Available at www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Journal of Pure and Applied Algebra 189 (2004) 61–87
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Mod q cohomology and Tate–Vogel cohomology of groups

Daniel Conduché[a,*], Hvedri Inassaridze[b], Nick Inassaridze[b]

[a] *U.F.R. de Mathématiques, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France*

[b] *A. Razmadze Mathematical Institute, Georgian Academy of Sciences, M. Alexidze St. 1, Tbilisi 380093, Georgia*

Received 7 May 2003; received in revised form 2 September 2003
Communicated by M.-F. Roy

Abstract

The notions of mod q cohomology and Tate–Farrell–Vogel cohomology of groups are introduced, where q is a positive integer. The first and the second mod q cohomology groups are described in terms of torsors and extensions respectively. The mod q cohomology of groups is expressed as cotriple cohomology. The reduction of mod q Tate–Farrell–Vogel cohomology theory to the case $q = p^m$ with p a prime is shown. For finite groups with periodic cohomology the periodicity of mod q Tate cohomology is proved.

MSC: Primary: 18G60; secondary: 18G15; 20E22; 20J06

0. Introduction and summary

During the last 20 years, many important works appeared investigating the mod q versions of algebraic and topological topics.

For instance Neisendorfer [20] studied a homotopy theory with $\mathbb{Z}/q$ coefficients (primary homotopy theory) having important applications to K-theory and homotopy theory.

* Corresponding author.
E-mail addresses: daniel.conduche@univ-rennes1.fr (D. Conduché), hvedri@rmi.acnet.ge (H. Inassaridze), inas@rmi.acnet.ge (N. Inassaridze).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2003.10.025

Browder [3] defined and investigated a mod q algebraic K-theory called the algebraic K-theory with $\mathbb{Z}/q$ coefficients.

Suslin and Voevodsky [22] calculated the mod 2 algebraic K-theory of the integers as a result of Voevodsky's solution of the Milnor conjecture [25].

Conduché and Rodriguez-Fernández [8] introduced and studied non-abelian tensor and exterior products modulo q of crossed modules (see also [6,9]) having properties similar to the Brown–Loday non-abelian tensor product of crossed modules [7].

Karoubi and Lambre [17] introduced the mod q Hochschild homology as the homology of the mapping cone of the morphism given by the q multiplication on the standard Hochschild complex. Then they constructed the Dennis trace map from mod q algebraic K-theory to mod q Hochschild homology and found an unexpected relationship with number theory.

N. Inassaridze [16] pursuing investigations of non-abelian mod q tensor products found applications to mod q algebraic K-theory and homology of groups. The study of non-abelian left derived functors [14] of the mod q tensor product of groups inspired the definition of the mod q homology of a group G with coefficients in a G-module A:

$$H_n(G,A;\mathbb{Z}/q) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}/q,A), \quad n \geqslant 0.$$

This mod q group homology is also the homology of the mapping cone of the q multiplication on the standard homological complex as in the case of the mod q Hochschild homology [17].

The aim of this paper is to give and investigate a cohomological version of the mod q homology theory of groups and to unify both theories into a mod q Tate–Farrell–Vogel cohomology of groups.

The article is organised as follows. In Section 1, given a chain complex, we provide the definition of its mod q homology and Φ-cohomology (Definition 1.1). We prove the Universal Coefficient Formulas (Proposition 1.2 and Corollary 1.3) and show that mod q (co)homology of chain complexes reduce to the case $q = p^m$ with p a prime by showing that, the groups $H_*(C_*;\mathbb{Z}/q)$ and $H^*_\Phi(C_*;\mathbb{Z}/q)$ are canonically isomorphic to the products $H_*(C_*;\mathbb{Z}/r) \times H_*(C_*;\mathbb{Z}/s)$ and $H^*_\Phi(C_*;\mathbb{Z}/r) \times H^*_\Phi(C_*;\mathbb{Z}/s)$, respectively, for $q = rs$, r and s relatively prime (Theorem 1.5 and Corollary 1.6).

We begin Section 2 by introducing our definition of the mod q cohomology $H^*(G,A;\mathbb{Z}/q)$ (Definition 2.1). Given a group G we introduce the notion of a (G,q)-torsor over a G-module A (Definition 2.8) and describe the first mod q cohomology group in terms of (G,q)-torsors over A (Theorem 2.9). Using our notions of pointed q-extension and q-extension (Definitions 2.10 and 2.13) we describe the second mod q cohomology of groups (Theorems 2.11 and 2.15).

In Section 3, we express the mod q cohomology of groups in terms of cotriple derived functors of the kernels of higher dimensions of the mapping cone of the q multiplication on the standard cohomological complex (Theorem 3.1).

In Section 4, we give an account of Vogel cohomology theory [26]. Goichot [11] gave a detailed exposition of Vogel's homology theory and its relations to Tate and Farrell theories. We shall give here the cohomological approach (see also [27, Section 5]). At least, in the case of finite groups it is the same but the point of view is slightly different.

In Section 5, the mod q Tate–Farrell–Vogel cohomology of groups is introduced (Definition 5.1). Finally, we show how periodicity properties of finite periodic groups extend to mod q Tate cohomology (Theorem 5.8) and give a property of cohomogically trivial G-modules for G a p-group (Theorem 5.12).

Notations and conventions: A ring R is always associative and unitary; an R-module A is a left R-module. $\mathscr{D}_R$ is the category of (unbounded) complexes of projective R-modules and $\mathscr{C}_R$ is the category of complexes of R-modules. Considering a group G, and given two G-modules A and A' we write $\mathrm{Hom}_G(A,A')$ for $\mathrm{Hom}_{\mathbb{Z}[G]}(A,A')$. We mean q as a positive integer and its product on any module A is represented by qA and $A/q = A/qA$. We denote by IG the augmentation ideal of $\mathbb{Z}[G]$. The groups $\mathbb{Z}$ and $\mathbb{Z}/q$ are trivial G-modules. We mean the group $H^{-1}(G,A)$ trivial.

1. Mod q homology and cohomology of chain complexes

In this section, we suppose given a contravariant functor $\Phi : \mathscr{D}_R \to \mathscr{C}_{\mathbb{Z}}$. We introduce the mod q homology and Φ-cohomology of a complex over R, show the universal coefficient theorem in both cases and show they reduce to the case $q = p^m$ with p a prime.

Given an object of the category $\mathscr{D}_R$

$$C_* \equiv \cdots \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \cdots, \quad n \in \mathbb{Z},$$

the product by q defines a morphism ${}_\times q : C_* \to C_*$ of chain complexes and the mapping cone of this morphism

$$\cdots \to C_{n+1} \oplus C_n \overset{\widetilde{\partial_{n+1}}}{\to} C_n \oplus C_{n-1} \overset{\widetilde{\partial_n}}{\to} C_{n-1} \oplus C_{n-2} \overset{\widetilde{\partial_{n-1}}}{\to} \cdots,$$

$\widetilde{\partial}_n(x_n, x_{n-1}) = (\partial_n(x_n) + qx_{n-1}, -\partial_{n-1}(x_{n-1}))$, denoted by $\mathrm{Mc}(C_*, q)_*$.

