the operator A in scheme (1.5) are uniformly bounded. The operator $A^{-1}K$ is fully continuous in H. If we take the normalized base functions $\widetilde{\varphi}_k = (l_1^{(n)})^{-\frac{1}{2}}\varphi_k$, $k = 1, 2, \ldots, n$, the conditions of Theorem 2 from [6] will be fulfilled. Therefore the following theorem is valid.

Theorem 2. *The suggested projective-iterative scheme is stable from $l_2^{(n)}$ to H.*

Note that the definition of the stability of the projective method on the basis of strong minimality of the base system in H_A ([7], p. 62) admits the norm perturbation $\|\Gamma_n\| \le r$, $\forall n \in \mathcal{N}$ where $r > 0$ is the fixed number. In an ordinary method of finite elements (scheme (1.6)), eigen numbers of the basic matrices $([\varphi_k, \varphi_i])_{i,k=1}^{(n)}$ are not simultaneously bounded below and above as $n \to \infty (h \to 0)$.

5. Residual Estimate

From (1.1) we have

$$(I + KA^{-1})Au = f, \quad Au \in f, \quad f \in H. \tag{5.1}$$

An approximate solution $\widetilde{u}_h$ satisfies the equation

$$(I + A^{-1}KP_h)\widetilde{u}_h = A^{-1}f,$$

which can be verified directly by means of (2.1) and (2.3). Therefore

$$(I + KP_hA^{-1})A\widetilde{u}_h = f. \tag{5.2}$$

Equations (5.1) and (5.2) yield

$$(I + KP_hA^{-1})(Au - A\widetilde{u}_h) = -KP^hu. \tag{5.3}$$

If the operator $I + KA^{-1}$ is invertible in H and $\|KP^hA^{-1}\| \to 0$ as $h \to 0$, then just in the same way as in (2.6) we get

$$Au - A\widetilde{u}_{h,l} =$$
$$(I + KP_hA^{-1})^{-1}(-KP_hA^{-1}) \cdots (I + KP_hA^{-1})(-KP^hu), \tag{5.4}$$

i.e.,

$$\|Au - A\widetilde{u}_{h,l}\| \le \|(I + KP_hA^{-1})^{-1}\|^{l+1}\|KP^hA^{-1}\|\|\|KP^hu\|, \tag{5.5}$$
$$l = 0, 1, \ldots, (h \le h_0).$$

Let an energetic norm

$$\|\cdot\|_{H_A} \le \widetilde{C}\|\cdot\|_m \tag{5.6}$$

and

$$\|u\|_{2m} \le \widetilde{\widetilde{C}}\|Au\|, \quad u \in D(A). \tag{5.7}$$

Note that estimates (5.6) and (5.7) are fulfilled in elliptic problems when coefficients of the operator A have certain smoothness. Moreover, let

$$\|KP^h u\| \le \|KA^{-\frac{1}{2}}\|\|A^{\frac{1}{2}}P^h u\|. \tag{5.8}$$

Then the following theorem is valid.

Theorem 3. *If the operator $I+KA^{-1}$ is invertible in H, the domain Ω is a P-dimensional cube, the norm of the differential operator is $2m$, the exact solution $u \in W_2^k(\Omega)$, $k \ge 2m$, base functions $\varphi_k \in H_A$, $k = 1,2,\dots,n(h)$, a number of projective-iterative cycles l is fixed, and the mesh pitch $h \to 0$, then the following estimate is valid:*

$$\|Au - A\widetilde{u}_{h,l}\| = O(h^{k+m(l-1)}), \tag{5.9}$$

($k-1$ is the power of the subspace S_h), $l = 0,1,\dots$.

