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Semi-abelian categories

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Dedicated in great admiration to Saunders Mac Lane at the occasion of his 90th birthday

Abstract

The notion of semi-abelian category as proposed in this paper is designed to capture typical algebraic properties valid for groups, rings and algebras, say, just as abelian categories allow for a generalized treatment of abelian-group and module theory. In modern terms, semi-abelian categories are exact in the sense of Barr and protomodular in the sense of Bourn and have finite coproducts and a zero object. We show how these conditions relate to "old" exactness axioms involving normal monomorphisms and epimorphisms, as used in the fifties and sixties, and we give extensive references to the literature in order to indicate why semi-abelian categories provide an appropriate notion to establish the isomorphism and decomposition theorems of group theory, to pursue general radical theory of rings, and how to arrive at basic statements as needed in homological algebra of groups and similar non-abelian structures.

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1. Introduction

1.1. Perhaps the shortest way of defining a category $\mathscr{C}$ to be *abelian* is to require that (see, for example [28, 1.597]; [38, 41.A]).

(A*1) $\mathscr{C}$ has finite products, and a zero object,
(A*2) $\mathscr{C}$ has (normal epi, normal mono)-factorizations, i.e., every morphism in $\mathscr{C}$ factors into a cokernel followed by a kernel.

Among the many consequences of these two powerful conditions, we note here only that $\mathscr{C}$ must be *additive*, and that therefore the finite products are biproducts, i.e., serve also as coproducts. It is fair to say that the study of abelian categories dominated the first two decades of Category Theory; indeed, it figures prominently in many important papers and monographs of the time—see, for example, [16,21,34,37,31,54,27,58,15,55]. (Here and below all references are given in chronological order.) After the name "abelian (bi)category" had been used by Mac Lane [53] to denote a more restrictive concept (which involved a predecessor of Grothendieck's famous AB5 axiom), the notion in today's sense appeared first in [16], but under the name "exact category" (which later was reserved for categories satisfying just (A*2)).

Of course, *the* "role model" for all abelian categories is the category of abelian groups (a statement that has been made precise in [28, 1.59], while the category of (not necessarily abelian) groups is painfully non-abelian (since not all monomorphisms are normal). However, when reading once again from [53, p. 507] that

"A further development giving the first and second isomorphism theorem, and so on, can be made by introducing additional carefully chosen axioms. This will be done below only in the more symmetrical abelian case",

we are reminded that, right from the beginning of Category Theory, it was very much the intention to *find a list of axioms which reflect the properties of groups*, *rings and algebras as nicely as the abelian-category axioms do for abelian groups and modules*. This is the theme of this paper.

1.2. There seems to be no easy way of weakening (A*1), (A*2) and arriving at such a list. Indeed, there have been many proposals for axioms in order to give a categorical approach to the isomorphism and decomposition theorems of group theory, to the general theory of radicals, and to homological algebra of non-abelian structures, but no generally accepted list of axioms emerged from these investigations. Nevertheless, we find it useful to recall several of these early developments, especially since some of them seem to have been forgotten, if not ignored from the outset, but still have considerable bearing on this paper.

Let us recall then that one important line of early categorical research grew out of the desire to establish isomorphism and decomposition theorems for general varieties of universal algebras and then for categories satisfying certain axioms (see particularly [3,33,2,39,51,40,22,66], ultimately leading to rather efficient lists of the needed categorical hypotheses, as presented in [63] and especially in [67,29]. Another line of categorical research was marked by Amitsur [1] and Shul'geifer [62] which led to rather concise lists of axioms for categorical radical theory, as presented in [42] and, more compactly, in [56]. Thirdly, we mention a group of early categorical papers directed at non-abelian homological algebra, including the little-known articles [41,30], which were used later in [43] to deal abstractly with commutators, nilpotency, and solvability, and in [7,8] with primary ideals. Better known is the important paper [32] which introduces a set of axioms suitable for general Baer extension theory (see also [59]). Other than the existence of certain limits and colimits, the axioms given in each of these papers require "good behaviour" of normal epi- and monomorphisms. In this paper, we generally refer to these types of requirements as to *old* (*-style*) *axioms*.

1.3. A distinctly new era began with [4]. *Barr-exact* categories no longer require the existence of a zero object and replace normal epimorphisms by regular epimorphisms, which had been studied systematically in [49]. Their subtle exactness condition (that equivalence relations be effective) is satisfied by *all* varieties of universal algebras; yet, Barr's notion is strong enough to satisfy Tierney's "equation" for pointed categories:

$$(\text{Barr-exact}) + (\text{additive}) = (\text{abelian}).$$

But their generality also means that Barr-exact categories are not restrictive enough to capture typical properties which would distinguish groups, rings and algebras from pointed sets, monoids and lattices, say. Universal algebraists have therefore pointed at the modularity of the congruence lattices in group-like varieties. This property is usually deduced from the stronger property that every reflexive (homomorphic) relation is already an equivalence relation, a condition which is equivalent to congruence permutability and which defines *Malcev varieties*. Among many other things, they allow for a satisfactory commutator theory (see [64]). For their categorical generalization, the Reader is referred to [19], which combines and elaborates on crucial observations by Klein [50], Meisen [57], Fay [24], Fay [25], Burgess–Caicado [17], Johnstone [47], Faro [23], Carboni–Lambek–Pedicchio [20], and others. The paper convincingly establishes the notion of *Malcev category* and characterizes *Barr-exact Malcev categories*. Commutator theory was extended from Malcev varieties to congruence-modular varieties in [36] (see also [35,26], and, for recent developments in the non-modular case, [48]) and then treated categorically in [60,61,45]; however, a good categorical definition appeared already in [43], under general conditions close to ours.

1.4. The notion of semi-abelian category as proposed in this paper is slightly stronger than that of a Barr-exact Malcev category which seems to suit, in many aspects, the needs of universal algebra perfectly, but those of homological algebra of group- and ring-like structures much less so. It combines Barr's exactness property with a crucial property that Mac Lane at the very end of [53] calls the *ABC extension equivalence theorem* and which we formulate here equivalently as the:

Short Five Lemma. *For every commutative diagram*

$$\begin{array}{ccccc} L & \xrightarrow{l} & F & \xrightarrow{q} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} \\ K & \xrightarrow{k} & E & \xrightarrow{p} & B \end{array} \qquad (1)$$

with regular epimorphisms p, q *and* k, l *their kernels*, *respectively*, w *is an isomorphism if* u *and* v *are isomorphisms*.