Definition 1.1. For $n \in \mathbb{Z}$

(i) the mod q homology of the complex C_* is given by

$$H_n(C_*; \mathbb{Z}/q) := H_n(\mathrm{Mc}(C_*, q)_*);$$

(ii) the Φ-cohomology of C_* is given by

$$H^n_\Phi(C_*) := H_{-n}(\Phi(C_*))$$

and the mod q Φ-cohomology of C_* is given by

$$H^n_\Phi(C_*; \mathbb{Z}/q) := H_{-n+1}(\Phi(C_*); \mathbb{Z}/q) = H_{-n+1}(\mathrm{Mc}(\Phi(C_*), q)_*).$$

Proposition 1.2 (Universal Coefficient Formula for mod q homology). *Given $C_* \in \mathscr{D}_R$ we have an exact sequence*

$$0 \to H_n(C_*) \otimes \mathbb{Z}/q \to H_n(C_*; \mathbb{Z}/q) \to \mathrm{Tor}(H_{n-1}(C_*), \mathbb{Z}/q) \to 0, \quad n \in \mathbb{Z}.$$

Proof (It is essentially [16, Proposition 3.5]). The mapping cone gives rise to an exact homology sequence

$$\cdots \to H_n(C_*) \xrightarrow{\times q} H_n(C_*) \to H_n(C_*;\mathbb{Z}/q) \to H_{n-1}(C_*) \xrightarrow{\times q} H_{n-1}(C_*) \to \cdots.$$

Now the product by q in a module A has cokernel $A/qA \cong A \otimes \mathbb{Z}/q$ and kernel $\mathrm{Tor}(A,\mathbb{Z}/q)$. The exactness of the homology sequence gives the result. □

Corollary 1.3 (Universal coefficient formula for mod q Φ-cohomology). *Given $C_* \in \mathcal{D}_R$ we have an exact sequence*

$$0 \to H_\Phi^{n-1}(C_*) \otimes \mathbb{Z}/q \to H_\Phi^n(C_*;\mathbb{Z}/q) \to \mathrm{Tor}(H_\Phi^n(C_*),\mathbb{Z}/q) \to 0, \quad n \in \mathbb{Z}.$$

Remark 1.4. In the examples we shall consider the morphism $\Phi({}_\times q)$ is the product by q and, up to isomorphism, $\mathrm{Mc}(\Phi(C_*),q)_{n+1} = \Phi(\mathrm{Mc}(C_*,q))_n$. This motivates the index shift in the definition of mod q Φ-cohomology.

Let $q = rs$, then there are canonical morphisms of chain complexes

$$\alpha_{r,*}: \mathrm{Mc}(C_*,q)_* \to \mathrm{Mc}(C_*,r)_* \quad \text{and} \quad \alpha_{s,*}: \mathrm{Mc}(C_*,q)_* \to \mathrm{Mc}(C_*,s)_*$$

given by $\alpha_{r,n}(x_n,x_{n-1}) = (x_n, sx_{n-1})$ and $\alpha_{s,n}(x_n,x_{n-1}) = (x_n, rx_{n-1})$ for all $n \in \mathbb{Z}$, respectively. It follows that one gets a canonical homomorphism

$$\alpha_n : H_n(C_*;\mathbb{Z}/q) \to H_n(C_*;\mathbb{Z}/r) \times H_n(C_*;\mathbb{Z}/s), \quad n \in \mathbb{Z}.$$

Theorem 1.5. *If $q = rs$ and the integers r, s are relatively prime, we have a canonical isomorphism*

$$H_n(C_*;\mathbb{Z}/q) \cong H_n(C_*;\mathbb{Z}/r) \times H_n(C_*;\mathbb{Z}/s)$$

for all $n \in \mathbb{Z}$.

Proof. The inverse homomorphism to α_n, $n \in \mathbb{Z}$, will be constructed. Since r and s are relatively prime, there exist $k, l \in \mathbb{Z}$ such that

$$kr + ls = 1. \tag{1}$$

Define two morphisms of chain complexes

$$\beta_{r,*}: \mathrm{Mc}(C_*,r)_* \to \mathrm{Mc}(C_*,q)_*$$

and

$$\beta_{s,*}: \mathrm{Mc}(C_*,s)_* \to \mathrm{Mc}(C_*,q)_*$$

by

$$\beta_{r,n}(x_n,x_{n-1}) = (lsx_n, lx_{n-1}) \quad \text{and} \quad \beta_{s,n}(x_n,x_{n-1}) = (krx_n, kx_{n-1})$$

for $n \in \mathbb{Z}$. These maps are compatible with boundary operators. We check it for $\beta_{r,*}$. In fact,

$$\widetilde{\partial}_n \beta_{r,n}(x_n, x_{n-1}) = \widetilde{\partial}_n(lsx_n, lx_{n-1}) = (ls\partial_n(x_n) + lqx_{n-1}, -l\partial_{n-1}(x_{n-1}))$$
$$= \beta_{r,n-1}(\partial_n(x_n) + rx_{n-1}, -\partial_{n-1}(x_{n-1})) = \beta_{r,n-1}\widetilde{\partial}_n(x_n, x_{n-1}).$$

Therefore, one gets a homomorphism

$$\beta_n : H_n(C_*; \mathbb{Z}/r) \times H_n(C_*; \mathbb{Z}/s) \to H_n(C_*; \mathbb{Z}/q), \quad n \in \mathbb{Z},$$

induced by $\beta_{r,n}$ and $\beta_{s,n}$. It remains to prove that $\alpha_*\beta_*$ and $\beta_*\alpha_*$ are identity maps.

Let (x_n, x_{n-1}) be an nth chain of $\mathrm{Mc}(C_*, q)_*$. Then, using (1), we have

$$\beta_n\alpha_n(x_n, x_{n-1}) = \beta_n((x_n, sx_{n-1}), (x_n, rx_{n-1}))$$
$$= (lsx_n, lsx_{n-1}) + (krx_n, krx_{n-1}) = (x_n, x_{n-1}),$$

thus $\beta_*\alpha_* = 1$.

Let (x_n, x_{n-1}) be an nth cycle of $\mathrm{Mc}(C_*, r)_*$, i.e.

$$\partial_n(x_n) + rx_{n-1} = 0, \quad \partial_{n-1}(x_{n-1}) = 0. \tag{2}$$

We have

$$\alpha_n\beta_n(x_n, x_{n-1}) = \alpha_n(lsx_n, lx_{n-1}) = (lsx_n, lsx_{n-1}) + (lsx_n, lrx_{n-1}).$$

Whence the equality

$$(x_n, x_{n-1}) - \alpha_n\beta_n(x_n, x_{n-1}) = (krx_n, krx_{n-1}) + (-lsx_n, -lrx_{n-1})$$

in the R-module $\mathrm{Mc}(C_*, r)_n \times \mathrm{Mc}(C_*, s)_n$.

