Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Pure and Applied Algebra 183 (2003) 27–43
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Smash product of pointed objects in lextensive categories

Aurelio Carboni[a,*,1], George Janelidze[b,2]

[a] Dipartimento di Scienze CC, FF e MM, Università dell' Insubria, Como 22100, Italy
[b] Mathematical Institute of the Georgian Academy of Sciences, Tbilisi, Georgia

Received 27 March 2002; received in revised form 7 January 2003
Communicated by G. Rosolini

Abstract

We describe a sufficient condition on a finitely complete and cocomplete lextensive category $\mathbf{X}$, under which the categorical smash product provides a canonical (symmetric, distributive with respect to finite coproducts) monoidal structure on the category $(1 \downarrow \mathbf{X})$ of its pointed objects. We also show that the ground category can be reconstructed as the category of *objects with counit* in $(1 \downarrow \mathbf{X})$.

MSC: 18A40; 18A30; 18B99

1. Introduction

There is a straightforward way to extend the definition of smash product from pointed topological spaces to pointed objects in an abstract category $\mathbf{X}$ with finite limits and colimits: for objects (X, p) and (Y, q) in $(1 \downarrow \mathbf{X})$, the smash product $(X, p) \wedge (Y, q) = (Z, r)$

* Corresponding author. Tel.: +39-031-238-6310; fax: +39-031-238-6119.
E-mail address: aurelio.carboni@uninsubria.it (A. Carboni).
[1] Supported by the Italian MIUR National Group 'Metodi Costruttivi in Topologia, Algebra e Analisi dei Programmi (MCTAAP).
[2] Partially supported by the Italian MIUR National Group MCTAAP, by the Australian Research Council, and by INTAS-97-31961.

0022-4049/03/$ - see front matter
doi:10.1016/S0022-4049(03)00034-3

is defined as the pushout

$$\begin{array}{ccc} X +_1 Y & \longrightarrow & 1 \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ X \times Y & \xrightarrow[s]{} & Z, \end{array} \tag{1.1}$$

where d is the canonical morphism $(X,p)+(Y,q) \to (X,p)\times(Y,q)$ in the pointed category $(1 \downarrow \mathbf{X})$. Equivalently it can be defined via the diagram

$$(X,p)+(Y,q) \xrightarrow{d} (X,p)\times(Y,q) \xrightarrow{s} (X,p)\wedge(Y,q) \tag{1.2}$$

in $(1 \downarrow \mathbf{X})$ by requiring the s to be the cokernel of d; accordingly $(X,p)\wedge(Y,q)$ is supposed to be equipped with a canonical morphism $s:(X,p)\times(Y,q)\to(X,p)\wedge(Y,q)$.

However it turns out that beyond the well-known cartesian closed case (e.g. compactly generated spaces) there is no obvious categorical reason for $\wedge$ to determine a canonical monoidal structure, and even the notion of "canonical" is to be chosen carefully. In Sections 2–5 (plus Section 6, where we prove distributivity of $\wedge$ with respect to $+$) we study this problem and show that it has a perfectly satisfactory solution under the first and the third of following three conditions on $\mathbf{X}$:

Condition 1.1. Finite sums (=coproducts) in $\mathbf{X}$ are pullback stable; in particular for every object A in $\mathbf{X}$, the product functor $A\times(-):\mathbf{X}\to\mathbf{X}$ preserves finite sums.

Condition 1.2. Sums are disjoint, i.e. for every two objects X and Y in $\mathbf{X}$, the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow[i]{} & X+Y, \end{array} \tag{1.3}$$

in which i and j are the coproduct injections, is a pullback.

Condition 1.3. For every object A in $\mathbf{X}$, the functor $A\times(-):\mathbf{X}\to\mathbf{X}$ preserves the following types of pushouts for any two objects X and Y in $\mathbf{X}$:

(a) of the form $X +_1 Y$,
(b) of the form (1.1).

The first two of these conditions are well known already from old work of A. Grothendieck; presently the categories satisfying them are called *extensive*, or *lextensive*—referring to the existence of finite limits. An elegant equivalent formulation of Conditions 1.1 and 1.2 are given by the requirement that the sum functor

$$\mathbf{X}\times\mathbf{X} \approx (\mathbf{X}\downarrow 1)\times(\mathbf{X}\downarrow 1) \to (\mathbf{X}\downarrow(1+1))$$

is an equivalence, as discussed in [2], to which we also will refer for the basic properties of lextensive categories. The third condition needs a special explanation:

Remark 1.4. If **X** is cartesian closed, then the functor $A \times (-) : \mathbf{X} \to \mathbf{X}$ preserves all colimits, and so Condition 1.3 holds trivially. If **X** is the opposite category of commutative algebras with unit over a field K, then $\times$ in **X** is the tensor product over K and the functor $A \times (-) : \mathbf{X} \to \mathbf{X}$ preserves all finite colimits simply because every module over a field is flat; hence in this case Condition 1.3 holds trivially again. If instead of a field we have an arbitrary commutative ring R (with unit), then $A \times (-)$ still preserves at least those coequalizers

$$U \underset{g}{\overset{f}{\rightrightarrows}} V \xrightarrow{h} W$$

in which h considered as an R-module homomorphism from W to V is a split monomorphism. Still, it is easy to see that this special colimit-preservation property implies Condition 1.3. In fact the example of the dual category of commutative R-algebras over a commutative ring R, i.e. the *category of affine R-schemes*, is the motivating example of a lextensive category for our search of conditions (much weaker than preservation of pushouts by the functors $A \times (-)$) to ensure associativity (and distributivity) of the smash product.

In Section 7 (under Conditions 1.1 and 1.2) we will construct a natural equivalence

$$\mathbf{X} \sim (1 \downarrow \mathbf{X})_* = ((1 \downarrow \mathbf{X}), (1+1, i), \wedge, \alpha, \lambda, \rho, \sigma)_*, \tag{1.4}$$

where the right-hand side denotes the category of objects with counit in the symmetric monoidal category $((1 \downarrow \mathbf{X}), (1+1, i), \wedge, \alpha, \lambda, \rho, \sigma)$ described in Section 5 (under Conditions 1.1 and 1.3); we also write it as $(1 \downarrow \mathbf{X})_*$ since that kind of monoidal structure is uniquely determined up to the choice of finite limits and finite colimits in **X**. This means that **X** can be reconstructed from $(1 \downarrow \mathbf{X})$, and it is at this point we will use the full strength of extensivity.