In fact, in the presence of Barr-exactness it suffices to require this property just for split epimorphisms, and it can be formulated without reference to a zero object and kernels, just using pullbacks. This then is Bourn's elegant notion of *protomodular category* (see [9]), a term coined after Carboni's modular categories [18], although the connection with the Short Five Lemma is also emphasized in Bourn's paper. Despite the fact of being a very elementary concept, protomodularity provides very powerful group-like tools. For example, in a Barr-exact category with pushouts of split monomorphisms it is equivalent to the existence of *semi-direct products* (see [13]), with the latter property also referred to as *semi-additivity* when $\mathscr{C}$ has finite coproducts and a zero object. Hence, when we define a category to be *semi-abelian* if it is Barr-exact and Bourn-protomodular with finite coproducts and a zero object, there is (almost by definition) the "equation"

$$(\text{Barr-exact}) + (\text{semi-additive}) = (\text{semi-abelian})$$

for pointed categories. Unlike abelianness, semi-abelianness is of course not self-dual; however, the conjunction with the dual concept is easily seen to give abelianness:

$$(\text{semi-abelian}) + (\text{semi-abelian})^{\text{op}} = (\text{abelian}).$$

These facts are presented in Sections 2 and 4 of the paper.

1.5. The main part of this paper is Section 3 where we describe the new notion of semi-abelian category (which grows out of Barr-exactness and Bourn-protomodularity) with old-style axioms, in terms of normal monomorphisms and normal epimorphisms. Apart from Mac Lane's original work, the "old" counterpart of protomodularity seems to appear as an axiom first in the practically unknown paper [41], in the following form:

Hofmann's Axiom. For every commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{q} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow{p} & B \end{array} \qquad (2)$$

with normal epimorphisms p, q and monomorphisms v, w, the monomorphism w is normal if v is normal and $\ker p \leqslant w$.

The only other old axioms needed to characterize semi-abelian categories concern the existence of finite limits and colimits as well as of images and inverse images of (normal) subobjects. In this context, Barr's exactness condition is hidden under the slogan that images of normal subobjects under normal epimorphisms are normal.

1.6. We finally wish to point out that all axioms considered in this paper are of *first-order type*, whereas many "old papers" contain second-order requirements (such as "the subobjects of each object form a set"). Nevertheless, the references given in Section 1 show that semi-abelian categories provide a good foundation for a meaningful categorical treatment of

- isomorphism and decomposition theorems,
- radical and commutator theory,
- homology theory of non-abelian structures,

and therefore seem to capture precisely Mac Lane's fundamental ideas of half a century ago. Further evidence for this has been provided by Bourn in two recent preprints: see [11,12].

2. Semi-abelian categories in terms of "new" axioms

2.1. For a category $\mathscr{C}$ to be *Barr-exact* or *effective regular* (see, for example, [28,66]) one usually postulates that

(Ex1) $\mathscr{C}$ has finite limits,
(Ex2) $\mathscr{C}$ has a pullback-stable (regular epi, mono)-factorization system,
(Ex3) all equivalence relations in $\mathscr{C}$ are effective;

conditions (Ex1), (Ex2) make $\mathscr{C}$ *regular*. Axiom (Ex1) is somewhat arbitrary; indeed, Barr himself did not require it (see [4]).

Recall that *regular epimorphisms* are morphisms occurring as coequalizers of pairs of parallel morphisms; *pullback stability* means that every pullback of a (regular epi, mono)-factorization is again such a factorization. An *equivalence relation* on an object A in $\mathscr{C}$ is given by a pair of morphisms $r_1, r_2 : R \to A$ for which the maps $\hom(X, r_1)$, $\hom(X, r_2)$ are (up to isomorphism) the projections of an equivalence relation on the set $\hom(X, A)$, for every object X in $\mathscr{C}$ (see [5,6]); it is *effective* if it is *induced* by some morphism $f : A \to B$, i.e., if it is the kernelpair of some f.

For a (regular epi, mono)-factorization $f = m \cdot e$ in any category $\mathscr{C}$, the morphism e is necessarily the coequalizer of the kernelpair of f, provided that the latter exists. Conversely, letting e be the coequalizer of the kernelpair of f and m be the morphism with $f = m \cdot e$, then m is certainly a monomorphism if all pullbacks of e are epic, since then the kernelpair of m must be trivial.

Taking into account also the well-known fact that the diagrams

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array} \qquad\qquad \begin{array}{ccc} & & A\times B \\ & & \downarrow{\scriptstyle 1_A\times\langle g,1_B\rangle} \\ A\times B & \xrightarrow{\langle l_A,f\rangle\times 1_B} & A\times C\times B \end{array} \tag{3}$$

have isomorphic limits, we arrive at the following equivalent formulation of (Ex1), (Ex2):

(Ex1a) $\mathscr{C}$ has finite products,
(Ex1b) $\mathscr{C}$ has pullbacks of pairs of (split) monomorphisms,
(Ex2a) $\mathscr{C}$ has coequalizers of kernelpairs,
(Ex2b) regular epimorphisms are stable under pullback.

We mention that in (Ex2) and (Ex2b), regular epimorphisms may be traded for strong ones, i.e., those which are orthogonal to monomorphisms (see [65]). However, the formally weaker strong epimorphisms (even extremal ones, see 2.4 below) then turn out to be regular.

2.2. For a category $\mathscr{C}$ with a terminal object 1, put $\mathrm{Pt}\mathscr{C} = (1 \downarrow \mathscr{C})$. For an object B in any category $\mathscr{C}$, call $\mathrm{Pt}_{\mathscr{C}}(B) = \mathrm{Pt}(\mathscr{C} \downarrow B)$ the *category of points of B in* $\mathscr{C}$; its objects are triples (E, p, s) with morphisms $p : E \to B$, $s : B \to E$ in $\mathscr{C}$ with $p \cdot s = 1_B$, and a morphism $f : (E, p, s) \to (E', p', s')$ in $\mathrm{Pt}_{\mathscr{C}}(B)$ satisfies $p' \cdot f = p$, $f \cdot s = s'$.

A category $\mathscr{C}$ is (*Bourn-*)*protomodular* (see [9]) if $\mathscr{C}$ has pullbacks and if

(PM) for every morphism $v : C \to B$ in $\mathscr{C}$, the pullback functor $v^* : \mathrm{Pt}_{\mathscr{C}}(B) \to \mathrm{Pt}_{\mathscr{C}}(C)$ (which pulls back p of (E, p, s) along v) reflects isomorphisms.

It is easy to see that in the presence of pullbacks, (PM) is equivalent to the condition that in every commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad r \quad} & D \\ {\scriptstyle y}\downarrow & \boxed{1} & \downarrow{\scriptstyle x} \\ F & \xrightarrow{\quad q \quad} & C \\ {\scriptstyle w}\downarrow & \boxed{2} & \downarrow{\scriptstyle v} \\ E & \xrightarrow{\quad p \quad} & B \end{array}$$

for which both $\boxed{1}$ and $\boxed{\substack{1\\2}}$ are pullback diagrams, also $\boxed{2}$ is a pullback diagram, provided that q is a split epimorphism (just consider the pullback $E \times_B C$; see Proposition 8 of [9]). As shown in [13], the provision may be changed to the requirement that q be a regular epimorphism and p an effective descent morphism in $\mathscr{C}$. If $\mathscr{C}$ is Barr-exact, so that every regular epimorphism is effective for descent (see [46]), it suffices to let p, q be regular epimorphisms.