By (2) we get

$$\widetilde{\partial_{n+1}}(0, kx_n) = (krx_n, -k\partial_n(x_n)) = (krx_n, krx_{n-1})$$

and

$$\widetilde{\partial_{n+1}}(0, -lx_n) = (-lsx_n, l\partial_n(x_n)) = (-lsx_n, -lrx_{n-1}).$$

Therefore

$$(x_n, x_{n-1}) - \alpha_n\beta_n(x_n, x_{n-1}) = \widetilde{\partial_{n+1}}((0, kx_n), (0, -lx_n)).$$

Obviously, the same is true for an nth cycle of $\mathrm{Mc}(C_*, s)_*$. Thus $\alpha_*\beta_* = 1$. □

Corollary 1.6. *Let $C_* \in \mathscr{D}_R$ and $q = rs$ with r and s relatively prime integers. Then there is a canonical isomorphism*

$$H^n_\Phi(C_*; \mathbb{Z}/q) \cong H^n_\Phi(C_*; \mathbb{Z}/r) \times H^n_\Phi(C_*; \mathbb{Z}/s)$$

for all $n \in \mathbb{Z}$.

As the product by q is obviously functorial the homotopy properties of Φ, if any, induce homotopy properties on the mod q Φ-cohomology.

Lemma 1.7. (i) *Let $C_* \in \mathscr{D}_R$, $\Phi' : \mathscr{D}_R \to \mathscr{C}_\mathbb{Z}$ be a second contravariant functor and $\theta : \Phi \to \Phi'$ a natural transformation, such that $\theta(C_*)$ is a weak equivalence between $\Phi(C_*)$ and $\Phi'(C_*)$. Then θ induces isomorphisms*

$$H^n_\Phi(C_*; \mathbb{Z}/q) \cong H^n_{\Phi'}(C_*; \mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$.

(ii) *Suppose Φ is a homotopy functor, i.e. homotopic complexes are sent to homotopic complexes. Let $C_*, C'_* \in \mathscr{D}_R$ be homotopic. Then we have isomorphisms*

$$H^n_\Phi(C_*; \mathbb{Z}/q) \cong H^n_\Phi(C'_*; \mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$.

Proof. (i) As the mapping cone construction is functorial we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H^{n-1}_\Phi(C_*) & \longrightarrow & H^{n-1}_\Phi(C_*) & \longrightarrow & H^n_\Phi(C_*; \mathbb{Z}/q) & \longrightarrow & H^n_\Phi(C_*) & \longrightarrow & H^n_\Phi(C_*) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n-1}_{\Phi'}(C_*) & \longrightarrow & H^{n-1}_{\Phi'}(C_*) & \longrightarrow & H^n_{\Phi'}(C_*; \mathbb{Z}/q) & \longrightarrow & H^n_{\Phi'}(C_*) & \longrightarrow & H^n_{\Phi'}(C_*).
\end{array}$$

By hypothesis the two vertical maps on the left and the two on the right are isomorphisms. The five lemma gives the result.

(ii) It works the same with the diagram

$$\begin{array}{ccccccccc}
H^{n-1}_\Phi(C'_*) & \longrightarrow & H^{n-1}_\Phi(C'_*) & \longrightarrow & H^n_\Phi(C'_*; \mathbb{Z}/q) & \longrightarrow & H^n_\Phi(C'_*) & \longrightarrow & H^n_\Phi(C'_*) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n-1}_\Phi(C_*) & \longrightarrow & H^{n-1}_\Phi(C_*) & \longrightarrow & H^n_\Phi(C_*; \mathbb{Z}/q) & \longrightarrow & H^n_\Phi(C_*) & \longrightarrow & H^n_\Phi(C_*).
\end{array} \qquad \square$$

Example 1.8. Let $K_* \in \mathscr{C}_R$ and $\Phi : \mathscr{D}_R \to \mathscr{C}_\mathbb{Z}$ be defined by $\Phi(C_*) = \mathscr{H}om(C_*, K_*)_*$, where

$$\mathscr{H}om(C_*, K_*)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(C_i, K_{i+n})$$

with the differential Δ given by

$$(\Delta f)_i(x) = \mathrm{d}(f_i(x)) + (-1)^{n+1} f_{i-1}(\mathrm{d}(x))$$

for $f = (f_i) \in \mathscr{H}om(C_*, K_*)_n$ and $x \in C_i$. Then we write $H^n(C_*, K_*) = H^n_\Phi(C_*)$ and $H^n(C_*, K_*; \mathbb{Z}/q) = H^n_\Phi(C_*; \mathbb{Z}/q)$. If the complex K_* is concentrated in degree 0 we get with $H^n(C_*, K_*)$ the usual cohomology with coefficients in K_0. If A is an R-module and K_* a resolution of A, the morphism $K_* \to A$ defined by the map $K_0 \to A$ induces an isomorphism

$$H^n(C_*, K_*; \mathbb{Z}/q) \to H^n(C_*, A; \mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$ by Lemma 1.7(i).

The “internal Hom functor” in the category of chain complexes of R-modules was first studied by Brown [5].

Lemma 1.9. *Let $C_* \in \mathscr{D}_R$ and $K_* \in \mathscr{C}_R$. We have, for all $n \in \mathbb{Z}$, a canonical isomorphism*

$$\mathscr{H}om(\mathrm{Mc}(C_*, q)_*, K_*)_n \cong \mathrm{Mc}(\mathscr{H}om(C_*, K_*), q)_{n+1}.$$

Proof. We have

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{Mc}(C_*, q)_i, K_{n+i}) &= \mathrm{Hom}_R(C_i \oplus C_{i-1}, K_{n+i})\\ &\cong \mathrm{Hom}_R(C_i, K_{n+i}) \oplus \mathrm{Hom}_R(C_{i-1}, K_{n+i}),\end{aligned}$$

which gives, taking the product over $\mathbb{Z}$ and exchanging the factors in the right side of the equality,

$$\begin{aligned}\mathscr{H}om(\mathrm{Mc}(C_*, q)_*, K_*)_n &\cong \mathscr{H}om(C_*, K_*)_n \oplus \mathscr{H}om(C_*, K_*)_{n+1}\\ &= \mathrm{Mc}(\mathscr{H}om(C_*, K_*), q)_{n+1}. \qquad \square\end{aligned}$$

A second example of the functor Φ will be considered in Section 4. Note that all results of this section are true when $\mathscr{D}_R$ is an additive subcategory of $\mathscr{C}_R$.

2. Mod q cohomology of groups

In this section, we shall define a mod q cohomology of groups by using Definition 1.1 and then express it as the Ext^* functors in the same way as the mod q homology of groups is expressed as the Tor_* functors [16]. The first and the second mod q cohomology of groups will be described in terms of q-torsors and q-extensions of groups, respectively.

Let G be a group, A a G-module and $P_* \to \mathbb{Z}$ a projective G-resolution of $\mathbb{Z}$. According to Example 1.8, $\mathscr{H}om(-, A)$ is a contravariant functor from $\mathscr{D}_{\mathbb{Z}[G]}$ to $\mathscr{C}_{\mathbb{Z}}$.