2. A simplified description of $(X, p) \wedge (1 + Y, i_Y)$

For a smash product of the form $(X, p) \wedge (1 + Y, i_Y)$, where $i_Y : 1 \to 1 + Y$ is the (first) coproduct injection, the pushout (1.1) can be rewritten as

$$\begin{array}{ccc} X + Y & \longrightarrow & 1 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle r} \\ X \times (1 + Y) & \xrightarrow[s]{} & Z, \end{array} \tag{2.1}$$

where now d is the composite $X + Y \approx X +_1 (1 + Y) \to X \times (1 + Y)$, and it is a straightforward calculation to prove

Lemma 2.1. *The morphism d in (2.1) coincides with the composite*

$$X + Y \approx (X \times 1) + (1 \times Y) \to (X \times 1) + (X \times Y) \to X \times (1 + Y), \tag{2.2}$$

in other words

$$d = \begin{bmatrix} \mathrm{id}_X & i!_X \\ p!_Y & j \end{bmatrix} : X + Y \to X \times (1 + Y), \tag{2.3}$$

where j is the (second) coproduct injection $Y \to 1 + Y$.

Corollary 2.2. *For $Y = 1$, the morphism d in (2.1) coincides with the following composite of canonical morphisms*:

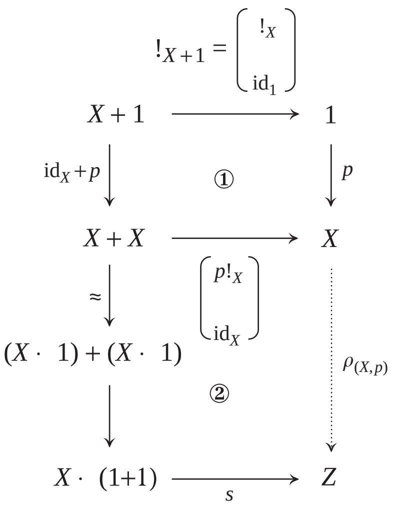
$$X + Y = X + 1 \to X + X \approx (X \times 1) + (X \times 1) \to X \times (1 + 1)$$
$$= X \times (1 + Y). \tag{2.4}$$

3. Unit object for the smash product

The familiar case of pointed spaces suggests that the unit for $\wedge$ should be the object $(1 + 1, i)$, where $i : 1 \to 1 + 1$ is (say) the first coproduct injection. We are going to show that this is true under Condition 1.1; more precisely, we are going to describe a canonical morphism $\rho_{(X,p)} : (X, p) \to (X, p) \wedge (1 + 1, i)$ and then prove that it is an isomorphism whenever so is the canonical morphism

$$(X \times 1) + (X \times 1) \to X \times (1 + 1). \tag{3.1}$$

Indeed, just consider the diagram



(3.2)

in which:

- the square ① is (obviously) a pushout;
- the left hand vertical composite is the same as (2.4), and hence there is a unique $\rho_{(X,p)}$ making ① + ② a pushout, which is the same as (1.1) for $Y = 1$;
- since ① and ② are pushouts, so is ②;
- therefore whenever the morphism (3.1) is an isomorphism, so is $\rho_{(X,p)}$.

Conclusion 3.1. Clearly $\rho_{(X,p)}$ is in fact a morphism in $(1 \downarrow \mathbf{X})$ from (X,p) to $(X,p)\wedge(1+1,i)$ natural in (X,p). With the canonical symmetry isomorphism σ for $\wedge$ and $\lambda = \sigma\rho$, this makes $((1 \downarrow \mathbf{X}), (1+1,i), \wedge, \lambda, \rho, \sigma)$ a colax symmetric magma (="non-associative symmetric monoidal category"), strong if the morphism (3.1) is an isomorphism for each (X,p) in $(1 \downarrow \mathbf{X})$. Note also, that since the middle horizontal arrow in (3.2) composed with the second coproduct injection $X \to X + X$ gives id_X, the morphism $\rho_{(X,p)}$ can be described as the appropriate composite $X \to X + X \approx (X \times 1) + (X \times 1) \to X \times (1+1) \to Z$; however this description exists only in $\mathbf{X}$ – not in $(1 \downarrow \mathbf{X})$.

Remark 3.2. If the ground category $\mathbf{X}$ is pointed, and hence $(1 \downarrow \mathbf{X})$ can be identified with $\mathbf{X}$ and $(1+1,i)$ with $0 = 1$, then (1.2) tells us that $(X,p)\wedge(1+1,i) \approx (1+1,i)$, i.e. $(1+1,i)$ becomes the zero object for $\wedge$ instead of being the unit.

4. Associativity

Defining the smash product in $(1 \downarrow \mathbf{X})$ we could equivalently begin with an arbitrary pointed category $\mathbf{A}$ with, say, finite limits and colimits, and define smash product in it via

$$A + B \xrightarrow{d} A \times B \xrightarrow{s} A \wedge B \tag{4.1}$$

instead of (1.2). Let us also introduce multiple smash products as follows:

Definition 4.1. The smash product $\wedge_{i=1}^{n} A_i = A_1 \wedge \cdots \wedge A_n$ of objects $A_1, \ldots, A_n$ in a pointed category $\mathbf{A}$ is defined via the diagram

$$\sum_{i=1}^{n} \prod_{j \neq i} A_j \xrightarrow{d} \prod_{i=1}^{n} A_i \xrightarrow{s} \bigwedge_{i=1}^{n} A_i, \tag{4.2}$$

by requiring s to be the cokernel of d. If $\mathbf{A}$ has (finite products, finite coproducts, and) all cokernels of this form, we will say that $\mathbf{A}$ is a category with smash products.

