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A NEW COMPUTATIONAL ALGORITHM OF SPECTRAL FACTORIZATION FOR POLYNOMIAL MATRIX-FUNCTIONS

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Abstract. In the present paper we describe the calculation procedure for approximate spectral factorization of polynomial matrix-functions. The algorithm depends on a new general method of multivariate spectral factorization developed in Studia Math. 137, 1999, 93-100. For simplicity, we consider only the second order matrix case.

რეზიუმე. წინამდებარე სტატიაში ალვწერთ პოლინომური მატრიცის სპექტრული ფაქტორიზაციის აპროქსიმაციის პროცედურას. ალგორითმი ეფუძნება [1] ნაშრომში განვითარებულ მრავალგანზომილებიანი სპექტრული ფაქტორიზაციის ახალ ზოგად მეთოდს. სიმარტივისათვის განვიხილავთ მხოლოდ მეორე რიგის მატრიცებს.

1. Introduction

Spectral factorization is the representation of a positive definite matrix-function $S(t) = (f_{ij})_{i,j=\overline{1,r}}$, $|t| = 1$, as a product

$$S(t) = \chi^+(t) \cdot (\chi^+(t))^*, \tag{1}$$

where χ^+ is an outer analytic matrix-function defined inside the unit circle and $(\chi^+)^*$ is its adjoint. This is the procedure arising in many system-theoretic applications.

In general, it is assumed that the entries of $S(t)$ are integrable functions, $f_{ij} \in L_1(\mathbb{T})$, and the entries of the factor matrix-function χ^+ belong to the Hardy space H_2. In that case, the condition

$$\log\det(S(t)) \in L_1 \tag{2}$$

is necessary and sufficient for factorization (1) to exist.

Factor matrix-functions are defined up to a constant right unitary multiplier. χ^+ is unique if we require that $\chi^+(0)$ be positive definite and in this case the factorization is called canonical.

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In the one-dimensional case, where $S(t)$ is a usual positive function, the factorization can be explicitly written by

$$\chi^+(z) = \exp\left\{\frac{1}{4\pi}\int_{\mathbb{T}} \frac{t+z}{t-z}\log S(t)\,dm(t)\right\}, \quad |z|<1. \tag{3}$$

No analog of this formula is valid in the multi-dimensional case since the logarithm of a matrix-valued function cannot be defined in any meaningful sense.

Formula (3) is inconvenient for practical calculations even when the density function $S(t)$ is a trigonometric polynomial, $S(t)=\sum_{k=-n}^{n} q_k t^k$. Therefore a variety of methods have been developed for approximate calculations of coefficients of the spectral factor (see [3]). The last breakthrough in this area has been made recently by using the Fast Fourier Transform (see [2]). Some of these methods can be extended to the more demanding vector case. However, the restriction on the matrix-function $S(t)$ to be rational is essential in these situations. Neither of these algorithms is considered to be exhaustive since some of the difficulties arise in dealing with specific cases. Namely, even for polynomial matrix-functions, if the determinant has zero close to the boundary, then the existing algorithms have bad convergent properties.

A completely new approach to the factorization problem was introduced in [1]. Without imposing any additional restrictions on the matrix-function $S(t)$, apart from the necessary and sufficient condition (2), the authors propose an effective spectral factorization algorithm. The algorithm is not of iterative type, but the achievement of any preassigned accuracy is guaranteed by the strict mathematical statements. The procedure involves calculations of a number of Fourier coefficients, which still needs certain computational time. Hence, in general, the algorithm is more appropriate to achieve an exact rather than a fast result. In spite of the evident improvement by the proposed algorithm – that it solves problem in the most general setting – the question naturally arises if the same algorithm has some additional advantages when it is applied to the cases where the spectral density is of more simple form.