Definition 2.1. The nth mod q cohomology, $H^n(G, A; \mathbb{Z}/q)$, of the group G with coefficients in the G-module A is

$$H^n(G, A; \mathbb{Z}/q) := H^n_{\mathscr{H}om(-,A)}(P_*; \mathbb{Z}/q), \quad n \geqslant 0.$$

Note that by Lemma 1.7(ii) these cohomology groups are well defined and do not depend on the choice of the projective G-resolution of $\mathbb{Z}$.

The next proposition immediately follows from Corollary 1.3.

Proposition 2.2 (Universal Coefficient Formula). *Let G be a group and A a G-module. Then there is a short exact sequence of abelian groups*

$$0 \to H^{n-1}(G, A) \otimes \mathbb{Z}/q \to H^n(G, A; \mathbb{Z}/q) \to \mathrm{Tor}(H^n(G, A), \mathbb{Z}/q) \to 0 \tag{3}$$

for $n \geqslant 0$.

We recall from [16, Proposition 3.2] that, given a projective G-resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ and $q > 0$, the morphism $\mathrm{Mc}(P_*, q)_* \to \mathbb{Z}/q$ defined by the composed map $\mathrm{Mc}(P_*, q)_0 = P_0 \to \mathbb{Z} \to \mathbb{Z}/q$ is a projective G-resolution of $\mathbb{Z}/q$. Therefore, applying Lemma 1.9, we have the following.

Theorem 2.3. $H^n(G, A; \mathbb{Z}/q) \cong \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}/q, A)$, $n \geqslant 0$.

Let us consider the standard bar resolution of the G-module $\mathbb{Z}$

$$C_*(G)\colon \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where C_n is the free G-module generated by all symbols $[x_1, \ldots, x_n]$, $n \geqslant 1$, $x_i \in G$, and C_0 is a free G-module generated by only one symbol $[\ ]$. The differential is defined by the formula

$$\partial[x_1|\cdots|x_n] = x_1[x_2|\cdots|x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1|\cdots|x_i x_{i+1}|\cdots|x_n] + (-1)^n[x_1|\cdots|x_{n-1}]$$

and $\varepsilon[\ ] = 1$.

According to Theorem 2.3, using also the classical convention converting chain complexes into cochain complexes, we call $\mathscr{H}om(\mathrm{Mc}(C_*(G)_*, q), A)_*$ the standard cochain complex for the mod q cohomology of G with coefficients in A and denote it by $D^*(G, A; \mathbb{Z}/q)$. We denote its cocycles by $Z^*(G, A; \mathbb{Z}/q)$ and its coboundaries by $B^*(G, A; \mathbb{Z}/q)$, while $Z^*(G, A)$ and $B^*(G, A)$ denote the cocycles and coboundaries of the standard cochain complex respectively.

As usual, we identify $\mathrm{Hom}_G(C_n, A)$ with the G-module $\mathrm{Set}(G^n, A)$ of maps from G^n to A for $n \geqslant 1$ and with A for $n = 0$. In the complex $D^*(G, A; \mathbb{Z}/q)$ we get, for $(f, g) \in \mathrm{Set}(G^n, A) \times \mathrm{Set}(G^{n-1}, A)$

$$\tilde{\delta}(f, g) = (\delta(f), qf - \delta(g)), \tag{4}$$

where δ is the classical differential given by

$$\delta(f)(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n}(-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n).$$

In the following example, $H^*(G, A; \mathbb{Z}/q)$ is neither isomorphic to $H^*(G, A)/q$ nor to $H^*(G, A/q)$.

Example 2.4. Let $\mathbb{Z}$ be the group of integers and $\mathbb{Q}/\mathbb{Z}$ the quotient of the group of rational numbers by $\mathbb{Z}$. Suppose that $\mathbb{Z}$ acts trivially on $\mathbb{Q}/\mathbb{Z}$. We have, for $n \geqslant 2$, $H^n(\mathbb{Z}, A) = 0$ for any G-module A, especially $\mathbb{Q}/\mathbb{Z}$, and $H^0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = H^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Since the group $\mathbb{Q}/\mathbb{Z}$ is divisible, $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/q = 0$. Whence the exact sequence

(3) gives $H^n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}; \mathbb{Z}/q) = 0$ for $n \geqslant 2$ and one has

$$H^0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}; \mathbb{Z}/q) = H^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}; \mathbb{Z}/q) = \mathbb{Z}/q.$$

While, for $n \geqslant 0$, $H^n(\mathbb{Z}, (\mathbb{Q}/\mathbb{Z})/q) = 0$ and $(H^n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}))/q = 0$.

Proposition 2.5. *Let G be a group and A a G-module.*

(a) *If A has exponent q, then*

$$H^n(G,A;\mathbb{Z}/q) \cong H^n(G,A) \oplus H^{n-1}(G,A), \quad n \geqslant 0.$$

(b) *If A is q-torsion-free, then*

$$H^0(G,A;\mathbb{Z}/q) = 0 \quad \textit{and} \quad H^n(G,A;\mathbb{Z}/q) \cong H^{n-1}(G,A/q), \quad n \geqslant 1.$$

Proof. (a) Follows from the triviality of the homomorphism ${}_{\times}q$ in equality (4).

(b) Obviously $H^0(G,A;\mathbb{Z}/q) = \mathrm{Tor}(H^0(G,A), \mathbb{Z}/q) = 0$. The short exact sequence

$$0 \to A \xrightarrow{\times q} A \to A/q \to 0 \tag{5}$$

induces a long exact cohomology sequence and we have only to construct the homomorphism $H^{n-1}(G,A/q) \to H^n(G,A;\mathbb{Z}/q)$, $n \geqslant 1$, compatible with the exact cohomology sequences and then apply the five lemma at each level. Using the short exact sequence of standard cochain complexes

$$0 \to \mathscr{H}om(C_*,A)_* \xrightarrow{\times q} \mathscr{H}om(C_*,A)_* \to \mathscr{H}om(C_*,A/q)_* \to 0$$

induced by the exact sequence (5), for any $(n-1)$-cocycle of $\mathscr{H}om(C_*,A/q)_*$ we find in a natural way an n-cocycle of $\mathscr{H}om(\mathrm{Mc}(C_*,q)_*,A)_*$. This map of cocyles induces the required homomorphism $H^{n-1}(G,A/q) \to H^n(G,A;\mathbb{Z}/q)$, $n \geqslant 1$. □

Proposition 2.5 provides a general reason why the mod q cohomology and homology of groups play a distinguished role especially for G-modules having torsion elements.

A q-derivation from G to A is a pair (f,a) consisting of a derivation $f: G \to A$ and an element $a \in A$ such that $qf(x) = xa - a$ for all $x \in G$.

Let $\mathrm{Der}(G,A;\mathbb{Z}/q)$ denote the abelian group of q-derivations from G to A. We write $\mathrm{Der}(G,A)$ for the abelian group of derivation from G to A and $\mathrm{PDer}(G,A)$ for the subset of principal derivations.