For three objects A, B, C in $\mathbf{A}$, consider the diagram

$$\begin{array}{ccccc}
(A\times B)+(A\times C)+(B\times C) & \xrightarrow{d} & A\times B\times C & \xrightarrow{s} & A\wedge B\wedge C \\
{\scriptstyle u}\downarrow & \text{①} & {\scriptstyle v}\downarrow & \text{②} & \downarrow{\scriptstyle w} \\
A\wedge(B+C) & \xrightarrow[\mathrm{id}_A\wedge d]{} & A\wedge(B\times C) & \xrightarrow[\mathrm{id}_A\wedge s]{} & A\wedge(B\wedge C),
\end{array} \tag{4.3}$$

where:

- u is induced by the composite $(A\times B)+(A\times C)\to A\times(B+C)\to A\wedge(B+C)$ and the zero morphism $B\times C\to A\wedge(B+C)$;
- v is the composite $A\times B\times C\approx A\times(B\times C)\to A\wedge(B\times C)$, which makes the square ① commutative since so is the diagram

$$\begin{array}{ccc}
(A\times B)+(A\times C) & \longrightarrow & A\times B\times C \\
\downarrow & & \downarrow \\
A\times(B+C) & \longrightarrow & A\times(B\times C)
\end{array}$$

  and the composite $B\times C\to A\times(B\times C)\to A\wedge(B\times C)$ is zero;
- w is the induced morphism making the square ② commute.

Definition 4.2. A category $\mathbf{A}$ with smash products is said to be $\wedge$-associative if for every three objects A, B, C in $\mathbf{A}$, the morphism $A\wedge B\wedge C\to A\wedge(B\wedge C)$ of (4.3) and the similar morphism $A\wedge B\wedge C\to(A\wedge B)\wedge C$ are isomorphisms.

Some important examples will be mentioned at the beginning of Section 8; the ones presented here are rather simple but "strange":

Example 4.3. There are many categories in which all smash products are zero, and hence those categories are trivially $\wedge$-associative. This applies to all categories (with finite products and coproducts) enriched in abelian monoids and in particular to additive categories, to Bourn protomodular categories, to pointed (quasi-)varieties of universal algebras having a binary term t with $t(0,x)=x=t(x,0)$, etc.

Example 4.4. Let $\mathbf{A}$ be a variety of universal algebras having a distinguished null-ary operation 0 and satisfying the identities $\omega(0,x_2,\ldots,x_n)=\omega(x_1,0,\ldots,x_n)=\cdots=\omega(x_1,\ldots,0,x_n)=\omega(x_1,\ldots,x_{n-1},0)=0$ for each basic n-ary operation $\omega(n=0,1,\ldots)$. It is then easy to see that the forgetful functor from $\mathbf{A}$ to the category of pointed sets preserves smash products, and then to deduce that $\mathbf{A}$ is $\wedge$-associative.

Example 4.5. Let $\mathbf{X}$ be a category with finite limits and colimits satisfying Condition 1.1. The forgetful functor $(1\downarrow\mathbf{X})\to\mathbf{X}$ has a left adjoint, and the corresponding Kleisli category $\mathbf{X}_+$ can be identified with the full subcategory in $(1\downarrow\mathbf{X})$ with objects all pairs (isomorphic to) $(1+X, i_X)$ for some X in $\mathbf{X}$. Using Condition 1.1, the diagram (2.1), and Lemma 2.1 it is easy to show that $\mathbf{X}_+$ is closed under finite products (and coproducts) and smash products in $(1 \downarrow \mathbf{X})$ by proving that $(1+X, i_X) \wedge (1+Y, i_Y) \approx (1+X\times Y, i_{X\times Y})$ for all X, Y in $\mathbf{X}$. Another simple calculation shows that $\mathbf{X}_+$ is $\wedge$-associative.

Instead of (4.3), consider now the diagram

$$\begin{array}{ccccc}
A+(B+C) & \underset{\mathrm{id}_A+0}{\overset{\mathrm{id}_A+d}{\rightrightarrows}} & A+(B\times C) & \xrightarrow{\mathrm{id}_A+s} & A+(B\wedge C) \\
\downarrow d & & \downarrow d & & \downarrow d \\
A\times(B+C) & \underset{\mathrm{id}_A\times 0}{\overset{\mathrm{id}_A\times d}{\rightrightarrows}} & A\times(B\times C) & \xrightarrow{\mathrm{id}_A\times s} & A\times(B\wedge C) \\
\downarrow s & & \downarrow s & & \downarrow s \\
A\wedge(B+C) & \underset{\mathrm{id}_A\wedge 0=0}{\overset{\mathrm{id}_A\wedge d}{\rightrightarrows}} & A\wedge(B\times C) & \xrightarrow{\mathrm{id}_A\wedge s} & A\wedge(B\wedge C);
\end{array} \tag{4.4}$$

since its columns are cokernel diagrams, and the top row is a coequalizer diagram, we obtain:

Lemma 4.6. *Let A, B, C be objects in a category $\mathbf{A}$ with smash products. Then*:

(a) *if the second row in (4.4) is a coequalizer diagram, then so is the third one*;
(b) *if the third row in (4.4) is a coequalizer diagram, then the object $A\wedge(B\wedge C)$ together with the canonical morphism from $A\times(B\times C)$ into it can be described as the colimit of*

$$\begin{array}{ccc}
 & & A+(B\times C) \\
 & & d \downarrow\downarrow 0 \\
A\times(B+C) & \underset{\mathrm{id}_A\times 0}{\overset{\mathrm{id}_A\times d}{\rightrightarrows}} & A\times(B\times C),
\end{array} \tag{4.5}$$

or, equivalently, as the cokernel of $(A\times(B+C))+(B\times C)\to A\times(B\times C)$.

Let us explain the last sentence in 4.6(b): Since $A+(B\times C)$ vanishes in the colimit above and hence so does A, to make $\mathrm{id}_A\times d$ equal to $\mathrm{id}_A\times 0$ is the same as to make it equal to zero. Therefore the colimit coincides with the cokernel of $(A\times(B+C))+A+(B\times C)\to A\times(B\times C)$; but then the middle A can be omitted since it is “smaller” than $A\times(B+C)$.