2.3. In the presence of a zero object 0 in $\mathscr{C}$, condition (PM) may be simplified further. Then it suffices to consider the morphism $i_B : 0 \to B$ instead of an arbitrary morphism $v : C \to B$ (since $i_B = v \cdot i_C$, so that reflection of isomorphisms by $i_B^* = i_C^* \cdot v^*$ implies the same for v^*). Pulling back $p : E \to B$ along i_B is taking the kernel of p; hence, (PM) now becomes:

(PM_0) for every object B in $\mathscr{C}$, the kernel functor $\ker_B : \mathrm{Pt}_{\mathscr{C}}(B) \to \mathscr{C}$ ($\cong \mathrm{Pt}_{\mathscr{C}}(0)$) reflects isomorphisms.

More elaborately, (PM_0) means that the *Split Short Five Lemma* holds true in $\mathscr{C}$: consider any commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\ l\ } & F & \xrightarrow{\ q\ } & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} & \boxed{2} & \downarrow{\scriptstyle v} \\ K & \xrightarrow{\ k\ } & E & \xrightarrow{\ p\ } & B \end{array}$$

in $\mathscr{C}$ with $k = \ker p$ and $l = \ker q$; then w is an isomorphism if u, v are isomorphisms and p, q are split epimorphisms (see [9]). Equivalently: $\boxed{2}$ is a pullback diagram, if u is an isomorphism and q is a split epimorphism; or if u is an isomorphism, q a regular epimorphism and p an effective descent morphism (see [13]). Hence, if $\mathscr{C}$ is Barr-exact with a zero object, (PM_0) *is equivalent to the Short Five Lemma* as stated in Section 1, as well as to the pullback cancellation property discussed in 2.2, where now just q (not necessarily p) is assumed to be a regular epimorphism.

2.4. If $\mathscr{C}$ has binary *sums* (= coproducts), in addition to pullbacks and a zero object, then the functor $\ker_B : \mathrm{Pt}_{\mathscr{C}}(B) \to \mathscr{C}$ of 2.3 has a left adjoint, given by

$$K \to \left(K + B, \begin{pmatrix} 0 \\ 1_B \end{pmatrix}, \text{coproduct injection} \right).$$

Consequently, $\ker_B$ reflects isomorphisms if and only if the counits of this adjunction do not factor through proper subobjects of their codomains; since $\mathscr{C}$ has equalizers, this simply means that the counits are *extremal epimorphisms*.

Calling a diagram

$$K \xrightarrow{k} E \underset{s}{\overset{p}{\rightleftarrows}} B, \tag{6}$$

with $k = \ker p$ and $p \cdot s = 1_B$ a *split short exact sequence*, we now see that (PM) is equivalent to:

(PM_0^+) for every split short exact sequence (6), the morphism $\begin{pmatrix} k \\ s \end{pmatrix} : K + B \to E$ is an extremal epimorphism.

Note that if $\mathscr{C}$ is regular, "extremal" is equivalent to "regular". Analysing this condition further, one obtains:

Proposition. *A category $\mathscr{C}$ with pullbacks, binary sums and a zero object is protomodular if and only if*:

(PM_0^+1) *for every split short exact sequence* (6), *the morphism* $\binom{k}{s}: K+B \to E$ *is an epimorphism*,

(PM_0^+2) *for every commutative diagram* (2) *with monomorphisms w, v and* $\ker p \leqslant w$, *regularity of v implies regularity of w, provided that q is a split epimorphism*.

The provision in (PM_0^+2) *may be changed to p being effective for descent and q a regular epimorphism*; *even to p, q being regular epimorphisms when $\mathscr{C}$ is Barr-exact*.

Proof. (PM_0^+) trivially implies (PM_0^+1), but also (PM_0^+2). In fact, given (2) (=⊡ as in (5)) with monomorphisms v, w and $k = \ker p \leqslant w$, the morphism l with $w \cdot l = k$ is the kernel of q. Hence, if q splits, (2) is a pullback diagram, so that regularity of v implies the same for w.

Conversely, given the split short exact sequence (6), assuming $\binom{k}{s} = w \cdot f$ with a monomorphism w, we may put $v = 1_B$, $q = p \cdot w$, $l = f \cdot i$ and $t = f \cdot j$ (with i, j the injections of $K + B$) and obtain $q \cdot t = 1_B$, $w \cdot l = k$, hence $\ker p \leqslant w$. Now, w is an epimorphism, by (PM_0^+1), and a regular monomorphism, by (PM_0^+2), hence an isomorphism. This shows extremality of $\binom{k}{s}$, as needed in (PM_0^+).

Remark. In the Proposition above, we may relativize (PM_0^+2) by trading regular monomorphisms for any class $\mathscr{M}$ of morphisms in $\mathscr{C}$ which is stable under pullback and satisfies $\mathscr{M} \cap \mathrm{Epi}\mathscr{C} = \mathrm{Iso}\mathscr{C}$, as follows:

for every commutative diagram (2) with monomorphisms w, v with $\ker p \leqslant w$, $v \in \mathscr{M}$ implies $w \in \mathscr{M}$, provided that q is a split epimorphism.

The choice $\mathscr{M} = \mathrm{Iso}\mathscr{C}$ gives the Short Five Lemma of 2.3, while $\mathscr{M} = \{\text{normal monos}\}$ gives Hofmann's Axiom 1.5.

2.5. We now combine the conditions discussed so far:

Definition. $\mathscr{C}$ is called *semi-abelian* if $\mathscr{C}$ is Barr-exact and Bourn-protomodular and if $\mathscr{C}$ has a zero object and finite sums (=coproducts).

Phrased less redundantly, a semi-abelian category $\mathscr{C}$ is a category which satisfies:

(SA1) $\mathscr{C}$ has binary products and sums and a zero object;

(SA2) $\mathscr{C}$ has pullbacks of (split) monomorphisms;

(SA3) $\mathscr{C}$ has coequalizers of kernelpairs;

(SA4) the Split Short Five Lemma holds true in $\mathscr{C}$ (see 2.3);

(SA5) regular epimorphisms in $\mathscr{C}$ are stable under pullback;
(SA6) equivalence relations in $\mathscr{C}$ are effective.

2.6. Examples (1) *Abelian categories.* Barr-exact categories which are additive (=enriched over the category of abelian groups) give precisely the abelian categories. They are semi-abelian, since additive categories with kernels satisfy (PM_0) (see [12]), and since products in them are biproducts. The category of (not necessarily abelian) groups gives an example of a semi-abelian category which is not abelian.

(2) *Ω-groups*. Every variety of universal algebras (for a finitary or infinitary theory $\mathscr{T}$) gives a Barr-exact category $\mathscr{T}$-Alg(*Set*). Furthermore, every variety of Ω-groups (i.e., a variety of universal algebras with underlying group structure such that the trivial subgroup is a subalgebra, see [39]) is protomodular and therefore semi-abelian. Hence, in addition to the category of groups, many "classical" algebraic categories (rings (not necessarily unital), Lie algebras, Jordan algebras (over a ring), etc.) are semi-abelian.

