Proc. A. Razmadze Math. Inst. 131 (2003), 123–125

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Non-abelian (Co)Homology of Lie Algebras

The main goal of current work is to construct and investigate non-abelian homology in all dimensions and second cohomology of Lie algebras, basically using ideas from [6, 5, 3], and generalizing the classical Chevalley–Eilenberg (co)homology and extending Guin's low-dimensional non-abelian (co)homology of Lie algebras [2].

Throughout this note all Lie algebras are considered over a commutative ring Λ with identity. We denote byLie the category of Lie algebras.

Let $\mathfrak{A}_N$ denote, for a fixed Lie algebra N, the category whose objects are all Lie algebras M together with an action of M on N and an action of N on M and introduce the free Lie algebra (over Λ-modules) cotriple $\mathbb{F}$ in this category in a reasonable way (see [7]).

The non-abelian tensor product of Lie algebras introduced in [1] is a basic tool for constructing of non-abelian homology of Lie algebras. Let us consider the non-abelian left derived functors $\mathcal{L}_k^{\mathbb{F}}(-\otimes N)$, $k \geq 0$ of the covariant functor $-\otimes N : \mathfrak{A}_N \to$Lie relative to the cotriple $\mathbb{F}$.

Proposition. *Let M be a Lie algebra and N be a module over M thought as an abelian Lie algebra acting trivially on the Lie algebra M, then there are natural isomorphisms*

$$\mathcal{L}_k^{\mathbb{F}}(-\otimes N)(M) \approx H_{i+1}^{CE}(M,N)\ , \qquad k \geq 1\ ,$$
$$\operatorname{Ker}\nu \approx H_1^{CE}(M,N)\ , \qquad \operatorname{Coker}\nu \approx H_0^{CE}(M,N)\ ,$$

where $\nu : M \otimes N \to N$ is a Lie homomorphism given by $\nu(m\otimes n) = {}^m n$, $m \in M$, $n \in N$ and H_*^{CE} denotes the Chevalley–Eilenberg homology of Lie algebras.

Using this proposition we make the following

Definition. Let M and N be Lie algebras acting on each other. Then the non-abelian homology of M with coefficients in N are defined by the following way:

$$H_k(M,N) = \mathcal{L}_{k-1}^{\mathbb{F}}(-\otimes N)(M)\ , \qquad k \geq 2\ ,$$
$$H_1(M,N) = \operatorname{Ker}\nu\ , \qquad H_0(M,N) = \operatorname{Coker}\nu\ ,$$

where $\nu : M\otimes N \to N/H$, $\nu(m\otimes n) = |{}^m n|$, and H is the ideal of the Lie algebra N generated by the elements ${}^{({}^n m)}n' - [n', {}^m n]$ for all $m \in M$, $n, n' \in N$.

Main result obtained on non-abelian homology of Lie algebras is the long exact homology sequence relating cyclic homology and Milnor cyclic homology, correcting the result of [2]. In particular, for a unital associative Λ-algebra A we introduce the definition of the first Milnor cyclic homology $HC_1^M(A)$ of A as the quotient of $A \otimes_\Lambda A$ by the relations

$$a\otimes b + b\otimes a = 0\ ,$$
$$ab\otimes c - a\otimes bc + ca\otimes b = 0\ ,$$
$$a\otimes bc - a\otimes cb = 0$$

2000 *Mathematics Subject Classification:* 17B40, 17B56, 18G10, 18G50.

Key words and phrases. Non-abelian tensor product, non-abelian (co)homology of Lie algebras, cyclic homology.

for all $a,\ b,\ c \in A$ (this coincides with the definition in the sense of [8] when Λ is a field of characteristic zero) and have the following

Theorem. *Let A be a non-commutative associative Λ-algebra with identity. Then there is an exact sequence of Λ-modules*

$$\cdots \longrightarrow H_2(A, V(A), [A,A]) \longrightarrow H_2(A, V(A)) \longrightarrow H_2(A, [A,A]) \longrightarrow$$
$$\longrightarrow H_1(A, V(A), [A,A]) \longrightarrow H_1(A, V(A)) \longrightarrow H_1(A, [A,A]) \longrightarrow$$
$$\longrightarrow HC_1(A) \longrightarrow HC_1^{M(A)} \longrightarrow [A,A]/[A,[A,A]] \longrightarrow 0\ ,$$

where $[A,A]$ is the additive commutator of A, $V(A)$ is defined as the quotient Lie algebra of the non-abelian tensor product $A \otimes A$ by the ideal generated by the elements $a \otimes b + b \otimes a$, $ab \otimes c - a \otimes bc + ca \otimes b$, $a,b,c \in A$ and $HC_1(A)$ is the first cyclic homology of A.