Theorem 4.7. *Let $\mathbf{A}$ be a category with smash products satisfying the following conditions for every triple A, B, C of its objects*:

(a) *The canonical morphism $(A\times B)+(A\times C)\to A\times(B+C)$ is an epimorphism*;

(b) *The functor* $A \wedge (-): \mathbf{A} \to \mathbf{A}$ *preserves the cokernel of the canonical morphism* $B + C \to B \times C$.

Then the category $\mathbf{A}$ *is* $\wedge$*-associative; the same is also true if we replace* (b) *with the following condition:*

(c) *The functor* $A \times (-): \mathbf{A} \to \mathbf{A}$ *preserves the coequalizer of the canonical and the zero morphisms from* $B + C$ *to* $B \times C$, *or, equivalently preserves the pushout*

$$\begin{array}{ccc} B + C & \longrightarrow & 0 \\ {\scriptstyle d}\downarrow & & \downarrow \\ B \times C & \xrightarrow[s]{} & B \wedge C, \end{array} \tag{4.6}$$

Proof. By Lemma 4.6, any of the conditions (b) and (c) imply that the canonical morphism $A \times (B \times C) \to A \wedge (B \wedge C)$ is a cokernel of $(A \times (B + C)) + (B \times C) \to A \times (B \times C)$; and since (a) tells us that $A \times (B + C)$ can be replaced with $(A \times B) + (A \times C)$, we conclude that the canonical morphism $A \wedge B \wedge C \to A \wedge (B \wedge C)$ is an isomorphism. Using similar arguments we can also prove that $A \wedge B \wedge C \to (A \wedge B) \wedge C$ is an isomorphism (for all A, B, C) and so $\mathbf{A}$ is $\wedge$-associative.

Corollary 4.8. *Let* $\mathbf{X}$ *be a category with finite limits and finite colimits satisfying Condition* 1.3. *Then* $(1 \downarrow \mathbf{X})$ is $\wedge$*-associative*.

Proof. Condition 1.3(b) implies that $(1 \downarrow \mathbf{X})$ satisfies the condition 4.7(c). Therefore we only need to show that $(1 \downarrow \mathbf{X})$ satisfies the condition 4.7(a), i.e. to show that for every triple $(X, p), (Y, q), (Z, r)$ of objects in $(1 \downarrow \mathbf{X})$, the canonical morphism $(X \times Y) +_1 (X \times Z) \to X \times (Y +_1 Z)$ is an epimorphism. However this follows from the fact it is equal to the composite of the regular epimorphism $(X \times Y) +_1 (X \times Z) \to (X \times Y) +_X (X \times Z) = (X \times Y) +_{X \times 1} (X \times Z)$ and the canonical morphism $(X \times Y) +_{X \times 1} (X \times Z) \to X \times (Y +_1 Z)$, which is an isomorphism by 1.3(a).

Remark 4.9. This approach to associativity in $(1 \downarrow \mathbf{X})$ is quite different from the standard one that works in the cartesian closed case and goes back at least to Eilenberg and Kelly [3] (see also Section VII.9 in S. Mac Lane's book [4]). In the general case there could also be other possibilities, for instance defining the triple smash product via

$$(A \times (B + C)) + (B \times (A + C)) + (C \times (A + B)) \to A \times B \times C \to A \wedge B \wedge C$$

instead of the top row in (4.3), which would coincide with our definition in the distributive case.

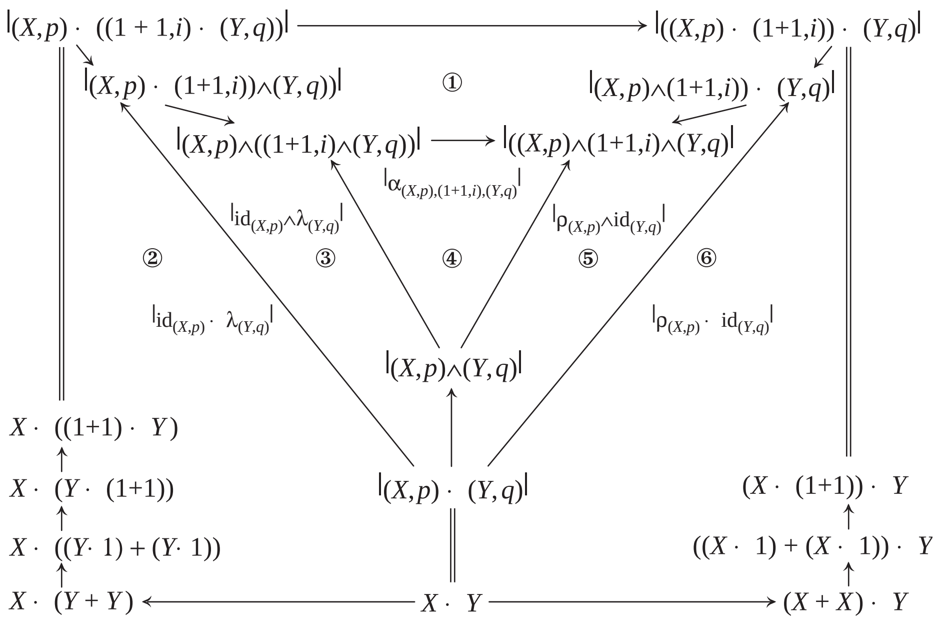
5. Coherence

Let $\mathbf{X}$ be a category with finite limits and finite colimits satisfying Conditions 1.1 and 1.3 as in Corollary 4.8. We make $(1 \downarrow \mathbf{X})$ a symmetric monoidal category

$((1 \downarrow \mathbf{X}), (1+1, i), \wedge, \alpha, \lambda, \rho, \sigma)$ as follows:

- $\wedge$ is the smash product as defined in Introduction (see (1.1) and (1.2)); this definition of course agrees with the one used in Section 4 (see (4.1)) for $\mathbf{A} = (1 \downarrow \mathbf{X})$.
- $(1+1, i)$ and ρ are as in Section 3 (see Conclusion 3.1). The readers should forgive us for using the "wrong direction" for ρ; those who will not, should use ρ^{-1} instead of it.
- $\alpha_{A,B,C} : A \wedge (B \wedge C) \to (A \wedge B) \wedge C$ is the composite of canonical isomorphisms $A \wedge (B \wedge C) \approx A \wedge B \wedge C \approx (A \wedge B) \wedge C$; it is well defined since $(1 \downarrow \mathbf{X})$ is $\wedge$-associative by Corollary 4.8.
- σ, the symmetry isomorphism for $\wedge$, is induced by the symmetry isomorphisms for $+$ and $\times$ in $(1 \downarrow \mathbf{X})$.
- $\lambda = \sigma\rho$ as in Conclusion 3.1, i.e. $\lambda_A : A \to (1+1, i) \wedge A$ is the composite of ρ_A and the symmetry isomorphism $A \wedge (1+1, i) \approx (1+1, i) \wedge A$ (again, those readers who will use ρ^{-1} instead of ρ, should also use λ^{-1} instead of λ of course).

We have to check that these data satisfies the coherence conditions required in the definition of monoidal category. For, we first observe that since α and σ are in fact *induced* by the associativity and symmetry isomorphisms for $\times$, the coherence conditions on $\wedge$ involving only α and σ follow from the similar conditions on $\times$. What remains is the commutativity of the triangle ④ in the diagram



where the bottom arrows are induced by the second coproduct injections $Y \to Y + Y$ and $X \to X + X$, respectively, the other unnamed arrows are the appropriate canonical isomorphisms, and $|(X, p) \times (Y, q)|$ denotes the underlying object in $\mathbf{X}$ of the object

$(X,p)\times(Y,q)$ of $(1\downarrow \mathbf{X})$, etc. However we have:

- the commutativity of the enveloping square can be shown by a straightforward calculation;
- ① commutes because α is induced by the associativity isomorphism for $\times$ as already mentioned;
- the commutativity of ② and ⑥ follows from the description of $\rho_{(X,p)}$ (in $\mathbf{X}$) as the composite $X\to X+X\approx(X\times 1)+(X\times 1)\to X\times(1+1)\to Z$ mentioned in Conclusion 3.1;
- ③ and ⑤ commute because the s in (1.2) is obviously natural in both arguments;
- hence the triangle ④ also commutes.

6. Distributivity

In a category $\mathbf{A}$ with smash products, consider the commutative diagram

$$\begin{array}{ccccc}
(A+B)+_A(A+C) & \xrightarrow{d+d} & (A\times B)+_A(A\times C) & \xrightarrow{s+s} & (A\wedge B)+_A(A\wedge C)\\
\downarrow & & \downarrow & & \downarrow\\
A+(B+C) & \xrightarrow[d]{} & A\times(B+C) & \xrightarrow[s]{} & A\wedge(B+C)
\end{array}$$

where the vertical arrows are appropriate canonical morphisms. Since its rows are cokernel diagrams and the first vertical arrow is an isomorphism, we conclude: if the second vertical arrow is an isomorphism, then so is the third one. Since the morphisms from A to $(A\wedge B)$ and to $(A\wedge C)$ used in the pushout $(A\wedge B)+_A(A\wedge C)$ are zeros, we have $(A\wedge B)+_A(A\wedge C)\approx(A\wedge B)+(A\wedge C)$, and for $\mathbf{A}=(1\downarrow\mathbf{X})$ we obtain:

Proposition 6.1. *Let* $\mathbf{X}$ *be a category with finite limits and finite colimits satisfying Condition* 1.3(a). *Then the smash product in* $(1\downarrow\mathbf{X})$ *is distributive with respect to* (*finite*) *coproducts.*

7. Recovering X from (1 ↓ X)

Definition 7.1. Let $\mathbf{A}$ be a category with smash products, I an object in $\mathbf{A}$, and for each object A in $\mathbf{A}$, ρ_A a fixed isomorphism from A to $A\wedge I$, natural in A. A morphism $e:A\to I$ is said to be a (the) counit of A if the diagram

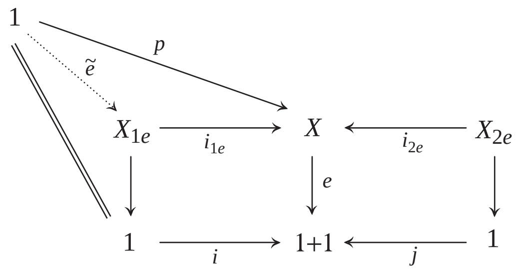
$$\begin{array}{ccc}
A\wedge A & \xrightarrow{\mathrm{id}_A\wedge e} & A\wedge I\\
{\scriptstyle\Delta_A}\uparrow & \nearrow{\scriptstyle\rho_A} & \\
A & &
\end{array}\tag{7.1}$$

where Δ_A is the composite $A \to A \times A \to A \wedge A$ of the diagonal morphism of A with the canonical morphism into $A \wedge A$, commutes. The full subcategory of $(I \downarrow \mathbf{A})$ with objects all pairs (A, e), with e a counit of A, will be denoted by $\mathbf{A}_*$.

Remark 7.2. (a) Since the "comultiplication" Δ_A is obviously cocommutative, the right counit condition we in fact require is equivalent to the left one—and this is why we do not need to require both of them. This also tells us that $\mathbf{A}_*$ can be identified with a subcategory in $\mathbf{A}$.

(b) Of course the category $\mathbf{A}_*$ is determined by $\mathbf{A}$ alone uniquely up to an isomorphism.