In fact (finitary) varieties $\mathscr{T}$-Alg(*Set*) of $\mathscr{T}$-algebras which are semi-abelian have been completely characterized in [14], by the following syntactical condition:

there are a nullary operation $e \in \Omega$, binary terms $t_1, \ldots, t_n$ and an $(n+1)$-*ary term t such that*

$$t(x, t_1(x, y), \ldots, t_n(x, y)) = y,$$

$$t_1(x,x) = \cdots = t_n(x,x) = e,$$

$$\omega(e, \ldots, e) = e \text{ for all } \omega \in \Omega$$

are identities in $\mathscr{T}$-Alg(*Set*).

(3) *Internal varieties.* Recall that the abelian group objects in an exact category form an abelian category (see [28, 1.595]); in fact, every abelian category is of this form, since the abelian group objects of an abelian category give the same category. The characterization of semi-abelian varieties in (2) can be used to prove:

Theorem. *Let the Lawvere theory $\mathscr{T}$ be such that $\mathscr{T}$-Alg(Set) is semi-abelian. Then the category $\mathscr{T}$-Alg($\mathscr{C}$) of internal $\mathscr{T}$-algebras in a Barr-exact category $\mathscr{C}$ is semi-abelian if and only if it has finite coproducts.*

Proof. Only the "if" part needs proof. With $\mathscr{C}$ also $\mathscr{T}$-Alg($\mathscr{C}$) is Barr-exact. Furthermore, using the syntactical features of $\mathscr{T}$ given in (2), one proves protomodularity of $\mathscr{T}$-Alg($\mathscr{C}$) as in the case $\mathscr{C} = Set$, i.e., the argumentation is Yoneda invariant. □

(4) *Crossed modules.* The category of crossed modules (= internal category objects of the category of groups) can be considered as a variety of Ω-groups and is therefore semi-abelian; likewise for pre-crossed modules, crossed complexes, etc. This fact is based on the observation that Loday's categorical groups (see [52]) may be described as groups equipped with two idempotent endomorphisms satisfying additional identities (see also [44]).

(5) *Relation with Malcev categories.* Recall that a Malcev operation on a set X is a map $\pi : X^3 \to X$ satisfying the equations $\pi(x,y,y) = x$ and $\pi(x,x,y) = y$ (see [65]). A finitely complete category $\mathscr{C}$ is *naturally Malcev* if it admits a natural transformation $\pi_X : X^3 \to X$ satisfying the identities of a Malcev operation (see [47]). Pointed naturally Malcev categories are additive (see [9]), hence trivially semi-abelian, in case they are Barr-exact with finite coproducts.

Conversely, protomodular categories have the important property that every reflexive relation (i.e., every monic pair $r_1, r_2 : R \to A$ with a common splitting) is already an equivalence relation (see [10]); in other words: *every protomodular* (*and*, *a fortiori*, *every semi-abelian category*) *is Malcev* in the sense of [20]. Moreover, the converse implication fails, even in the presence of a zero object: consider the variety of algebras with one Malcev operation and one nullary operation.

(6) *Further examples*. The category of *Heyting algebras* (=cartesian closed posets with finite joins) is protomodular ([10]) and therefore semi-abelian. The category $\mathit{Set}^{\mathrm{op}}$ and, in fact, the dual category of any elementary topos is exact and protomodular (see [10]) but in general not semi-abelian, because of the missing zero object. However, the dual of the category of pointed sets (or of poined objects in any topos) is semi-abelian. More generally, if $\mathscr{C}$ is an exact protomodular category with finite colimits, then $\mathrm{Pt}_{\mathscr{C}}(A)$ is clearly semi-abelian, for every object A.

2.7. We conclude this section with two useful observations on quotient objects in protomodular categories (the second of which was already established in [9]). First, recall that for every object A in any category $\mathscr{C}$ with limits and colimits as needed, there is a Galois equivalence

$$\mathrm{EER}(A) \underset{\text{kerpair}}{\overset{\text{coequ}}{\rightleftarrows}} \mathrm{Q}(A) \tag{7}$$

between the ordered class of quotients of A (represented by regular epimorphisms with domain A) and the ordered class of effective equivalence relations on A. If $\mathscr{C}$ has a zero object, there is also the Galois correspondence

$$\mathrm{Q}(A) \underset{\text{coker}}{\overset{\text{ker}}{\rightleftarrows}} \mathrm{S}(A), \tag{8}$$

with $\mathrm{S}(A)$ the ordered class of subobjects of A (represented by monomorphisms with codomain A); this restricts to an equivalence

$$\mathrm{NQ}(A) \underset{\text{coker}}{\overset{\text{ker}}{\rightleftarrows}} \mathrm{NS}(A) \tag{9}$$

between *normal* quotients and *normal* subobjects of A. Since in any pullback diagram arrows on opposite sides have isomorphic kernels, the composite map $\ker \cdot \mathrm{coequ}$ of

(7) and (8) can be displayed as

$$\begin{array}{ccc} EER(A) & \xrightarrow{\ker} & S(A) \\ r = (R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A) & \longmapsto & \ker r_1 \ (= \ker r_2). \end{array} \tag{10}$$

Proposition. *Let $\mathscr{C}$ be protomodular. Then*:

(1) *If equivalence relations are effective in $\mathscr{C}$ and if $\mathscr{C}$ has* (*regular epi*, *mono*) *-factorizations*, *then pushouts of regular epimorphisms exist in $\mathscr{C}$.*
(2) *If $\mathscr{C}$ has a zero object and cokernels of kernels*, *then every regular epimorphism in $\mathscr{C}$ is normal.*

Proof. (1) Since a protomodular category is Malcev (see 2.6(3)), binary suprema exist in the ordered class ER(A) of equivalence relations on A which, under (Ex3), is equal to EER(A). Hence, suprema exist in Q(A). (Note that the factorization system provides coequalizers of effective equivalence relations.) Now it only remains to verify that the supremum of two regular epimorphisms with domain A satisfies the universal property of a pushout, which follows elementarily with the given factorization system.

(2) For $p \in \mathrm{Q}(A)$ and $q = \mathrm{coker}(\ker p) \leqslant p$, we have $\ker p = \ker q$. Hence, letting r, s be the induced equivalence relations of p, q, respectively, in order to obtain $p = q$ we just need to show:

$$s \leqslant r \quad \text{and} \quad \ker s_1 = \ker r_1 \quad \text{implies } s = r,$$

which is an immediate consequence of (PM_0). □

2.8. In the presence of finite coproducts, pushouts of split epimorphisms suffice to obtain the existence of all finite colimits (see 2.1, dual). Hence, Proposition 2.7 gives immediately:

Corollary. *A semi-abelian category $\mathscr{C}$ has all finite colimits and a stable* (normal epi, mono)*-factorization system.*

3. Semi-abelian categories in terms of "old" axioms.

3.1. Consider the following old-style conditions on a category $\mathscr{C}$:

(SA^*1) = (SA1) $\mathscr{C}$ has binary products and sums and a zero object.
(SA^*2) $\mathscr{C}$ has binary intersections of monomorphisms (that is: S(A) has binary infima, for every object A in $\mathscr{C}$).
($\mathrm{SA}^*3\mathrm{a}$) Every product projection is a normal epimorphism.