Proof. For $k = 2m$ and $s = m$ estimate (3.2) results in

$$\|A^{-1}g - (A^{-1}g)_I\|_m \le C_m h^{2m-m}\|A^{-1}g\|_{2m}. \tag{5.10}$$

Next, from (3.2) and (5.7), with regard for the inequality

$$\|A^{-1}g - P_h A^{-1}g\|_0 \le \|A^{-1}g - (A^{-1}g)_I\|,$$

we obtain

$$\|(A^{-1}g)_I - P_h A^{-1}g\|_0 \le \|(A^{-1}g)_I - A^{-1}g\|_0 + \|A^{-1}g - P_h A^{-1}g\|_0 \le$$
$$\le 2\|A^{-1}g - (A^{-1}g)_I\| \le 2C_0\widetilde{\widetilde{C}}h^{2m}\|g\|_0. \tag{5.11}$$

The functions u_I, $P_h u \in S_h$ and therefore inequality (2.14) from [8] (p. 37)

$$|u_I - P_h u|^2_{\alpha+1} \le P\cdot 4C^2(k-1)h^{-2}|u_I - P_h u|^2_\alpha,$$
$$1 \le \alpha+1 \le q, \quad C^2(k-1) \equiv \big(C(k-1)\big)^2$$

is valid for these functions. The norm $\|\cdot\|^2_\alpha = \sum\limits_{k=0}^{\alpha} |\cdot|^2_k$, $|\cdot|_k$ is the half-norm. Therefore

$$\|u_I - P_h u\|^2_{\alpha+1} \le \big(h^2 + p4C^2(k-1)\big)h^{-2}\|u_I - P_h u\|^2_\alpha,$$

whence

$$\|u_I - P_h u\|_{\alpha+l} \le \big(h^2 + 4C^2(k-1)p\big)^{l/2}h^{-l}\|u_I - P_h u\|_\alpha,$$

which for $\alpha = 0$, $l = m$ gives

$$\|u_I - P_h u\|_m \le D_m h^{-m}\|u_I - P_h u\|_0, \tag{5.12}$$

where the pitch

$$D_m \equiv (2^{-2} + 4pC^2(k-1))^{\frac{m}{2}}, \quad h \le 2^{-1}.$$

Estimates (5.12) and (5.11) yield

$$\|(A^{-1}g)_I - P_h A^{-1}g\|_m \le D_m \cdot 2C_0\widetilde{\widetilde{C}}h^m\|g\|_0. \tag{5.13}$$

Further, on the basis of (5.6), (5.10) and (5.13) we have

$$\|KP^hA^{-1}g\| \le \|KA^{-\frac{1}{2}}\|\|A^{\frac{1}{2}}P^hA^{-1}g\| \le \|KA^{-\frac{1}{2}}\|\widetilde{C}\|P^hA^{-1}g\|_m \le$$
$$\le \|KA^{-\frac{1}{2}}\|\widetilde{C}\big(\|A^{-1}g-(A^{-1}g)_I\|_m + \|(A^{-1}g)_I - P_hA^{-1}g\|_m\big) \le$$
$$\le \|KA^{-\frac{1}{2}}\|\widetilde{C}(C_mh^m\widetilde{\widetilde{C}} + D_m\cdot 2C_0\widetilde{\widetilde{C}}h^m)\|g\|, \tag{5.14}$$

i.e.,

$$\|KP^hA^{-1}\| \le E_mh^m,$$

where

$$E_m \equiv \|KA^{-1/2}\|\widetilde{C}\widetilde{\widetilde{C}}(C_m + D_m\cdot 2C_0).$$

If the exact solution $u \in W_2^k(\Omega)$, $k \ge 2m$ then again by virtue of (3.2) we have

$$\|u_I - P_hu\|_0 \le \|u-u_I\|_0 + \|u-P_hu\|_0 \le 2C_0h^k\|u\|_k. \tag{5.15}$$

By (5.6),

$$\|KP^hu\| \le \|KA^{-\frac{1}{2}}\|\|A^{\frac{1}{2}}P^hu\| \le \|KA^{-\frac{1}{2}}\|\widetilde{C}\|P^hu\|_m. \tag{5.16}$$

Inequalities (5.12) and (5.15) yield

$$\|u_I - P_hu\|_m \le D_mh^{-m}\cdot 2C_0h^k\|u\|_k. \tag{5.17}$$

Thus we have

$$\|P^hu\|_m \le \|u-u_I\|_m + \|u_I - P_hu\|_m. \tag{5.18}$$

On the basis of (5.18), (3.2) and (5.17), inequality (5.16) leads to

$$\|KP^hu\| \le F_mh^{k-m}\|u\|_k, \tag{5.19}$$

where

$$F_m \equiv \|KA^{-\frac{1}{2}}\|\widetilde{C}(C_m + 2D_mC_0).$$

Finally, by virtue of (5.14) and (5.19), from inequality (5.5) we obtain estimate (5.9). Thus Theorem 3 is proved. □