Plainly, any pair of the form (f_a, qa), with f_a the principal derivation from G to A induced by $a \in A$, is a q-derivation. We call it a *principal q-derivation*. The set $\mathrm{PDer}(G,A;\mathbb{Z}/q)$ of principal q-derivations is a subgroup of $\mathrm{Der}(G,A;\mathbb{Z}/q)$.

Clearly, using the identification of $\mathrm{Hom}_G(C_1,A)$ with $\mathrm{Set}(G,A)$ and of $\mathrm{Hom}_G(C_0,A)$ with A, a pair $(f,a) \in D^1(G,A;\mathbb{Z}/q)$ is a cocycle if and only if it is a q-derivation. Furthermore, it is a coboundary if and only if it is a principal q-derivation. Hence, the identification induces a natural isomorphism

$$H^1(G,A;\mathbb{Z}/q) \cong \mathrm{Der}(G,A;\mathbb{Z}/q)/\mathrm{PDer}(G,A;\mathbb{Z}/q).$$

Note that the map $\mathrm{PDer}(G,A;\mathbb{Z}/q) \to \mathrm{PDer}(G,A)$ given by $(f_a, qa) \mapsto f_a$, $a \in A$, is an isomorphism if and only if $H^0(G,A)$ is a group of exponent q.

Proposition 2.6. *The group* $\mathrm{Der}(G,A;\mathbb{Z}/q)$ *is isomorphic to the group of pairs* (α, a), *where* α *is an automorphism of the semidirect product* $A \rtimes G$ *inducing identity maps on* A *and* G, *and* a *is an element of* A *such that* α^q *is equal to the inner automorphism* β_a *of* $A \rtimes G$ *induced by* a. *Moreover*, $\mathrm{PDer}(G,A;\mathbb{Z}/q)$ *is isomorphic to the group of pairs* (β_a, qa).

Proof. Similar to the classical case. □

It is well known [18] that any derivation f can be extended to the abelian group homomorphism $\gamma : \mathbb{Z}[G] \to A$ given by $\gamma\left(\sum_i n_i g_i\right) = \sum_i n_i f(g_i)$ satisfying the condition $\gamma(rs) = r\gamma(s) + \varepsilon(s)\gamma(r)$ for all $r,s \in \mathbb{Z}[G]$. The restriction of γ to IG induces a G-module homomorphism $\beta : IG \to A$ and one gets the well-known isomorphism $\mathrm{Der}(G,A) \xrightarrow{\vartheta} \mathrm{Hom}_G(IG,A)$ with $\vartheta(f) = \beta$. Let $I(G,q) = \mathrm{Ker}\,\tilde{\varepsilon}$ with $\tilde{\varepsilon} : \mathbb{Z}[G] \to \mathbb{Z}/q$, $\tilde{\varepsilon}\left(\sum_i n_i g_i\right) = \left[\sum_i n_i\right]$. It is easy to see that an element $\sum_i n_i g_i$ of $\mathbb{Z}[G]$ belongs to $I(G,q)$ if and only if q divides $\sum_i n_i$ for $q > 0$.

The set K of elements $(f,a) \in \mathrm{Der}(G,A;\mathbb{Z}/q)$ for which there exists a G-module homomorphism $\alpha : I(G,q) \to A$ such that $\alpha(x) = \vartheta(f)(x)$ for $x \in IG$ and $\alpha(q1) = a$, is a subgroup of $\mathrm{Der}(G,A;\mathbb{Z}/q)$. Let $\alpha_a : I(G,q) \to A$ be the G-module homomorphism given by $\alpha_a(u) = ua$, $u \in I(G,q)$. Since, for any principal derivation f_a and for $x \in IG$, $\vartheta(f_a)(x) = xa$, one gets $\alpha_a(x) = \vartheta(f_a)(x)$, $x \in IG$ and $\alpha_a(q1) = q1a = qa$. Therefore $K \supseteq \mathrm{PDer}(G,A;\mathbb{Z}/q)$.

Proposition 2.7. *There is a short exact sequence of abelian groups*

$$0 \to \mathrm{Hom}_G(I(G,q),A) \xrightarrow{\varphi} \mathrm{Der}(G,A;\mathbb{Z}/q) \to \mathrm{Der}(G,A;\mathbb{Z}/q)/K \to 0.$$

Proof. Define the homomorphism φ by $\varphi(\alpha) = (f,a)$ for $\alpha \in \mathrm{Hom}_G(I(G,q),A)$, where $\vartheta(f) = \alpha|_{IG}$ and $a = \alpha(q1)$. The pair (f,a) is a q-derivation. Indeed we have $q\alpha(x) = \alpha(xq1) = xa$ for $x \in IG$. Since $\{q1\} \cup \{g-1 \mid g \in G\}$ is a generating set of $I(G,q)$ as a G-module, $\varphi(\alpha) = \varphi(\alpha')$ implies $\alpha = \alpha'$. Clearly, the image of φ is the subgroup K. □

Now the group $H^1(G,A;\mathbb{Z}/q)$ will be expressed by torsors. Recall [21] that a principal homogeneous space over A is a non-empty G-set P with right action $(p,a) \mapsto pa$ of A compatible with G-action such that, given $p, p' \in P$, there exists a unique $a \in A$ such that $p' = pa$. We introduce the following notion.

Definition 2.8. A (G,q)-torsor over a G-module A is a pair (P,f), where P is a principal homogeneous space over A and f is a map from P to A subject to the following conditions:

(i) $f(xb) = f(x) + qb$ for $x \in P$, $b \in A$;
(ii) $qa_s = sf(x) - f(x)$ with a_s defined by $sx = xa_s$, $s \in G$, $x \in P$.

Two (G,q)-torsors (P,f) and (P',f') over a G-module A are said to be equivalent if there is a bijection $\vartheta: P \to P'$ such that ϑ is compatible with the actions of G and A, and $f = f'\vartheta$.

Denote by $P(G,A;\mathbb{Z}/q)$ the set of equivalence classes of (G,q)-torsors over A. One can construct a natural sum on $P(G,A;\mathbb{Z}/q)$ given by $(P,f)+(P',f')=(P'',f'')$, where P'' is a quotient of $P\times P'$ by the relation $(x,x')=(xa,x'(-a))$ for $x\in P$, $x'\in P'$, $a\in A$, and $f''=f+f'$. Under this sum, $P(G,A;\mathbb{Z}/q)$ is an abelian group with zero element $(A,{}_{\times}q)$.

Theorem 2.9. *For any G-module A there is a canonical isomorphism*

$$P(G,A;\mathbb{Z}/q)\cong H^1(G,A;\mathbb{Z}/q).$$

Proof. One has a natural homomorphism

$$\alpha: P(G,A;\mathbb{Z}/q)\to H^1(G,A;\mathbb{Z}/q)$$

defined as follows: given a (G,q)-torsor (P,f) take an element $x\in P$. Then the equality $sx=xa_s$ defines a derivation $\varphi_x: G\to A$ given by $\varphi_x(s)=a_s$. It is easily checked that the pair $(\varphi_x, f(x))$ is a q-derivation (use the equality (ii) of Definition 2.8). The element $[(\varphi_x,f(x))]$ does not depend on the element $x\in P$ and therefore the map α given by $\alpha[(P,f)]=[(\varphi_x,f(x))]$ is well defined (use the equality (i) of Definition 2.8).