(SA*3b) (Images under normal epimorphisms) For every normal epimorphism $p:E \to B$ and every monomorphism $w:F \to E$, there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{q} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow{p} & B \end{array} \tag{11}$$

with a monomorphism v and a normal epimorphism q.

(SA*4) (Hofmann's Axiom) For every commutative diagram (11) with normal epimorphisms p, q and monomorphisms v, w, the morphism w is normal, provided that v is normal and that every normal monomorphism $k:K \to E$ with $p \cdot k = 0$ factors through w.

(SA*5) (Inverse images under normal epimorphisms) For every normal epimorphism $p:E \to B$ and every monomorphism $v:C \to B$, there is a commutative diagram (11) with a monomorphism w and a normal epimorphism q.

(SA*6) For every commutative diagram (11) with normal epimorphisms p, q and monomorphisms v, w, the morphism v is normal if w is normal.

The purpose of this section is to show that *conditions* (SA 1–6) *are equivalent to* (SA* 1–6). More precisely, we shall often work with the (formally weaker) conditions

(SA**2) *$\mathscr{C}$ has binary intersections of split monomorphisms*,
(SA**3) = (SA*3a)&(SA*3b), but with p a normal split epimorphism,
(SA**4) = (SA*4), but with p, q normal split epimorphisms,
(SA**5) = (SA*5) (no change),
(SA**6) = (SA*6), but with p, q normal split epimorphisms,

and then prove:

Theorem. *Let $\mathscr{C}$ satisfy* (SA1) *and* (SA**2). *Then*:

(1) *$\mathscr{C}$ has* (normal epi, mono)-*factorizations if and only if* (SA**3) *is satisfied*; *in this case $\mathscr{C}$ has all finite limits.*

(2) *$\mathscr{C}$ is protomodular with* (normal epi, mono)-*factorizations if and only if* (SA**3) *and* (SA**4) *are satisfied*; *in this case*, *the canonical morphism*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A + B \to A \times B$$

is a normal epimorphism, *for all objects A, B.*

(3) *$\mathscr{C}$ is protomodular with stable* (normal epi, mono)-*factorizations if and only if* (SA**3–5) *are satisfied.*

(4) *$\mathscr{C}$ is protomodular and Barr exact if and only if* (SA**3–6) *or*, *equivalently*, *if* (SA*3–6) *are satisfied*; *in this case*, *$\mathscr{C}$ has all finite colimits.*

3.2. First we note that a regular category $\mathscr{C}$ trivially satisfies (SA2), hence (SA*2), but also (SA*3b) and (SA*5), provided that regular epimorphisms are normal. This provision is certainly given when $\mathscr{C}$ is semi-abelian (see 2.7); here we show its link with (SA*3a):

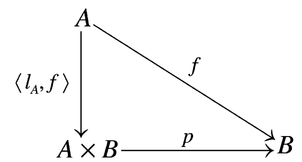
Proposition. *Let the category* $\mathscr{C}$ *satisfy* (SA1) *and* (SA**2). *Then*:

(1) *The following conditions are equivalent*:
  (i) $\mathscr{C}$ *has* (normal epi, mono)-*factorizations*;
  (ii) $\mathscr{C}$ *satisfies* (SA**3);
  (iii) $\mathscr{C}$ *satisfies* (SA*3b), *and split epimorphisms are normal*;
  (iv) $\mathscr{C}$ *satisfies* (SA*3b), *and regular epimorphisms are normal.*

(2) *Each of* (i)–(iv) *implies that* $\mathscr{C}$ *has all finite limits*, *and that*
  (v) *kernels and their cokernels exist in* $\mathscr{C}$, *and every morphism* f *with* $\ker f = 0$ *is a monomorphism.*

(3) *In the presence of* (SA*5), *condition* (v) *is equivalent to* (i)–(iv).

Proof. (1) The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are trivial, and for (ii) $\Rightarrow$ (i) factor the given morphism $f : A \to B$ through its graph as in



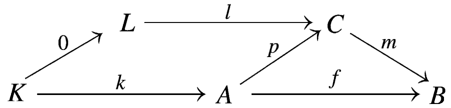
(12)

Then note that (SA*3a) allows us to apply (the split version of) (SA*3b) in this situation and obtain the desired factorization of f.

(i) $\Rightarrow$ (iv) Condition (i) trivially gives (SA*3b), and also that extremal epimorphisms in $\mathscr{C}$ must be normal, which is then a fortiori true for regular epimorphisms.

(2) Having (normal epi, mono)-factorizations one shows immediately that infima in S(A) enjoy the universal property of pullbacks. Hence, using 2.1, from (SA1) and (SA**2) we obtain the existence of all finite limits in $\mathscr{C}$. In particular, any morphism f must have a kernel, the cokernel of which is the normal-epi part of f. In case $\ker f = 0$ the cokernel is 1, and f coincides with the mono part of its factorization.

(3) For (v) $\Rightarrow$ (i), consider the commutative diagram



(13)

with $k = \ker f$, $p = \operatorname{coker} k$ and $l = \ker m$. With (SA*5) one can find a normal epimorphism $q : D \to L$ and a morphism $w : D \to A$ with $l \cdot q = p \cdot w$. Since $f \cdot w = m \cdot l \cdot q = 0$, w factors as $w = k \cdot x$, and since l is monic, one sees that the normal epimorphism q must be 0. Hence $L \cong 0$, and we obtain with (v) that m is a monomorphism. $\square$

Corollary. *Under conditions* (SA*1–3b), $\mathscr{C}$ *has all finite limits and* (normal epi, mono)-*factorizations*; *these are stable under pullback along monomorphisms if and only if* (SA*5) *is satisfied.*

Proof. Apply 2.1 and the proposition above. For the additional statement use the fact that, in the presence of the factorization system, a composite $q = r \cdot t$ is a normal epimorphism only if r is one. □

3.3. Next we wish to clarify the meaning of (SA*4) in modern terms. In fact, first we shall be able to do this only for (SA**4):

Proposition. *Let $\mathscr{C}$ have pullbacks, a zero object and* (normal epi, mono)*-factorizations. Then $\mathscr{C}$ is protomodular if and only if* (SA**4) *is satisfied.*

Proof. It is clear that (SA**4) follows from (PM$_0$), just as (PM$_0^+$2) was derived in Proposition 2.4: consider $\mathscr{M} = \{\text{normal monos}\}$ in Remark 2.4.