For the residual we have

$$\|f - A\widetilde{u}_{h,l} - K\widetilde{u}_{h,l}\| \le (I + \|KA^{-1}\|)\|Au - A\widetilde{u}_{h,l}\| = O(h^{k+m(l+1)}).$$

6. The Uniform Estimate

From the known multiplicative inequalities and embedding theorems we obtain the following inequality ([9], p. 46):

$$\|v\|_{C(\overline{\Omega})} \le C\|v\|_0^{1-\frac{p}{4m}}\|v\|_{2m}^{\frac{p}{4m}}, \quad \forall\, v \in W_2^{2m}(\Omega), \tag{6.1}$$

here $4m > p$, p is dimension of the domain Ω, $2m$ is the order of the differential operator A, $\|\cdot\|_0 \equiv \|\cdot\|_{L_2(\Omega)}$, $\|\cdot\|_{2m} \equiv \|\cdot\|^{2m}_{W_{2(\Omega)}}$.

Theorem 4. *In the conditions of Theorems* 1 *and* 3 *the uniform estimate*

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O\left(h^{k+m(l+1)-\frac{p}{2}}\right) \tag{6.2}$$

is valid, where $(k-1)$ *is power of the subspace of base functions,* l *is a number of projective-iterative cycles,* p *is dimension of the cube* Ω*, the exact solution* $u\in W_2^k(\Omega)$*,* $k\geq 2m$*,* $4m>p$*,* $l=0,1,\dots,$ *is fixed,* $h\to 0$*.*

Estimate (6.2) follows from (6.1) by virtue of estimates (3.7) and (5.9).

Corollaries of Theorem 4:

(1) if A is the second order differential operator, K is the first order differential operator, $\overline{\Omega}=[0,1]$, as the base system are taken piecewise linear finite functions ($m=1$, $p=1$, $k=2$), and the exact solution $u\in W_2^2(0,1)$, then

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O(h^{2,5+l}),\quad l=0,1,\dots,$$

l is fixed, $h\to 0$, for $\overline{\Omega}=[0,1]\times[0,1]$ ($p=2$)

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O(h^{2+l}),$$

for $\overline{\Omega}=[0,1]\times[0,1]\times[0,1]$ ($p=3$)

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O(h^{\frac{3}{2}+l});$$

(2) if as the base system are taken piecewise cubic Hermitian finite functions ($k=4$), then for $u\in W_2^2(\Omega)$, for $p=1$

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O(h^{4,5+l}),$$

for $p=2$

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O(h^{4+l}),$$

for $p=3$

$$\left\|u-\overline{u}_{h,l}\right\|_{C(\overline{\Omega})}=O(h^{3,5+l}).$$

7. Numerical Realization

Let us consider the boundary value problem

$$-u''(x)+p_1(x)u'(x)+p_2(x)u(x)=f(x),\quad 0<x<1, \tag{7.1}$$
$$u(0)=u(1)=0.$$

We take the space $H\equiv L_2(0,1)$. The operator $Au\equiv -u''(x)$, $u(0)=u(1)=0$, and the operator $Ku\equiv p_1u'+p_2u$. The scalar product in the energetic space H_A is $[u,v]=\int_0^1 u'v'dx$.

Let p_1', $p_2\in C[0,1]$. For $\forall\, u$, $v\in H_A$,

$$(Ku,v)=(p_1u'+p_2u'',v)=(-p_1v'+(p_2-p_1')v,u),$$

i.e.,

$$K^*v=-p_1v'+(p_2-p_1')v$$

(u and v satisfy the boundary conditions).

Further,

$$\|K^*v\| = \left[\left(\overline{p}_2 + \overline{p}_1'\right) \cdot 2^{-\frac{1}{2}}\right] \|v'\|, \quad \forall\, v \in H_A,$$

where $\overline{p}_1 \equiv \max\limits_{x\in[0,1]} |p(x)|$ are analogous to $\overline{p}_2$ and $\overline{p}_1'$. Therefore the operators $K^*A^{-\frac{1}{2}}$ and $A^{-\frac{1}{2}}K$ are bounded.