Conversely, if (φ,a) is a q-derivation, define a (G,q)-torsor (P,f) over A as follows: take $P=A$ and A acts on P by $xa=x+a$ for $x\in P$, $a\in A$. The group G acts on P by ${}^{s}x=\varphi(s)+sx$. The map $f: P\to A$ is given by $f(x)=a+qx$.

It is easily checked that the pair (P,f) is a (G,q)-torsor over A, that one gets a well-defined homomorphism

$$\beta: H^1(G,A;\mathbb{Z}/q)\to P(G,A;\mathbb{Z}/q)$$

given by $\beta([(\varphi,a)])=[(P,f)]$ and that $\alpha\beta$ and $\beta\alpha$ are identity maps. □

To describe the group $H^2(G,A;\mathbb{Z}/q)$ in terms of extensions some definitions will be introduced.

Definition 2.10. Let G be a group and A a G-module. A pointed q-extension of G by A is a triple (E,u,g) consisting of an extension $E: 0\to A\to B\to G\to 1$ of G by A, a section map $u: G\to B$ and a map $g: G\to A$, such that

$$qv(x,y)=(\delta g)(x,y)=xg(y)-g(xy)+g(x)$$

for all $x,y\in G$, where $v: G\times G\to A$ is the factorization system induced by the section u.

The pointed q-extension (E,u,g) is said to be equivalent to the pointed q-extension (E',u',g') if there exists a morphism $(1_A,\sigma,1_G): E\to E'$ and an element $a\in A$ such that

$$g'(x)-g(x)=q(u'(x)-\sigma u(x))-xa+a$$

for all $x\in G$.

This binary relation $\sim$ is an equivalence. The proof is left to the reader.

Let us denote by $E^1(G,A;\mathbb{Z}/q)$ the set of equivalence classes of pointed q-extensions of the group G by the G-module A.

Theorem 2.11. *Let G be a group and A a G-module. There is a natural bijection*

$$E^1(G,A;\mathbb{Z}/q) \xrightarrow{\omega} H^2(G,A;\mathbb{Z}/q).$$

Proof. Define a map ω by $\omega[(E,u,g)] = [(v,g)]$ for $[(E,u,g)] \in E^1(G,A;\mathbb{Z}/q)$, where $v: G \times G \to A$ is the factorization system induced by the section u.

Correctness: we have to show that if $(E,u,g) \sim (E',u',g')$, then $[(v,g)] = [(v',g')]$. It is well-known [18] that

$$v'(x,y) - v(x,y) = xh(y) - h(xy) + h(x)$$

for all $x \in G$, where $h(x) = u'(x) - \sigma u(x)$. But there exists an element $a \in A$ such that $g'(x) - g(x) = qh(x) - xa + a$ for all $x \in G$. It means that we have $(v',g') - (v,g) \in B^2(G,A;\mathbb{Z}/q)$.

Injectivity of ω: Let $[(E,u,g)], [(E',u',g')] \in E^1(G,A;\mathbb{Z}/q)$ and $[(v,g)] = [(v',g')]$, i.e. there exists $h \in \mathrm{Set}(G,A)$ and $a \in A$ such that $v'(x,y) - v(x,y) = (\delta h)(x,y)$ and $g'(x) - g(x) = qh(x) - xa + a$, for all $x, y \in G$. We can choose in the second extension a section u'' and a map g'' in such a way that $v'' = v$ and $g'' = g$. In effect let us define, the section $u''(x) = u'(x) - h(x)$, for $x \in G$, and the map $g'': G \to A$ by $g''(x) = g(x) - qh(x) + xa - a$. It is easy to show that

$$(E',u',g') \sim (E',u'',g'') \sim (E,u,g)$$

implying $[(E,u,g)] = [(E',u',g')]$.

Surjectivity of ω: Let $(v,g) \in Z^2(G,A;\mathbb{Z}/q)$. We take the extension

$$E: 0 \to A \to B \to G \to 0$$

induced by the 2-cocycle v and the section $u_0(x) = (0,x)$. Then we get the equality $\omega([E,u_0,g]) = [(v,g)]$. $\square$

Remark 2.12. If G is a group and A a G-module and satisfying the following condition:

$$\text{for any map } h: G \to A,\ \delta(qh) = 0 \Rightarrow qh \text{ is cohomologically trivial,} \tag{6}$$

then the group $H^2(G,A;\mathbb{Z}/q)$ can be described in terms of pairs (E,g) consisting of a map $g: G \to A$ and an extension E of G by A having a factorisation system v such that $qv = \delta g$. The relation $\sim$ between such pairs will be similar requiring that the sections u and u' inducing the 2-cocycles v and v', respectively, such that $qv = \delta g$ and $qv' = \delta g'$ must satisfy the equality of the equivalence relation. Clearly, the condition $H^1(G,A) = 0$ implies condition (6) and both conditions are equivalent to each other if A is a q-divisible group.

Moreover, for any G-module A, it is possible to introduce a “Baer sum” on the set $E^1(G,A;\mathbb{Z}/q)$ making the map ω an isomorphism.

Before defining q-extensions of groups we recall some properties about extensions of groups induced by derivations [12].

Let G be a group and

$$E: 0 \to A' \to A \to A'' \to 0,$$

an exact sequence of G-modules. Given a derivation $f: G \to A''$, one gets an induced extension $f^*(E)$. If f is a principal derivation the induced extension splits. If two derivations $f_1, f_2: G \to A$ are equivalent the induced extensions $f_1^*(E)$ and $f_2^*(E)$ are equivalent. Given two derivations $f_1, f_2: G \to A$ the extension $(f_1 + f_2)^*(E)$ is the Baer sum of the extensions $f_1^*(E)$ and $f_2^*(E)$.

Let G be a group, A a G-module and $f: G \to A/q$ a derivation. We consider the exact sequence $0 \to qA \to A \xrightarrow{c} A/q \to 0$ and call $\gamma^*(q)$ the induced extension of G by qA.

Given an extension $E: 0 \to A \to B \to G \to 1$, we call qE the extension induced by the q-multiplication from A to qA.

Definition 2.13. Let G be a group and A a G-module. A q-extension of G by A is a pair (E, f), where E is an extension of G by A and $f: G \to A/q$ a derivation, such that the induced extensions qE and $f^*(q)$ are equivalent. Two q-extensions (E, f) and (E', f') are equivalent if the extensions E and E' are equivalent and the derivations f and f' are equivalent.

Let $\mathrm{Ext}(G, A; \mathbb{Z}/q)$ be the set of equivalence classes of q-extensions of G by A. If (E, u, g) is a pointed q-extension then $(E, c \circ g)$ is a q-extension, since $qv = \delta g$ whence the extensions qE and $(c \circ g)^*(q)$ are equivalent. Furthermore, the map $(E, u, g) \mapsto (E, c \circ g)$ sends two equivalent pointed q-extensions onto two equivalent q-extensions. So it induces a map $\Phi: E^1(G, A; \mathbb{Z}/q) \to \mathrm{Ext}(G, A; \mathbb{Z}/q)$.