Conversely, checking (PM$_0$) we consider the commutative diagram (5) with split epimorphisms p, q, isomorphisms u, v, and $k = \ker p$, $l = \ker q$. Now w has a (normal epi, mono)-factorization $w = m \cdot e$. Any morphism x with $e \cdot x = 0$ factors as $x = l \cdot y$, and from $k \cdot u \cdot y = m \cdot e \cdot x = 0$ one derives $y = 0$ and then $x = 0$. Hence, the normal epimorphism e is the cokernel of 0 and therefore an isomorphism. Consequently, in (5) we may assume w to be monic. The implication (i) ⇒ (iii) of Proposition 3.2 shows that the split epimorphisms p, q are normal, so that an application of (SA**4) gives that w is a normal monomorphism. Dually to the argumentation employed previously one shows that any morphism x with $x \cdot u = 0$ must be 0. Hence, w is an isomorphism. □

Corollary. *Let $\mathscr{C}$ satisfy* (SA*1), (SA*2) *and* (SA*3b). *Then the following conditions are equivalent:*

(i) *$\mathscr{C}$ is protomodular*;

(ii) (SA**4) *is satisfied, and for every split short exact sequence* (6), $\binom{k}{s} : K + B \to E$ *is an epimorphism*;

(iii) (SA**4) *is satisfied, and for all objects A, B, the canonical morphism $e : A + B \to A \times B$ is an epimorphism*;

(iv) (SA*3a) *and* (SA**4) *are satisfied.*

Proof. (i) ⇒ (ii) follows from the Proposition and (PM$_0^*$1) of 2.4.

(ii) ⇒ (iii) is trivial, as one may consider the split short exact sequence

$$A \overset{\langle 1_A, 0\rangle}{\longrightarrow} A \times B \underset{\langle 0, 1_B\rangle}{\overset{p}{\rightleftarrows}} B. \tag{14}$$

(iii) ⇒ (iv) In (14), we must show that the projection p is the cokernel of $\langle 1_A, 0\rangle$. But for any x with $x \cdot \langle 1_A, 0\rangle = 0$ one routinely checks that $x \cdot \langle 0, 1_B\rangle \cdot p \cdot e = x \cdot e$, hence $x \cdot \langle 0, 1_B\rangle \cdot p = x$ under hypothesis (iii). Also, any morphism y with $y \cdot p = x$ necessarily satisfies $y = y \cdot p \cdot \langle 0, 1_B\rangle = x \cdot \langle 0, 1_B\rangle$.

(iv) ⇒ (i) follows from the proposition, in conjunction with Corollary 3.2. □

Remarks. (1) In the corollary, "epimorphism" may be replaced by "extremal epimorphism" (see (PM_0^+) of 2.4) and then in fact by "normal epimorphism" (since by Proposition 3.2 we have (normal epi, mono)-factorizations).

(2) The corollary in conjunction with Proposition 3.2 proves Theorem 3.1(1), (2).

(3) Once we have established Barr-exactness of $\mathscr{C}$ we may trade $(\mathrm{SA}^{**}4)$ for $(\mathrm{SA}^{*}4)$: see the equivalent formulations of (PM_0) in 2.3.

3.4. Next we give the *proof of Theorem* 3.1 (3). All that needs to be done after Corollary 3.2 and Remark 3.3 is to show that, in the presence of $(\mathrm{SA}^{**}4)$, condition $(\mathrm{SA}^{*}5)$ guarantees full pullback stability, not just along monomorphisms.

In any pullback diagram

$$\begin{array}{ccc} D & \xrightarrow{p'} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{p} & B \end{array} \tag{15}$$

f factors through a split epi followed by a split mono, as in (12).

Hence, we may assume f to be a split epimorphism. Considering now the (normal epi, mono)-factorization $p' = m \cdot e$, putting $g := f \cdot m$ we have a pullback diagram

$$\begin{array}{ccc} D & \xrightarrow{e} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow{p} & B \end{array} \tag{16}$$

Hence, $\ker g = \ker f' = \ker f$. Furthermore, since with f also f' is a split epimorphism, with p also $p \cdot f' = g \cdot e$ and then g are regular epimorphisms. Since f is effective for descent, (PM_0) applied to

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow{f} & B \end{array} \tag{17}$$

gives that m is an isomorphism; equivalently, that p' is a normal epimorphism.

3.5. Towards Theorem 3.1(4) we first prove:

Proposition. *In a protomodular category* $\mathscr{C}$ *with a zero object and* (normal epi, mono)-*factorizations satisfying* $(\mathrm{SA}^{**}6)$, *reflexive relations are effective equivalence relations.*

Proof. A reflexive relation is a monic pair $r_1, r_2 : R \to A$ with a common section d. Let $k = \ker r_1$ and $n = r_2 \cdot k$. Clearly, n is a monomorphism (since $r_1 \cdot k \cdot x = 0 = r_1 \cdot k \cdot y$ for all x, y); in fact, it is normal, as one sees applying $(\mathrm{SA}^{**}6)$ to

$$\begin{array}{ccc} K & \xrightarrow{1_K} & K \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle n} \\ R & \xrightarrow{r_2} & A \end{array} \tag{18}$$

We now let $q = \operatorname{coker} n : A \to B$ and s be its induced equivalence relation $s_1, s_2 : S \to A$. One has

$$q \cdot r_1 \cdot k = 0 = q \cdot n = q \cdot r_2 \cdot k, \; q \cdot r_1 \cdot d = q = q \cdot r_2 \cdot d,$$

hence $q \cdot r_1 = q \cdot r_2$, since k, d are jointly epic in the protomodular category $\mathscr{C}$ (cf. (PM_0^+1)). Consequently, there is a morphism $t : R \to S$ with $s_i \cdot t = r_i$ $(i = 1, 2)$.

We must show that t is an isomorphism. But since $h := \ker s_1 = \ker q$, we have the commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{k} & R & \xrightarrow{r_1} & A \\ {\scriptstyle 1_K}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle 1_A} \\ K & \xrightarrow{h} & S & \xrightarrow{s_1} & A \end{array} \tag{19}$$

in which the horizontal arrows form split short exact sequences. Hence, everything follows with (PM_0). □

3.6. In order to complete the proof of Theorem 3.1, we only need to show:

Proposition. *A semi-abelian category* $\mathscr{C}$ *satisfies* (SA*6).