Green's function of the operator $Au = -u''(x)$, $u(0) = u(1) = 0$,

$$G(x,t) = \begin{cases} (1-x)t, & t \le x, \\ x(1-t), & t \ge x, \quad x,t \in [0,1]. \end{cases}$$

Introduce the operator

$$Lv \equiv -KA^{\frac{1}{2}}v + A^{\frac{1}{2}}Kv, \quad \forall\, v \in D(A).$$

The operator

$$A^{1/2}v = -iv', \quad Av = A^{1/2}(A^{1/2}v) = -v'', \quad Lv = A$$

We have

$$A^{-1}Lv = \int_0^1 G(x,t)\big[-i(p_1'v' + p_2'v)\big]\,dt.$$

Taking into account that $G_t'(x,t)$ is discontinous of the first order for $x(t)$, $G(x,t)$ is continuous; they are bounded (almost everywhere) by the number 1, and hence we have

$$\|A^{-1}Lv\| \le (\overline{p}_1' + \overline{p}_1'' + \overline{p}_1')\|v\|, \quad \forall\, v \in D(A). \tag{7.2}$$

Therefore we find that the operator $A^{-1}KA^{1/2}$ is bounded for p_1'', $p_2' \in C[0,1]$.

Now we take the uniform mesh $h = \frac{1}{n}$ and the piecewise linear finite functions

$$\varphi_k^{(x)} \equiv \varphi_k^h = \begin{cases} \varphi_k^{(1)} = h^{-1}(x - x_{k-1}), & x \in [x_{k-1}, x_k], \\ \varphi_k^{(2)} = h^{-1}(x_{k+1} - x), & x \in [x_k, x_{k+1}], \\ 0, & x \,\overline{\in}[x_{k-1}, x_{k+1}]. \end{cases}$$

We need the functions

$$A^{-1}\varphi_k = \int_0^1 G(x,t)\varphi_k(t)\,dt = \int_{x_{k-1}}^{x_k} G(x,t)\varphi_k^{(1)}(t)\,dt +$$

$$+ \int_{x_k}^{x_{k+1}} G(x,t)\varphi_k^{(2)}(2)\,dt, \quad k = 1,2,\ldots,\ n-1, \quad n = h^{-1}. \tag{7.3}$$

Thus

(1) for $x \le x_{k-1}$,

$$A^{-1}\varphi_k = x\int\limits_{x_{k-1}}^{x_k}(1-t)h^{-1}(t-x_{k-1})\,dt + x\int\limits_{x_k}^{x_{k+1}}(1-t)h^{-1}(x_{k+1}-t)\,dt,$$

(2) for $x \in [x_{k-1}, x_k]$,

$$A^{-1}\varphi_k = (1-x)\int\limits_{x_{k-1}}^{x} th^{-1}(t-x_{k-1})\,dt +$$

$$+x\int\limits_{x}^{x_k}(1-t)h^{-1}(t-x_{k+1})\,dt + x\int\limits_{x_k}^{x_{k+1}}(1-t)h^{-1}(x_{k+1}-t)\,dt,$$

(3) for $x \in [x_k, x_{k+1}]$,

$$A^{-1}\varphi_k = (1-x)\int\limits_{x_{k-1}}^{x_k} th^{-1}(t-x_{k-1})\,dt +$$

$$+(1-x)\int\limits_{x_k}^{x} th^{-1}(x_{k+1}-t)\,dt + x\int\limits_{x}^{x_{k+1}}(1-t)h^{-1}(x_{k+1}-t)\,dt,$$

(4) for $x \ge x_{k+1}$,

$$A^{-1}\varphi_k = (1-x)\int\limits_{x_{k-1}}^{x_k} th^{-1}(t-x_{k-1})\,dt + (1-x)\int\limits_{x_k}^{x_{k+1}} th^{-1}(x_{k+1}-t)\,dt.$$