Lemma 2.14. *Let G be a group and A a G-module. The map Φ is surjective. Furthermore, for any q-extension (E, f) of G by A with $E: 0 \to A \to B \to G \to 1$ there exists a pair $(u, g) \in \mathrm{Set}(G, B) \times \mathrm{Set}(G, A)$ such that $c \circ g = f$ and $qv = \delta g$, where $v \in Z^2(G, A)$ is induced by u.*

Proof. The first sentence is a consequence of the second. Let $u \in \mathrm{Set}(G, B)$ be a section map of the morphism $B \to G$ and $v_1 \in Z^2(G, A)$ be given by

$$v_1(x, y) = u_1(x)u_1(y)u_1(xy)^{-1}.$$

Let $g: G \to A/q$ such that $f = c \circ g$ then, as the extensions $f^*(q)$ and qE are equivalent there is a map $h_0: G \to qA$ such that $\delta(g) = v_1 + \delta(h_0)$. Let $h: G \to A$ be such that $h_0 = qh$. Let $u = u_1 h$. There is a section map of $B \to G$ and the associated 2-cocycle is $v = v_1 + \delta h$. □

The morphism Φ is not, generally, injective: Consider a q-extension (E, f) and two pairs (u_1, g_1) and (u_2, g_2) as in Lemma 2.14, then we have $u_1 = u_2 + h$, $g_1 = g_2 + h'$ with $h, h': G \to A$. Now we must have $q\delta h = q\delta h'$ but this does not imply $h = h'$; we have only the condition $q\delta(h - h') = \delta(q(h - h')) = 0$.

Note that a map $k : G \to A$ with $q\delta k = 0$ corresponds to the inclusion

$$Z^1(G, qA) \hookrightarrow \mathrm{Hom}_G(C_1, qA) \hookrightarrow \mathrm{Hom}_G(C_1, A) \hookrightarrow \mathrm{Hom}_G(C_2 \oplus C_1, A).$$

Theorem 2.15. *Let G be a group and A a G-module. The group $\mathrm{Ext}(G, A; \mathbb{Z}/q)$ of equivalence classes of q-extensions of G by A is isomorphic to the quotient $H^2(G, A; \mathbb{Z}/q)/L$, where L is the image of $H^1(G, qA)$ induced by the composed map*

$$\mathrm{Set}(G, qA) \hookrightarrow \mathrm{Set}(G, A) \hookrightarrow \mathrm{Set}(G^2, A) \times \mathrm{Set}(G, A),$$

where the first map is induced by the inclusion $qA \hookrightarrow A$.

Proof. The remark we made for two pointed q-extensions inducing the same extension works as well for pointed q-extensions inducing equivalent q-extensions. Let (E, f) be a q-extension of G by A. Let (E, u, g) be a pointed q-extension such that $\Phi([(E, u, g)]) = [(E, f)]$. Let $\Psi([(E, f)]) = (c' \circ \omega)([(E, u, f)])$, where $c' : H^2(G, A; \mathbb{Z}/q) \to H^2(G, A; \mathbb{Z}/q)/L$ is the canonical map. By Remark 2.12 if (E', f') is equivalent to (E, f) and (E', u', g') such that $\Phi([(E', u', g')]) = [(E', g')]$, we have $(c' \circ \omega)([(E', u', g')]) = (c' \circ \omega)([(E, u, g)])$. Then $\Psi([(E, f)]) = \Psi([(E', f')])$ and the map Ψ is well defined. Furthermore it is surjective, since $(c' \circ \omega)$ is surjective.

Now suppose $\Psi([(E, f)]) = 0$. There is a q-extension (E, u, g) such that $\omega([(E, u, g)]) \in L$, that is $\omega([(E, u, g)]) = [(v, g')]$ with $v = 0$ and $q\delta g' = 0$. So the q-extension (E, f) is equivalent to $(E', 0)$, where E' is the trivial extension. □

3. Mod q cohomology of groups as cotriple cohomology

In this section, the mod q cohomology of groups will be described as cotriple cohomology.

Let $\mathfrak{Gr}_A$ denote, for a fixed abelian group A, the category whose objects are all groups G together with an action of G on A and morphisms are group homomorphisms $\alpha : G \to G'$ preserving the actions, namely ${}^{g}a = {}^{\alpha(g)}a$ for $a \in A$, $g \in G$.

Let $F : \mathfrak{Gr}_A \to \mathfrak{Gr}_A$ be the endofunctor defined as follows: for an object G of $\mathfrak{Gr}_A$, let $F(G)$ denote the free group on the underlying set of G with an action: ${}^{|g_1|^{\varepsilon_1} \cdots |g_s|^{\varepsilon_s}}a = {}^{g_1^{\varepsilon_1}}(\ldots({}^{g_s^{\varepsilon_s}}a)\ldots)$, where $a \in A$, $|g_1|^{\varepsilon_1} \cdots |g_s|^{\varepsilon_s} \in F(G)$ and $\varepsilon_i = \pm 1$; for a morphism $\alpha : G \to G'$ of $\mathfrak{Gr}_A$, let $F(\alpha)$ be the canonical homomorphism from $F(G)$ to $F(G')$ induced by α. Let $\tau : F \to 1_{\mathfrak{Gr}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $G \in \mathrm{ob}\,\mathfrak{Gr}_A$ by the injection $G \to F(G)$. We obtain a cotriple $\mathscr{F} = (F, \tau, \delta)$ on the category $\mathfrak{Gr}_A$. Let us consider the cotriple resolution $F_*(G) \xrightarrow{\tau_G} G$ of an object G of the category $\mathfrak{Gr}_A$, where

$$F_*(G) \equiv \cdots \begin{array}{c}\to\\ \vdots\\ \to\end{array} F_n(G) \begin{array}{c} d_0^n \\ \to\\ \vdots\\ \to \\ d_n^n\end{array} \cdots \begin{array}{c} d_0^2 \\ \to \\ \to \\ \to \\ d_2^2 \end{array} F_1(G) \begin{array}{c} d_0^1 \\ \to \\ \to \\ d_1^1 \end{array} F_0(G),$$

$F_n(G) = F^{n+1}(G) = F(F^n(G))$, $d_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$, $0 \leqslant i \leqslant n$.

Let $T : \mathfrak{G}r_A \to Ab\mathfrak{G}r$ be a contravariant functor to the category of abelian groups. Applying T dimension-wise to the simplicial group $F_*(G)$ yields an abelian cosimplicial group $TF_*(G)$. Then the nth cohomology group of the abelian cosimplicial group $TF_*(G)$ is called the nth right derived functor $R^n_{\mathscr{F}}T$ of the functor T with respect to the cotriple $\mathscr{F}$.