In the rest of this section we assume $\mathbf{X}$ to be a lextensive category with finite colimits. For a morphism $e:(X, p) \to (1+1, i)$ (where $(1+1, i)$ is the unit object for $\wedge$—see Section 3) in $(1 \downarrow \mathbf{X})$ consider the commutative diagram



(7.2)

where the rows are coproduct diagrams and hence the two squares are pullbacks, and $\tilde{e}$ is determined by the universal property of the first of them. Since (by lextensivity [2]) (X, e) corresponds to the pair (X_{1e}, X_{2e}) under the equivalence between the categories $(\mathbf{X} \downarrow (1+1))$ and $\mathbf{X} \times \mathbf{X} \approx (\mathbf{X} \downarrow 1) \times (\mathbf{X} \downarrow 1)$, we obtain:

Lemma 7.3. *There is an equivalence of categories* $((1 \downarrow \mathbf{X}) \downarrow (1+1, i)) \sim (1 \downarrow \mathbf{X}) \times \mathbf{X}$, *under which*:

(a) $((X, p), e) \mapsto ((X_{1e}, \tilde{e}), X_{2e})$ *in the notation of* (7.2);
(b) *an object* $((U, f), V)$ *in* $(1 \downarrow \mathbf{X}) \times \mathbf{X}$ *corresponds to the object* $((U+V, kf), !_U + !_V)$, *where* $k: U \to U+V$ *is the first coproduct injection.*

In order to prove our main result (Theorem 7.7 below) we need three more technical lemmas:

Lemma 7.4. *In the notation of* (7.2), *the composite*

$$X \xrightarrow{\text{diagonal}} X \cdot X \xrightarrow{\mathrm{id}_X \cdot e} X \cdot (1+1) \approx X + X \xrightarrow{\begin{pmatrix} p!_X \\ \mathrm{id}_X \end{pmatrix}} X \qquad (7.3)$$

has the following properties:

(a) *it can be described as the unique morphism* $\hat{e}: X \to X$ *with* $\hat{e}i_{1e} = p!_{X_{1e}}$ *and* $\hat{e}_{i_{2e}} = i_{2e}$;
(b) *it is an endomorphism of* (X, p) *in* $(1 \downarrow \mathbf{X})$.

Proof. (a) Use the obvious commutative diagram

$$
\begin{array}{ccccc}
X_{1e} & \longrightarrow & X & \longleftarrow & X_{2e} \\
\downarrow & & \downarrow & & \downarrow \\
X \times X_{1e} & \longrightarrow & X \times X & \longleftarrow & X \times X_{2e} \\
\downarrow & & \downarrow & & \downarrow \\
X + 1 & \longrightarrow & X + (1 \times 1) & \longleftarrow & X + 1 \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & X + X & \longleftarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & X & \longleftarrow & X
\end{array}
$$

(b) just needs another straightforward calculation. □

Lemma 7.5. *The morphism* $\hat{e}: (X, p) \to (X, p)$ *above makes the diagram*

$$
\begin{array}{ccc}
(X, p) \wedge (X, p) & \xrightarrow{\mathrm{id}_{(X,p)} \wedge e} & (X, p) \wedge (1 + 1, i) \\
{\scriptstyle \Delta_{(X,p)}} \uparrow & & \uparrow {\scriptstyle \rho_{(X,p)}} \\
(X, p) & \xrightarrow[\hat{e}]{} & (X, p),
\end{array}
\tag{7.4}
$$

where Δ *is as Definition* 7.1, *commute.*

Proof. Comparing the composite (7.3) with the square ② in (3.2), we conclude that $\rho_{(X,p)}\hat{e}$ coincide with the composite $(X, p) \to (X, p) \times (X, p) \to (X, p) \times (1 + 1, i) \to (X, p) \wedge (1 + 1, i)$ and hence with $(X, p) \to (X, p) \times (X, p) \to (X, p) \wedge (X, p) \to (X, p) \wedge (1 + 1, i)$, i.e. with $(\mathrm{id}_{(X,p)\wedge} e)\Delta_{(X,p)}$, as desired. □

Lemma 7.6. *The following conditions are equivalent*:

(a) *the diagram*

$$\begin{array}{ccc} (X,p)\wedge(X,p) & \xrightarrow{\mathrm{id}_{(X,p)}\wedge e} & (X,p)\wedge(1+1,i) \\ {\scriptstyle \Delta_{(X,p)}}\big\uparrow & \nearrow{\scriptstyle \rho_{(X,p)}} & \\ (X,p) & & \end{array} \qquad (7.5)$$

where $\Delta_{(X,p)}$ is as in (7.4), *commutes*; *that is*, $e:(X,p)\to(1+1,i)$ *is a counit of* (X,p) *in the sense of Definition* 7.1;

(b) $\hat{e}=\mathrm{id}_X$, *where* $\hat{e}$ *is as in Lemma* 7.4;

(c) $i_{1e}=p!_{X_{1e}}$, *in the notation of* (7.1) *and* 7.4(a);

(d) $\tilde{e}$ *in* (7.2) *is an isomorphism*;

(e) $X_{1e}\approx 1$, *where* X_{1e} *is as in* (7.2);

(f) *the diagram*

$$\begin{array}{ccc} 1 & \xrightarrow{i_{1e}} & X \\ \Big\| & & \Big\downarrow{\scriptstyle e} \\ 1 & \xrightarrow[i]{} & 1+1 \end{array}$$

is a pullback;

(g) *the morphism* $e:(X,p)\to(1+1,i)$ (*in* $(1\downarrow \mathbf{X})$) *has zero kernel.*

Proof. (b) $\Rightarrow$ (a) follows from Lemma 7.5, and since $\rho_{(X,p)}$ is an isomorphism, the same is true for (a) $\Rightarrow$ (b). (b) $\Rightarrow$ (c) follows from 7.4(a), and since $\mathrm{id}_X i_{2e}=i_{2e}$, the same is true for (c) $\Rightarrow$ (b).

(c) $\Rightarrow$ (d): Since i_{1e} is a monomorphism (being a coproduct injection in a lextensive category), (c) implies that $!_{X_{1e}}$ also is a monomorphism; since $!_{X_{1e}}\tilde{e}=\mathrm{id}_1$ (see (7.2)), this implies that $\tilde{e}$ is an isomorphism.

(d) $\Rightarrow$ (c): If $\tilde{e}$ is an isomorphism, then $i_{1e}=p!_{X_{1e}}$ follows from the equalities $!_{X_{1e}}\tilde{e}=\mathrm{id}_1$ and $i_{1e}\tilde{e}=p$ that determine $\tilde{e}$ in (7.2).