Our calculations show that

(1) for $x \le x_{k-1}$,

$$A^{-1}\varphi_k = x\big[h^2(-k)+h\big],$$

(2) for $x \in [x_{k-1}, x_k]$,

$$A^{-1}\varphi_k =$$

$$= h^{-1}\bigg\{-\frac{x^3}{6} + \frac{x_{k+1}}{2}x^2 + x\Big[h^3(-k) + \frac{h^2}{2}(-k^2+2k+1)\Big] + \frac{1}{6}x_{k-1}^3\bigg\},$$

(3) for $x \in [x_k, x_{k+1}]$,

$$A^{-1}\varphi_k =$$

$$= h^{-1}\bigg\{\frac{x^3}{6} - \frac{x_{k+1}}{2}x^2 + x\Big[h^3(-k) + \frac{h^2(k+1)^2}{2}\Big] - \frac{1}{3}h^3k^3 + \frac{1}{6}h^3(k-1)^3\bigg\},$$

(4) for $x \ge x_{k+1}$,

$$A^{-1}\varphi_k = (1-x)h^2k.$$

Next we consider a particular type of the operator K, $Ku = xu' + u$. The right-hand side $f = -4x^3 + 3x^2 + 6x - 2$ and the exact unique solution $u = x^2 - x^3$.

We have

$$\varphi_k' = \begin{cases} h^{-1}, & x \in]x_{k-1}, x_k[\,, \\ -h^{-1}, & x \in]x_k, x_{k+1}[\,, \\ 0, & x \overline{\in} [x_{k-1}, x_{k+1}]. \end{cases}$$

For $A^{-1}t\varphi_k'$, $k = 1, 2, \dots, n-1$, similarly to $A^{-1}\varphi_k$, we find that

(1) for $x \le x_{k-1}$,

$$A^{-1}t\varphi_k' = x(-h + h^2 \cdot 2k),$$

(2) for $x \in [x_{k-1}, x_k]$,

$$A^{-1}t\varphi_k' = -\frac{h^{-1}}{6}x^3 + x\Big[h^2 \cdot 2k + \frac{h}{2}(k^2 - 2k - 1)\Big] - \frac{h^{-1}}{3}x_{k-1}^3,$$

(3) for $x \ge x_{k+1}$,

$$A^{-1}t\varphi_k' = \frac{h^{-1}}{6}x^3 + x\Big[h^2 \cdot 2k - \frac{h}{2}(k+1)^2\Big] + \frac{h^2}{3}(k^3 + 3k^2 - 3k + 1),$$

(4) for $x \ge x_{k+1}$,

$$A^{-1}t\varphi_k' = (1 - x)h^2(-2k).$$

The expression

$$A^{-1}f = \frac{x^5}{5} - \frac{x^4}{4} - x^3 + x^2 + \frac{x}{20}.$$

We take the pitch $h = \frac{1}{4}$. Then:

$$A^{-1}\varphi_1 = \begin{cases} -\frac{2}{3}x^3 + \frac{3}{16}x, & x \in [0, h], \\ \frac{2}{3}x^3 - x^2 + \frac{7}{16}x - \frac{1}{48}, & x \in [h, 2h], \\ \frac{1}{16} - \frac{1}{16}x, & x \in [2h, 1], \end{cases}$$

$$A^{-1}\varphi_2 = \begin{cases} \frac{1}{8}x, & x \in [0, h], \\ -\frac{2}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{96}, & x \in [h, 2h], \\ \frac{2}{3}x^3 - \frac{3}{2}x^2 + x - \frac{5}{32}, & x \in [2h, 3h], \\ \frac{1}{8} - \frac{1}{8}x, & x \in [3h, 1]. \end{cases}$$

$$A^{-1}\varphi_3 = \begin{cases} \frac{1}{16}x, & x \in [0,2h], \\ -\frac{2}{3}x^3 + x^2 - \frac{7}{16}x + \frac{1}{12}, & x \in [2h,3h], \\ \frac{2}{3}x^3 - 3x^2 + \frac{29}{16}x - \frac{23}{48}, & x \in [3h,1]3 \end{cases}$$

$$A^{-1}t\varphi_1' = \begin{cases} -\frac{2}{3}x^3 - \frac{x}{8}, & x \in [0,h], \\ \frac{2}{3}x^3 - \frac{3}{8}x + \frac{1}{24}, & x \in [h,2h], \\ \frac{1}{8}x - \frac{1}{8}, & x \in [2h,1], \end{cases}$$