The aim of the present paper is to demonstrate the fitness of the algorithm [1] when it is applied to polynomial matrix-functions. In this situation, the Fourier coefficients required by the algorithm can be found without calculating any integral. Moreover, one can immediately obtain as many number of them as necessary to achieve the desired accuracy. Below we describe the steps of the procedure for an approximate calculation of the spectral factor in the case of polynomial matrix-valued density function. We avoid any mathematical justification for which the interested reader is referred to [1].

We restrict ourselves to considering the second order matrices

$$S(t)=\begin{pmatrix} A(t) & B(t) \\ B^*(t) & C(t) \end{pmatrix}. \tag{4}$$

This particular case illustrates all the difficulties arising when one deals with matrix-valued densities rather than scalar valued ones. Dimensions higher than two do not cause any additional difficulty, just the volume of necessary calculations increases proportionally to the dimension.

2. Notations and Definitions

For a positive trigonometrical polynomial $A(t)=\sum\limits_{k=-n}^{n} a_k t^k$, $A(t)\geq 0$ when $|t|=1$, there exists a unique analytic polynomial

$$P(t)=\sum_{k=0}^{n} p_k t^k \tag{5}$$

positive at the origin and without zeros inside the unit circle, such that

$$A(t)=P(t)\cdot P^*(t).$$

We denote $P(t):=\sqrt{A}(t)$ and, having the coefficients of A, we refer to the process of finding the coefficients of $\sqrt{A}$ as the one-dimensional polynomial spectral factorization.

Since the 0-th coefficient of (5) is different from 0, we can define the formal infinite polynomial

$$P^{-1}(t)=\sum_{k=0}^{\infty} q_k t^k, \tag{6}$$

which satisfies the condition $P(t)\cdot P^{-1}(t)=1$. (It is always assumed that the coefficients of the product are well defined if at least one factor is a finite polynomial.) Obviously, the coefficients q_k of (6) can be easily determined by a simple algebraic expression in the recurrent way. When we know that (5) does not have zeros inside the unit circle, we can conclude that the formal series (6) is convergent inside the unit circle. Furthermore, to make our calculation procedure described below consistent with the theoretical justification given in [1], we have to bear in mind the validity of the following simple

Lemma. *Let $Q(t)$ and $P(t)$ be two trigonometric polynomials where $P(t)$ is analytic, not containing zeros inside the unit circle. If $Q(t)/P(t)$ is integrable on the unit circle* (*to clarify, the latter condition means that if t_0 is zero of the polynomial $P(t)$ of multiplicity m_0 and $|t_0|=1$, then it is also zero of the polynomial $Q(t)$ of multiplicity $m_1\geq m_0$*). *Then the Fourier* *coefficients of function $Q(t)/P(t)$, $|t|=1$, can be determined by the formal product*

$$Q(t)\cdot P^{-1}(t).$$

If we have a formal infinite polynomial $R(t)=\sum\limits_{k=-\infty}^{\infty} c_k t^k$, then $[R(t)]_0^n$ denotes the analytic polynomial $\sum\limits_{k=0}^{n} c_k t^k$ and $R^*(t)=\sum\limits_{k=-\infty}^{\infty} \overline{c_{-k}}\, t^k$.

I_N stands for the identity matrix of order N.

3. Data of the Problem

It is well-known that if the spectral density $S(t)$ is a trigonometric polynomial of order n, then the factor matrix-function $\chi^+(t)$ is an analytic polynomial of the same order n. Thus, having as initial data the coefficients of three polynomials $A(t)=\sum\limits_{k=-n}^{n} a_n t^n$, $B(t)=\sum\limits_{k=-n}^{n} b_k t^k$ and $C(t)=\sum\limits_{k=-n}^{n} c_k t^k$ (see (4)), which satisfy the conditions $A(t)\geq 0$ and $(AC-BB^*)(t)\geq 0$, $|t|=1$, we have to obtain the coefficients of four analytical polynomials $\chi_{i,j}(t)=\sum\limits_{k=0}^{n} l_k^{i,j} t^k$, $1\leq i,j\leq 2$, as a final result. Obviously, the desired coefficients can be calculated only approximately.