Proof. Considering once again the commutative diagram (2) with normal epimorphisms p, q, a monomorphism v and a normal monomorphism w, we let $e = \operatorname{coker} w$. In the exact Malcev category $\mathscr{C}$ the pushout diagram

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{g} & Q \end{array} \tag{20}$$

exists, and the canonical morphism $h : E \to P \times_Q B$ is a normal epimorphism: see [19], Theorem 5.7. Let $k = \ker f$, and let $t : C \to K$ be the monomorphism with $k \cdot t = v$. Since the pullback $f' : P \times_Q B \to P$ of f along g has the same kernel as f, $k' = \ker f'$ satisfies $g' \cdot k' = k$, where $g' : P \times_Q B \to B$ is the pullback of g along f. Now the commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{w} & E & \xrightarrow{e} & P \\ {\scriptstyle t\cdot q}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_P} \\ K & \xrightarrow{k'} & P \times_Q B & \xrightarrow{f'} & P \end{array} \tag{21}$$

shows that its left-hand side is a pullback. Consequently, $t \cdot q$ is a normal epimorphism, whence also t is one. Hence, t is in fact an isomorphism, and v must be normal. □

3.7. Combining 2.5, 3.1 and 3.2 we see that the following list of "old" conditions also characterizes semi-abelian categories $\mathscr{C}$:

(SA'1)=(SA1) $\mathscr{C}$ has binary products and sums and a zero object;
(SA'2)=(SA2) $\mathscr{C}$ has pullbacks of (split) monomorphisms;

(SA'3) $\mathscr{C}$ has cokernels of kernels, and every morphism with zero kernel is a monomorphism;
(SA'4)=(SA4) the (Split) Short Five Lemma holds true in $\mathscr{C}$;
(SA'5) normal epimorphisms are stable under pullback;
(SA'6) images of normal monomorphisms under normal epimorphisms are normal monomorphisms.

4. Additional remarks

4.1. We mentioned in Section 1.4 that *a category $\mathscr{C}$ is abelian if and only if both $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are semi-abelian*. In fact, with 1.59 of [28], this follows from the following rather obvious statement:

Proposition. *If for the pointed protomodular category $\mathscr{C}$ also its dual category $\mathscr{C}^{\mathrm{op}}$ is protomodular, then finite products in $\mathscr{C}$ are biproducts.*

Proof. We must show that, for all objects A, B in $\mathscr{C}$, the canonical morphism $A+B \to A\times B$ is an isomorphism. Indeed, applying (PM_0^*) to (14) one sees that it is an extremal epimorphism and, dually, also an extremal monomorphism.

Remark. For the proposition it suffices that $\mathscr{C}^{\mathrm{op}}$ satisfies (PM_0^+1) (see 2.4).

4.2. We wish to explain the notion of semi-additive category as mentioned in Section 1.4. In [13], a category $\mathscr{C}$ with (split) pullbacks is said to *have semidirect products* if the functor $v^*: \mathrm{Pt}_{\mathscr{C}}(B) \to \mathrm{Pt}_{\mathscr{C}}(E)$ is monadic for all $v: E \to B$ in $\mathscr{C}$; it is shown that when $\mathscr{C}$ is Barr-exact and protomodular, $\mathrm{Pt}_{\mathscr{C}}(B)$ has coequalizers of reflexive pairs and v^* preserves them. Hence, *if $\mathscr{C}$ is Barr-exact, $\mathscr{C}$ has semi-direct products if and only if it is protomodular and has pushouts of split monomorphisms* (with the latter guaranteeing the existence of left adjoints to the functors v^*; see [9]).

Corollary. *If $\mathscr{C}$ is Barr-exact with finite coproducts and a zero object, $\mathscr{C}$ is semi-abelian if and only if $\mathscr{C}$ has semidirect products.*

To explain what semi-direct products *are* in this situation, consider $v = i_B: 0 \to B$, so that $v^* = \ker_B: \mathrm{Pt}_{\mathscr{C}}(B) \to \mathscr{C}$. With T_B the monad induced by $\ker_B$, the *semi-direct product* $(X,\xi) \rtimes B$ of a T_B-algebra (X,ξ) with B is simply the $\mathrm{Pt}_{\mathscr{C}}(B)$-object corresponding to (X,ξ) under the category equivalence $\mathrm{Pt}_{\mathscr{C}}(B) \sim \mathscr{C}^{T_B}$. The paper [13] explains that this definition gives indeed the usual semi-direct product in the category of groups. If $\mathscr{C}$ is an additive category, T_B is simply (isomorphic to) the identity monad on $\mathscr{C}$, so that (X,ξ) is given by the object X alone, and the semi-direct product becomes the direct sum: $X \rtimes B = X \times B = X \oplus B$.

If we, therefore, call a category $\mathscr{C}$ *semi-additive* if $\mathscr{C}$ has finite sums, a zero object and kernels such that each kernel functor $\mathrm{Pt}_{\mathscr{C}}(B) \to \mathscr{C}$ is monadic, we obtain:

Corollary. *$\mathscr{C}$ is semi-abelian if and only if $\mathscr{C}$ is Barr-exact and semi-additive.*

4.3. As an epilogue we wish to reflect once again on the choice of axioms for semi-abelianness. We hope that the Reader will agree that the requirements of Barr-exactness and protomodularity are natural and indispensable. Also the requirement for the existence of finite coproducts seems natural since they provide fundamental algebraic constructions. There then remains the condition for the existence of a zero object, which really breaks down into the existence requirement for an initial object 0 and a terminal object 1 on the one hand, and the condition $0 \cong 1$ on the other. Under the mission outlined in Section 1, it is this very last condition which appears to be dispensable. Abandoning it would certainly give additional interesting examples but would also come with a considerable price tag in terms of technical complications (as the Reader of [11,12] will realize immediately). But more importantly for our paper, a direct comparison with old-style axioms would no longer be possible.

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References

[1] S.A. Amitsur, A general theory of radicals, II. Radicals in rings and bicategories, Amer. J. Math. 76 (1954) 100–125.
[2] M. Atiyah, On the Krull–Schmidt Theorem with applications to sheaves, Bull. Soc. Math. France 84 (1956) 307–317.
[3] R. Baer, Direct decompositions, Trans. Amer. Math. Soc. 62 (1947) 62–98.
[4] M. Barr, Exact Categories, in: Lecture Notes in Mathematics, Vol. 236, Springer, Berlin, 1971, pp. 1–120.
[5] M. Barr, C. Wells, Toposes, Triples and Theories, Springer, Berlin, 1985.
[6] F. Borceux, Handbook of Categorical Algebra, Vol. 1–3, Cambridge University Press, Cambridge, 1994.
[7] S.G. Botha, Primary ideals in categories, Preprint, UNISA, Pretoria, 1997.
[8] S.G. Botha, A. Buys, Prime ideals in categories, Comm. Algebra 13 (1985) 1171–1186.
[9] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, in: A. Carboni, M.C. Pedicchio, G. Rosolini (Eds.), Lecture Notes in Mathematics, Vol. 1488, Springer, Berlin, 1991, pp. 43–62.
[10] D. Bourn, Mal'cev categories and fibrations of pointed objects, Appl. Categorical Structures 4 (1996) 307–327.
[11] D. Bourn, Baer sums and fibred aspects of Mal'cev operations, preprint, Université du Littoral, Calais, 1999a.
[12] D. Bourn, Normal subobjects and abelian objects in protomodular categories, preprint, Université du Littoral, Calais, 1999b.
[13] D. Bourn, G. Janelidze, Protomodularity, descent, and semi-direct products, Theory Appl. Categories 4 (1998) 37–46.
[14] D. Bourn, G. Janelidze, Characterization of protomodular varieties of universal algebras, in preparation.
[15] H.-B. Brinkmann, D. Puppe, Abelsche und exakte Kategorien, Korrespondenzen, Lecture Notes in Mathematics, Vol. 96, Springer, Berlin, 1969.
[16] D.A. Buchsbaum, Exact categories and duality, Trans. Amer. Math. Soc. 80 (1955) 1–34.