$$A^{-1}t\varphi_2' = \begin{cases} 0, & x \in [0,h], \\ -\frac{2}{3}x^3 + \frac{1}{8}x - \frac{1}{48}, & x \in [h,2h], \\ \frac{2}{3}x^3 - \frac{7}{8}x + \frac{5}{16}, & x \in [2h,3h], \\ \frac{1}{4}x - \frac{1}{4}, & x \in [3h,1], \end{cases}$$

$$A^{-1}t\varphi_3' = \begin{cases} \frac{1}{8}x, & x \in [0,2h], \\ -\frac{2}{3}x^3 + \frac{5}{8}x - \frac{1}{6}, & x \in [2h,3h], \\ \frac{2}{3}x^3 - \frac{13}{8}x + \frac{23}{24}, & x \in [3h,1]. \end{cases}$$

To construct the algebraic system (1.5) we shall need the following matrices:

$$M_1 \equiv (\varphi_k, \varphi_i)_{i,k=1}^{(3)}, \quad M_2 \equiv (t\varphi_k', A^{-1}\varphi_i)_{i,k=1}^{(3)} = (A^{-1}t\varphi_k', \varphi_i)_{i,k=1}^{(3)},$$
$$M_3 \equiv (A^{-1}\varphi_k, \varphi_i)_{i,k=1}^{(3)}.$$

The matrix M_1 is known ([2], p. 104). We calculate the matrices M_2 and M_3 and obtain

$$M_1 = \begin{pmatrix} \frac{1}{6} & \frac{1}{24} & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & \frac{1}{24} & \frac{1}{6} \end{pmatrix}, \quad M_2 = \begin{pmatrix} -\frac{13}{5\cdot 16^2} & -\frac{1}{5\cdot 16^2} & \frac{1}{8\cdot 16} \\ -\frac{111}{30\cdot 16^2} & -\frac{13}{5\cdot 16^2} & \frac{109}{30\cdot 16^2} \\ -\frac{1}{8\cdot 16} & -\frac{53}{15\cdot 16^2} & \frac{7}{5\cdot 16^2} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \dfrac{31}{15\cdot 16^2} & \dfrac{59}{30\cdot 16^2} & \dfrac{1}{16^2} \\ \dfrac{59}{30\cdot 16^2} & \dfrac{46}{15\cdot 16^2} & \dfrac{59}{30\cdot 16^2} \\ \dfrac{1}{16^2} & \dfrac{59}{30\cdot 16^2} & \dfrac{31}{15\cdot 16^2} \end{pmatrix},$$

The matrix $M \equiv M_1 + M_2 + M_3$,

$$M = \begin{pmatrix} \dfrac{79}{30\cdot 16} & \dfrac{373}{30\cdot 16^2} & \dfrac{3}{16^2} \\ \dfrac{134}{15\cdot 16^2} & \dfrac{647}{15\cdot 16^2} & \dfrac{244}{15\cdot 16^2} \\ -\dfrac{1}{16} & \dfrac{273}{30\cdot 16^2} & \dfrac{692}{15\cdot 16^2} \end{pmatrix},$$

The right-hand side of the algebraic system (1.5)

$$b^{(3)} \equiv (b_1, b_2, b_3)^T, \quad b_i = (A^{-1}f, \varphi_i), \quad i = 1,2,3,$$
$$b_1 = 0,015137, \ \ b_2 = 0,033545, \ \ b_3 = 0,033396.$$

All calculations are performed to within the accuracy 10^{-6}.
The algebraic system (1.5) is written in the form

$$Ma^{(3)} = b^{(3)} \tag{7.4}$$

its solutions

$$a_1 = 0,042240, \quad a_2 = 0,129767, \quad a_3 = 0,160635.$$

By the method of Galerkin-Petrov, the third approximation

$$u_3(x) = \sum_{k=1}^{3} a_k \varphi_k(x). \tag{7.5}$$

One iteration

$$\widetilde{u}_3 = -A^{-1}Ku_3 + A^{-1}f = -\sum_{k=1}^{3} a_k(A^{-1}t\varphi_k' + A^{-1}\varphi_k) + A^{-1}f.$$