(d) $\Leftrightarrow$ (e) is obvious from (7.2), and (e) $\Leftrightarrow$ (f) follows from the fact that the left-hand square in (7.2) is a pullback. (f) $\Leftrightarrow$ (g) is trivial. □

Theorem 7.7. *Let* $\mathbf{X}$ *to be a lextensive category with finite colimits. Then the functor* $\mathbf{X}\to(1\downarrow\mathbf{X})_*$ *defined by* $X\mapsto((1+X,i_X),\mathrm{id}_1+!_X)$ (*where* i_X *is the first coproduct injection*) *is a category equivalence.*

Proof. The equivalence described in Lemma 7.3 induces an equivalence between the full subcategory in $((1\downarrow\mathbf{X})\downarrow(1+1,i))$ with objects all $((X,p),e)$ having $\tilde{e}:1\to X_{1e}$ an isomorphism and the category $\mathbf{X}$. The equivalence (a) $\Leftrightarrow$ (d) in Lemma 7.6 then tells us that full subcategory in $((1 \downarrow \mathbf{X}) \downarrow (1+1,i))$ is nothing but $(1 \downarrow \mathbf{X})_*$. The fact that the equivalence is given by $X \mapsto ((1+X, i_X), \mathrm{id}_1 + !_X)$ follows from 7.3(b). □

8. Final remarks

Both of our main results, namely the existence of the symmetric monoidal structure on $(1 \downarrow \mathbf{X})$ described in Section 5 (and satisfying Proposition 6.1) and Theorem 7.7, do apply whenever the ground category $\mathbf{X}$ satisfies Conditions 1.1, 1.2, and 1.3. And as we already mentioned, among categories satisfying these conditions we have

- any lextensive cartesian closed category with finite colimits, the category of compactly generated topological spaces, and the categories of all categories, of groupoids, of (pre)ordered sets, etc.; in particular, since not all colimits are needed for the smash product but just those appearing in the definition, the reader can verify that any pretopos has them and satisfies Conditions 1.1, 1.2, and 1.3;
- the opposite category $(R \downarrow \mathbf{CommRings})^{\mathrm{op}} \approx ((\mathbf{CommRings})^{\mathrm{op}\downarrow} R)$ of commutative R-algebras over a commutative ring R; here rings and algebras are supposed to be associative and to have the unit element, and homomorphisms are supposed to preserve it.

However we do not really know how far are our conditions from being necessary, and at the moment we are not able to answer many natural questions related to this–including those on general topological spaces. Still, we should certainly make the following:

Remark 8.1. (a) In contrast to Examples 4.3 and 4.4, it is easy to show that the opposite categories of groups, of (non-commutative) rings without unit, and many other similar categories are not $\wedge$-associative. The category of commutative R-algebras without unit is $\wedge$-associative, which follows from *any* of the following two observations:

- this category is equivalent to $(1 \downarrow \mathbf{X})$, where $\mathbf{X} = (R \downarrow \mathbf{CommRings})^{\mathrm{op}}$;
- in this category the smash product $\wedge$ is nothing but the tensor product over R.

Hence it looks like we have no "natural" example of a lextensive category $\mathbf{X}$ with finite colimits for which $(1 \downarrow \mathbf{X})$ is not $\wedge$-associative. However there are "unnatural ones":

Let $\mathbf{X} = \mathrm{Fam}(\mathbf{A})$ be the category of families (or of finite families) of objects in a category $\mathbf{A}$ that is pointed and has finite limits and finite colimits, but which is not $\wedge$-associative. Then $\mathbf{X}$ and $(1 \downarrow \mathbf{X})$ also have finite limits and finite colimits, and moreover, the canonical functor $\mathbf{A} \to (1 \downarrow \mathbf{X})$ preserves them. Therefore $(1 \downarrow \mathbf{X})$ cannot be $\wedge$-associative.

(b) Let $\mathbf{A}$ be as in Example 4.4. Then it is $\wedge$-associative, but does $\wedge$ have a unit object "similar" to $(1+1,i)$ in $(1 \downarrow \mathbf{X})$, which would make $\mathbf{A}$ a monoidal category?

In fact it is easy to see that such an object must have a two-element underlying set, say $\{0, a\}$, with

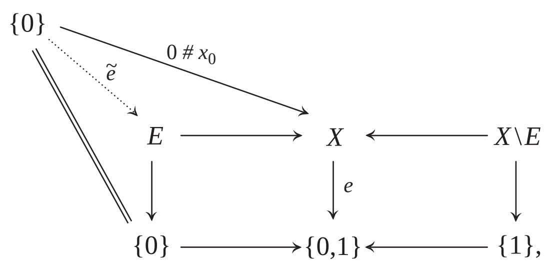
$$\omega(x_1, \ldots, x_n) = \begin{cases} 0 \text{ if either at least one of } x_1, \ldots, x_n \text{ is } 0 \text{ or } n = 0, \\ a \text{ if } n \neq 0 \text{ and } x_1 = \cdots = x_n = a \end{cases}$$

for each basic n-ary operation ω. And the answer is affirmative if and only if the variety **A** contains this two-element algebra.

(c) The equivalence described in Theorem 7.7 is of course a special type of descent construction. In the language of descent theory we could say that whenever **X** (has finite colimits and) is lextensive, the morphism $0 \to 1$ in it is an effective codescent morphism—or, equivalently that the left adjoint of the forgetful functor $(1 \downarrow \mathbf{X}) \to \mathbf{X}$ is comonadic. However if we only require that comonadicity (without lextensivity), there is no way to describe descent data as objects with counits as in Theorem 7.7, and to obtain various reformulations as in Lemma 7.6. It is interesting to compare this with another instances of those descent data described in [1]. Let us also recall that the existence of unit object in $(1 \downarrow \mathbf{X})$ needs only the preservation of finite sums (actually just of 1+1) mentioned in Condition 1.1, and the $\wedge$-associativity of $(1 \downarrow \mathbf{X})$ needs only Condition 1.3; it is interesting that this (obviously) includes the case of an additive **X**, and hence the situation of [1], eventhough the smash product is trivial there.