[17] W.D. Burgess, X. Caicedo, Congruences in regular categories, Rev. Columbiana de Matemáticas 15 (1981) 43–64.
[18] A. Carboni, Categories of affine spaces, J. Pure Appl. Algebra 61 (1989) 243–250.
[19] A. Carboni, G.M. Kelly, M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appl. Categorical Struct. 1 (1993) 385–421.
[20] A. Carboni, J. Lambek, M.C. Pedicchio, Diagram chasing in Mal'cev categories, J. Pure Appl. Algebra 69 (1991) 271–284.
[21] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.
[22] B. Eckmann, P.J. Hilton, Group-like structures in general categories I, Math. Ann. 145 (1962) 227–255.
[23] E. Faro, On a conjecture of Lawvere, preprint, SUNY, Buffalo, 1989.
[24] T.H. Fay, On commuting congruences in regular categories, Math. Colloq. Univ. Cape Town 11 (1977) 13–31.
[25] T.H. Fay, On categorical conditions for congruences to commute, Algebra Univ. 8 (1978) 173–179.
[26] R. Freese, R. McKenzie, Commutator Theory for Congruence Modular Varieties, London Mathematical Society Lecture Notes, Vol. 125, Cambridge, 1987.
[27] P. Freyd, Abelian Categories. An Introduction to the Theory of Functors, Harper & Row, New York, 1964.
[28] P. Freyd, A. Scedrov, Categories, Allegories, North-Holland, Amsterdam, 1990.
[29] R. Fritsch, O. Wyler, The Schreier Refinement Theorem for categories, Arch. Math. 22 (1971) 570–572.
[30] A. Fröhlich, Non-abelian homological algebra I, Derived functors and satelites, Proc. London Math. Soc. (3) 11 (1961) 239–275.
[31] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962) 323–448.
[32] M. Gerstenhaber, A categorical setting for the Baer extension theory, in: Proceedings of Symposia in Pure Mathematics, Vol. 17, American Mathematical Society, Providence, RI, 1970, pp. 50–64.
[33] A.W. Goldie, The Jordan–Hölder theorem for general abstract algebras, Proc. London Math. Soc. (2) 52 (1950) 107–113.
[34] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J. 2 (1957) 119–221.
[35] H.P. Gumm, Geometrical methods in congruence modular varieties, Mem. Amer. Math. Soc. 45 (1983).
[36] J. Hagemann, C. Herrmann, A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity, Arch. Math. 32 (1979) 234–245.
[37] A. Heller, Homological algebra in abelian categories, Ann. Math. 68 (1958) 484–525.
[38] H. Herrlich, G.E. Strecker, Category Theory, Allyn Bacon, Boston, 1973.
[39] P.J. Higgins, Groups with multiple operators, Proc. London Math. Soc. (3) 6 (1956) 366–416.
[40] P.J. Hilton, W. Ledermann, On the Jordan–Hölder theorem in homological monoids, Proc. London Math. Soc. 10 (1960) 321–334.
[41] F. Hofmann, Über eine die Kategorie der Gruppen umfassende Kategorie, Sitzungsberichte Bayerische Akad. Wissenschaften, Math. Naturw. Klasse (1960) 163–204.
[42] M. Holcombe, R. Walker, Radicals in categories, Proc. Edinburgh Math. Soc. 21 (1978) 111–128.
[43] S.A. Huq, Commutator, nilpotency and solvability in categories, Quart. J. Math. Oxford (2) 19 (1968) 363–389.
[44] G. Janelidze, M.C. Pedicchio, Internal categories and groupoids in congruence modular varieties, J. Algebra 193 (1997) 552–570.
[45] G. Janelidze, M.C. Pedicchio, Pseudogroupoids and commutators, Trieste, 1998, preprint.
[46] G. Janelidze, W. Tholen, Facets of descent I, Appl. Categorical Structures 2 (1994) 245–281.
[47] P.T. Johnstone, Affine categories and naturally Maltsev categories, J. Pure Appl. Algebra 61 (1989) 251–256.
[48] K. Kearnes, Á. Szendrei, The relationship between two commutators, Internat. J. Algebra and Comput. 8 (1998) 497–531.
[49] G.M. Kelly, Monomorphisms, epimorphisms and pullbacks, J. Austral. Math. Soc. 9 (1969) 124–142.
[50] A. Klein, Relations in categories, Ill. J. Math. 14 (1970) 536–550.
[51] A.G. Kurosh, Direct decompositions in algebraic categories, Trudy Mosk. Mat. Obshch. 8 (1959) 391–412 (in Russian).
[52] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, J. Pure Appl. Algebra 24 (1982) 179–202.
[53] S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. 56 (1950) 485–516.

[54] S. Mac Lane, Homology, Springer, Berlin, 1963.
[55] S. Mac Lane, Categories for the Working Mathematician, 2nd Edition, Springer, New York, 1998.
[56] L. Márki, R. Wiegandt, Remarks on radicals in categories, in: Lecture Notes in Mathematics, Vol. 962, Springer, Berlin, 1982, pp. 190–196.
[57] J. Meisen, On bicategories of relations and pullback spans, Comm. Algebra 1 (1974) 377–401.
[58] B. Mitchell, Theory of Categories, Academic Press, New York, 1965.
[59] G. Orzech, Obstruction theory in algebraic categories, I and II, J. Pure Appl. Algebra 2 (1972) 287–314, 315–340.
[60] M.C. Pedicchio, A categorical approach to commutator theory, J. Algebra 177 (1995) 647–657.
[61] M.C. Pedicchio, Arithmetical categories and commutator theory, Appl. Categorical Structures 4 (1996) 297–305.
[62] E.G. Shul'geifer, On the general theory of radicals in categories, Math. Sb. (N.S.) 51 (1960) 487–500 (in Russian).
[63] E.G. Shul'geifer, The Schreier Theorem in normal categories, Sib. Mat. Ž . 13 (1972) 688–697 (in Russian).
[64] J.D.H. Smith, Mal'cev Varieties, Lecture Notes in Mathematics, Vol. 554, Springer, Berlin, 1976.
[65] P.T. Taylor, Practical Foundations, Cambridge University Press, Cambridge, 1999.
[66] M.S. Tsalenko, Correspondences over a quasi-exact category, Mat. Sb. 73 (115) (1967) 564–584 (in Russian).
[67] O. Wyler, The Zassenhaus Lemma for categories, Arch. Math. 22 (1971) 561–569.