Taking into account the expressions $A^{-1}\varphi_k$, $A^{-1}t\varphi_k'$, $k = 1,2,3$ we obtain
(1) for $x \in [0,h]$,

$$A^{-1}Ku_3 = a_1\Big(-\frac{4}{3}x^3 + \frac{1}{16}x\Big) + a_2 \cdot \frac{x}{8} + a_3 \cdot \frac{3x}{16};$$

(2) for $x \in [h, 2h]$,

$$A^{-1}Ku_3 =$$
$$= a_1\Big(\frac{4}{3}x^3 - x^2 + \frac{1}{16}x + \frac{1}{48}\Big) + a_2\Big(-\frac{4}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{8}x - \frac{1}{96}\Big) + a_3 \cdot \frac{3x}{16};$$

(3) for $x \in [2h, 3h]$,

$$A^{-1}Ku_3 = a_1\Big(\frac{1}{16}x - \frac{1}{16}\Big) + a_2\Big(\frac{4}{3}x^3 - \frac{3}{2}x^2 + \frac{1}{8}x + \frac{5}{32}\Big) + \\ + a_3\Big(- \frac{4}{3}x^3 + x^2 + \frac{3}{16}x - \frac{1}{2}\Big),$$

(4) for $x \in [3h, 1]$,

$$A^{-1}Ku_3 = \\ = a_1\Big(\frac{1}{16}x - \frac{1}{16}\Big) + a_2\Big(\frac{1}{8}x - \frac{1}{8}\Big) + a_3\Big(\frac{4}{3}x^3 - 2x^2 + \frac{3}{16}x + \frac{23}{48}\Big).$$

Finally,

$$\widetilde{u}_3 = -A^{-1}Ku_3 + A^{-1}f, \tag{7.6}$$

where $A^{-1}Ku_3$ are defined above; it depends on the interval $[x_{k-1}, x_k]$, $k = 1, 2, 3, 4$, and the polynomial A^{-1} is added. Expressions (7.5) and (7.6) for $u_3(x)$ and $\widetilde{u}_3(x)$ depend on the interval. This naturally belongs to the ordinary method of finite elements.

The order of convergence in $L_2(0, 1)$ is

$$\|u - u_h\| = O(h^2), \quad \|u - \widetilde{u}_h\| = O(h^3).$$

If instead of $\{\varphi_k\}$ in the capacity of the base system we take $\{\widehat{\varphi}_k\}$, where $\widehat{\varphi}_k = h^{-1/2}$, then approximate solutions u_3 and $\widehat{u}_3$ do not vary. Instead of the matrix M_1 we have $h^{-1}M_1$ and analogously $h^{-1}M_2$ and $h^{-1}M_3$; in the right-hand side we have $h^{-1/2}b^{(3)}$ and hence there take place their perturbations. The system of elements $\{\widehat{\varphi}_k\}$ is uniformly linearly independent, and the condition of the stability of the projective-iterative method is fulfilled.

In the table below, in the discrete points x_i we present the values of the exact solution u, an approximate solution u_3, an approximate solution with one iteration $\widetilde{u}_3$, and the errors $u - u_3$ and $u - \widetilde{u}_3$. At the end of the interval they are zeros.

In the norm $L_2(0, 1)$ the relative errors are

$$\frac{\|u - u_3\|}{\|u\|} \approx 9,9\% \qquad \frac{\|u - \widetilde{u}_3\|}{\|u\|} \approx 0,6\%.$$

x_i	u	u_3	$\widetilde{u}_3$	$u-u_3$	$u-\widetilde{u}_3$
1/8	0,013672	0,021120	0,013854	−0,007448	−0,000182
1/4	0,046875	0,042240	0,047229	0,004635	−0,000354
3/8	0,087891	0,086004	0,088254	0,001887	−0,000363
1/2	0,125000	0,129767	0,125533	0,004767	−0,000533
5/8	0,146484	0,145202	0,147134	0,001282	−0,000650
3/4	0,140625	0,160635	0,141996	−0,020010	−0,001371
7/8	0,095703	0,080318	0,095756	0,015385	−0,000053

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