As Lemma 7.6 is mentioned in 8.1(c), the reader would probably ask, what are these $\hat{e}$ and $\tilde{e}$ involved in its formulation, and what does Lemma 7.6 really say about them in concrete examples? Let us consider two simple cases:

Example 8.2. For $\mathbf{X} = \mathbf{Sets}$, we have $(1 \downarrow \mathbf{X}) = \mathbf{Pointed\ Sets}$; we will write the objects of this category as pairs (X, x_0), where x_0 is an element in X. The object $(1+1, i)$ of $(1 \downarrow \mathbf{X})$ can then be written as $(\{0,1\}, 0)$, and giving a morphism $e : (X, x_0) \to (\{0,1\}, 0)$ is to give a subset E of X containing x_0 (which is the inverse image of 0 under e). In this notation the diagram (7.2) becomes

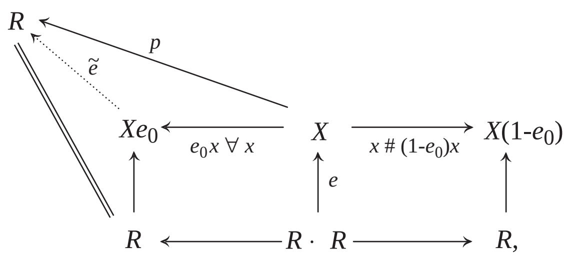


where all horizontal arrows are the inclusion maps, and so $\tilde{e}(0) = x_0$. Since from 7.4(a) we obtain

$$\hat{e}(x) = \begin{cases} x_0 \text{ if } x \text{ is in } E, \\ x \text{ if } x \text{ is not in } E, \end{cases}$$

it is easy to check that each of the conditions (a)–(g) in Lemma 7.6 says that E must be equal to $\{x_0\}$.

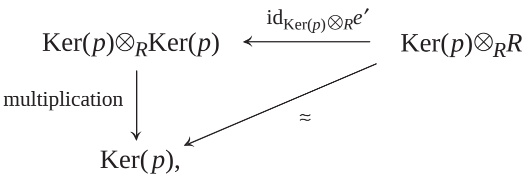
Example 8.3. For $\mathbf{X} = (R \downarrow \mathbf{CommRings})^{\mathrm{op}}$, we will still write (X, p) for the objects in $(1 \downarrow \mathbf{X})$, although here p is a (unit preserving) ring homomorphism from X to R. The diagram (7.2) displayed in the category of commutative rings becomes



where $e_0 = e((1, 0))$, and the bottom horizontal arrows are the product projections. Note that:

- the units in Xe_0 and $X(1 - e_0)$ are e_0 and $(1 - e_0)$, respectively;
- e_0 is an idempotent in X with $p(e_0) = 1$ and every such an idempotent uniquely determines the corresponding e via $e((\mathrm{r}_1, \mathrm{r}_2)) = \mathrm{r}_1 e_0 + \mathrm{r}_2(1 - e_0)$;
- for x in Xe_0 we have $\tilde{e}(x) = p(x) = p(e_0 x)$.

Now 7.2 gives $e_0\hat{e}(x) = p(x)e_0$ and $(1 - e_0)\hat{e}(x) = (1 - e_0)x$, and so $\hat{e}(x) = p(x)e_0 + (1 - e_0)x$. In order to translate Lemma 7.6 in the "language of elements" let us also observe that under the equivalence of $(1 \downarrow \mathbf{X})$ with the category of commutative R-algebras without unit the (X, p) corresponds to the kernel of p, and diagram (7.5) transforms into the following diagram of ring homomorphisms:



(8.1)

where e' is the homomorphism from $R \approx \mathrm{Ker}(R \times R \to R)$ to $\mathrm{Ker}(p)$ induced by e, and since R is identified with $\mathrm{Ker}(R \times R \to R)$ via $r\#(0, r)$, we have $e'(r) = r(1 - e_0)$. It is easy to see now that the equivalent conditions of Lemma 7.6 translate, respectively, as the following conditions on an idempotent e_0 in X satisfying $p(e_0) = 1$:

(a) the diagram (8.1) commutes, i.e. $x(r(1 - e_0)) = xr$ for all x in $\mathrm{Ker}(p)$ and r in R - or, equivalently, $xe_0 = 0$ for all x in $\mathrm{Ker}(p)$, which means that $1 - e_0$ is a (the) unit in the ring $\mathrm{Ker}(p)$;
(b) $x = p(x)e_0 + (1 - e_0)x$ for all x in X;
(c) $xe_0 = p(x)e_0$ for all x in X;
(d) the map $Xe_0 \to R$ induced by p is bijective;

(e) $Xe_0 \approx R$;

(f) the diagram

$$\begin{array}{ccc} R & \xleftarrow{\;\;p\;\;} & X \\ \Big\| & & \Big\uparrow {\scriptstyle \begin{array}{c} \mathrm{r}_1 e_0 + \mathrm{r}_2(1-e_0) \\ \& \\ (\mathrm{r}_1, \mathrm{r}_2) \end{array}} \\ R & \xleftarrow[\text{first projection}]{} & R \times R \end{array}$$

is a pushout;

(g) the morphism $R \to \mathrm{Ker}(p)$ defined by $r\#r(1-e_0)$ has zero cokernel.

Finally, note that most of the translations above can be repeated for commutative semirings (with unit) since their opposite category also is lextensive; however instead of specifying e_0 we would have to specify a pair (e_0, e_1) with $e_0 + e_1 = 1$, and we would have nothing like the equivalence with rings without unit mentioned in 8.1(a) and used for (8.1) and for the translation of 7.6(g).

References

[1] A. Carboni, G. Janelidze, Modularity and descent, J. Pure Appl. Algebra 99 (1995) 255–265.
[2] A. Carboni, S. Lack, R.F.C. Walters, Introduction to extensive and distributive categories, J. Pure Appl. Algebra 84 (1993) 145–158.
[3] S. Eilenberg, G.M. Kelly, Closed categories, Proceedings of the Conference on Categorical Algebra, La Jolla, 1965, Springer, Berlin, 1966, pp. 421–562.
[4] S. Mac Lane, Categories for the Working Mathematician, GTM 5, Springer, Berlin, 1971.