In order to introduce the second non-abelian cohomology of Lie algebras we define P-crossed R-module (M,μ) in the context of Lie algebras (the group theoretic analogue is given in [3], [4]) and construct the Lie algebra of derivations $\mathrm{Der}(P,(M,\mu))$, which coincides with the Lie algebra $\mathrm{Der}_R(R,M)$ from [2] when (M,μ) is a crossed R-module viewed as an R-crossed R-module.

Consider the following diagram of Lie algebras

$$P \quad \underset{l_1}{\overset{l_0}{\rightrightarrows}} F \xrightarrow{\tau} R, \qquad (*)$$

where F is a free Lie algebra over some Λ-module, R acts on F and also on itself by Lie multiplication, τ has Λ-module section preserving the actions, $P = \{(x,y) \in F \times F \mid \tau(x) = \tau(y)\}$, $l_0(x,y) = x$ and $l_1(x,y) = y$. Then any crossed R-module (M,μ) can be viewed as a P-crossed R-module induced by τl_i $(i = 0,\ 1)$ and a F-crossed R-module induced by τ. Denote by $\widetilde{Z^1}(P,(M,\mu))$ the subset of $\mathrm{Der}(P,(M,\mu)$ consisting of all elements of the form $(\alpha,0)$ satisfying the condition $\alpha(\Delta) = 0$, where $\Delta = \{(x,x) \mid x \in F\} \subseteq P$. Clearly $\widetilde{Z^1}(P,(M,\mu))$ is Λ-submodule of $\mathrm{Der}(P,(M,\mu))$.

Denote by $\widetilde{B^1}(P,(M,\mu))$ the Λ-submodule of $\widetilde{Z^1}(P,(M,\mu))$ consisting of all elements $(\alpha,0)$ for which there exists $(\beta,h) \in \mathrm{Der}(F,(M,\mu))$ with $\beta l_0 - \beta l_1 = \alpha$. Then $\widetilde{Z^1}(P,(M,\mu))/\widetilde{B^1}(P,(M,\mu))$ is unique up to isomorphism of choosing the diagram (*) for crossed R-module (M,μ). Moreover, one has the following

Proposition. *Let R be a Lie algebra and (M,μ) a crossed R-module.*

(i) *There is a canonical Λ-module epimorphism*

$$\vartheta : H^2_{CE}(R, \operatorname{Ker}\mu) \longrightarrow \widetilde{Z^1}(P,(M,\mu))/\widetilde{B^1}(P,(M,\mu))\ ,$$

(ii) *If $r \in Z(R)$ for any element $(\alpha, r) \in \mathrm{Der}(F,(M,\mu))$, then ϑ is an isomorphism.*

Note that the condition (ii) of this proposition is always satisfied when either R is an abelian Lie algebra or M is an R-module thought of as crossed R-module $(M,0)$. This assertion motivates the following

Definition. Let R be a Lie algebra and (M,μ) a crossed R-module. Then the Λ-module $\widetilde{Z^1}(P,(M,\mu))/\widetilde{B^1}(P,(M,\mu))$ will be called the second non-abelian cohomology of R with coefficients in (M,μ) and will be denoted by $H^2(R,M)$.

Finally using our second cohomology module of Lie algebras we prolong Guin's six-term exact cohomology sequence [2] to nine-term exact cohomology sequence.

Theorem. *Let R be a Lie algebra and*

$$0 \longrightarrow (L,0) \xrightarrow{\xi} (M,\mu) \xrightarrow{\theta} (N,\nu) \longrightarrow 0$$

an exact sequence of crossed R-modules, having a Λ-module section. Then there is an exact sequence of Λ-modules

$$0 \longrightarrow H^0(R,L) \xrightarrow{\xi^0} H^0(R,M) \xrightarrow{\theta^0} H^0(R,N) \xrightarrow{\delta^0} H^1(R,L) \xrightarrow{\xi^1} H^1(R,M)$$
$$\xrightarrow{\theta^1} H^1(R,N) \xrightarrow{\delta^1} H^2(R,L) \xrightarrow{\xi^2} H^2(R,M) \xrightarrow{\theta^2} H^2(R,N)\ .$$

where θ^1 is a Lie homomorphism and δ^1 is crossed homomorphism with the action of $H^1(R,N)$ on $H^2(R,L)$ induced by the action of R on L.

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