4. Calculation Procedure

First, we have to fix a large positive integer N which specifies the accuracy of the calculated coefficients. Theoretically, one can determine the dependence of the approximation on N. But, in practice, one can take N so large as to reliably solve the system of $N\times N$ linear algebraic equations with a positive definite coefficient matrix, all eigenvalues of which exceed 1.

Step 1. Perform the one-dimensional factorization to obtain the analytic polynomials

$$P(t)=\sqrt{A}(t) \quad \text{and} \quad F(t)=\sqrt{AC-BB^*}(t)$$

which are of order n and $2n$, respectively.

Step 2. Calculate $d_0, d_1, \dots, d_N$, the first $N+1$ coefficients of the (infinite) polynomial

$$P^{-1}F(t)=\sum_{k=0}^{\infty} d_k t^k\,,$$

and $\gamma_0,\gamma_1,\dots\gamma_N$, the first $N+1$ non-negative coefficients of

$$P^{-1}B(t)=\sum_{k=-n}^{\infty} \gamma_k t^k.$$

Step 3. Define the matrix $\Theta = (\theta[i,j])_{i,j=\overline{0,N}}$ by the formula

$$\theta[i,j] = \theta[j,i] = \begin{cases} \sum_{k=0}^{N-(i+j)} d_k \gamma_{i+j+k} & \text{when } \ i+j \leq N \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\Delta = \Theta \cdot \overline{\Theta} + I_{(N+1)}.$$

Note that $\Theta \cdot \overline{\Theta}$ is positive definite since Θ is symmetric and $\overline{\Theta} = \Theta^*$. Thus Δ is a positive definite matrix function with all eigenvalues exceeding 1.

Step 4. Solve the system of algebraic linear equations

$$\Delta \cdot X = \mathbf{1},$$

where X is the $N+1$ column vector with unknown variables and $\mathbf{1}$ is the $N{+}1$ column vector which has the first entry equal to 1 and all other entries to 0.

The solution is assumed to be $x_0, x_1, \dots, x_N$.

Then determine the column vector $Y = (y_0, y_1, \dots, y_N)^T$ by the formula

$$Y = \Theta \cdot X.$$

Step 5. Calculate $M = \Big(\big|\sum_{k=0}^{N} x_k\big|^2 + \big|\sum_{k=0}^{N} y_k\big|^2\Big)^{\frac{1}{2}}$ and define the analytic polynomials of order N

$$\alpha_N(t) = \frac{1}{M}\sum_{k=0}^{N} x_k t^k \quad \text{and} \quad \beta_N(t) = \frac{1}{M}\sum_{k=0}^{N} y_k t^k.$$

Step 6. Obtain the result

$$\chi_{1,1}(t) = [P\alpha_N(t)]_0^n, \quad \chi_{1,2}(t) = [P\beta_N(t)]_0^n,$$
$$\chi_{2,1}(t) = \big[(P^{-1}B)^*\alpha_N - F(\beta_N)^*\big]_0^n, \quad \chi_{2,2}(t) = \big[(P^{-1}B)^*\beta_N + F(\alpha_N)^*\big]_0^n.$$

We have to multiply the result by the corresponding constant unitary matrix if we need to get the canonical factorization.

5. Conclusion

It is proved in [1] that if we let N tend to infinity, then the limiting result will be exact. In practice, we have approximate results when performing Steps 1 and 4. Thus, in the polynomial case, the multivariate factorization problem is reduced to the one-dimensional case and to the problem of solving linear algebraic equations. Both of these techniques are extremely well developed as compared with the methods of solving of the original problem.

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We are grateful to IT specialist T. Kobachishvili who has implemented our algorithm using the *C*++ programming language. This program is available free at www.rmi.acnet.ge/SpFact. (The factorization of a scalar-valued polynomial, Step 1, in this program is performed by the Wilson's algorithm [4].) The factorization program for higher order matrices may be obtained by contacting the authors.

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