It is well known that the right derived functors of the contravariant functor of derivations $\mathrm{Der}(-,A) = Z^1(-,A) : \mathfrak{G}r_A \to Ab\mathfrak{G}r$ with respect to the cotriple $\mathscr{F}$ are isomorphic, up to dimension shift, to the group cohomology functors $H^*(-,A)$ [2]. Similar assertion is not true for mod q cohomology of groups, i.e. the cotriple derived functor $R^n_{\mathscr{F}}Z^1(-,A;\mathbb{Z}/q)$ of the contravariant functor of q-derivations $\mathrm{Der}(-,A;\mathbb{Z}/q) = Z^1(-,A;\mathbb{Z}/q) : \mathfrak{G}r_A \to Ab\mathfrak{G}r$ is not isomorphic to the mod q group cohomology functor $H^{n+1}(-,A;\mathbb{Z}/q)$ for some $n \geqslant 1$. In effect, if G is a free group acting on A, then $R^1_{\mathscr{F}}Z^1(G,A;\mathbb{Z}/q) = 0$, while, using Proposition 1.2, $H^2(G,A;\mathbb{Z}/q)$ is isomorphic to $H^1(G,A)/q$.

Theorem 3.1. *Let G be a group and A a G-module. Then there are natural isomorphisms*

$$R^0_{\mathscr{F}}Z^k(G,A;\mathbb{Z}/q) \cong Z^k(G,A;\mathbb{Z}/q),$$

$$R^n_{\mathscr{F}}Z^k(G,A;\mathbb{Z}/q) \cong H^{n+k}(G,A;\mathbb{Z}/q)$$

for $k > 1$ and $n > 0$.

Proof. The augmented simplicial group $\tau_G : F_*(G) \to G$ is simplicially exact and therefore is left (right) contractible as an augmented simplicial set. Since

$$D^k(L,A;\mathbb{Z}/q) = \mathrm{Set}(L^k,A) \oplus \mathrm{Set}(L^{k-1},A)$$

for any group L acting on A, the abelian cochain complex

$$0 \to D^k(G,A;\mathbb{Z}/q) \to D^k(F_0(G),A;\mathbb{Z}/q) \to D^k(F_1(G),A;\mathbb{Z}/q)$$
$$\to D^k(F_2(G),A;\mathbb{Z}/q) \to \cdots \to D^k(F_n(G),A;\mathbb{Z}/q) \to \cdots \quad (7)$$

becomes exact for $k \geqslant 0$, implying $R^0_{\mathscr{F}}D^k(G,A;\mathbb{Z}/q) \cong D^k(G,A;\mathbb{Z}/q)$ and $R^n_{\mathscr{F}}D^k(G,A;\mathbb{Z}/q) = 0$, $n > 0$.

For any $k \geqslant 0$ the short exact sequence of abelian cochain complexes

$$0 \to Z^k(F_*(G),A;\mathbb{Z}/q) \to D^k(F_*(G),A;\mathbb{Z}/q) \to B^{k+1}(F_*(G),A;\mathbb{Z}/q) \to 0 \quad (8)$$

induces a long exact sequence of cotriple derived functors

$$0 \to R^0_{\mathscr{F}}Z^k(G,A;\mathbb{Z}/q) \to R^0_{\mathscr{F}}D^k(G,A;\mathbb{Z}/q) \to R^0_{\mathscr{F}}B^{k+1}(G,A;\mathbb{Z}/q)$$
$$\to R^1_{\mathscr{F}}Z^k(G,A;\mathbb{Z}/q) \to R^1_{\mathscr{F}}D^k(G,A;\mathbb{Z}/q) \to \cdots.$$

The injection $R^0_{\mathscr{F}}B^{k+1}(G,A;\mathbb{Z}/q) \hookrightarrow R^0_{\mathscr{F}}D^{k+1}(G,A;\mathbb{Z}/q) \cong D^{k+1}(G,A;\mathbb{Z}/q)$ yields the exact sequence

$$0 \to R^0_{\mathrm{F}}Z^k(G,A;\mathbb{Z}/q) \to D^k(G,A;\mathbb{Z}/q) \to D^{k+1}(G,A;\mathbb{Z}/q),$$

showing that $R^0_{\mathrm{F}}Z^k(G,A;\mathbb{Z}/q) \cong Z^k(G,A;\mathbb{Z}/q)$.

It is easily checked that any short exact sequence of G-modules

$$0 \to A_1 \to A \to A_2 \to 0$$

induces a long exact cohomology sequence

$$0 \to Z^k(G,A_1;\mathbb{Z}/q) \to Z^k(G,A;\mathbb{Z}/q) \to Z^k(G,A_2;\mathbb{Z}/q) \to H^{k+1}(G,A_1;\mathbb{Z}/q)$$
$$\to H^{k+1}(G,A;\mathbb{Z}/q) \to H^{k+1}(G,A_2;\mathbb{Z}/q) \to H^{k+2}(G,A_1;\mathbb{Z}/q) \to \cdots.$$

It follows that for G a free group the sequence

$$0 \to Z^k(G,A_1;\mathbb{Z}/q) \to Z^k(G,A;\mathbb{Z}/q) \to Z^k(G,A_2;\mathbb{Z}/q) \to 0$$

is exact for $k > 1$, since in this case $H^{k+1}(G,A;\mathbb{Z}/q) = 0$. Hence for $k > 1$ there is a long exact sequence of cotriple right derived functors

$$0 \to Z^k(G,A_1;\mathbb{Z}/q) \to Z^k(G,A;\mathbb{Z}/q) \to Z^k(G,A_2;\mathbb{Z}/q) \to R^1_{\mathscr{F}}Z^k(G,A_1;\mathbb{Z}/q)$$
$$\to R^1_{\mathscr{F}}Z^k(G,A;\mathbb{Z}/q) \to R^1_{\mathscr{F}}Z^k(G,A_2;\mathbb{Z}/q) \to R^2_{\mathscr{F}}Z^k(G,A_1;\mathbb{Z}/q) \to \cdots.$$

Now it will be shown that $R^n_{\mathscr{F}}Z^k(G,A;\mathbb{Z}/q) = 0$ for $k \geqslant 1$ and $n > 0$, if A is an injective G-module.

The following complex of abelian cosimplicial groups

$$0 \to D^0(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta^0_*}} D^1(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta^1_*}} D^2(F_*(G),A;\mathbb{Z}/q)$$
$$\xrightarrow{\widetilde{\delta^2_*}} D^3(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta^3_*}} \cdots \xrightarrow{\widetilde{\delta^{k-1}_*}} D^k(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta^k_*}} \cdots \tag{9}$$

is exact at the terms $D^k(F_*(G),A;\mathbb{Z}/q)$, $k \geqslant 3$, since $H^k(F_*(G),A;\mathbb{Z}/q) = 0$, $k \geqslant 3$, by the universal coefficient formula (Proposition 2.2).

It is easy to show that any injective G-module is a q-divisible group and the proof is similar to the case of injective abelian groups.

Since $F_n(G)$, $n \geqslant 0$, is a free group, the group $Z^1(F_n(G),A)$ of 1-cocycles is isomorphic to a direct product $\prod_{i \in J} A_i$ of copies $A_i = A$, where the set J is a basis of $F_n(G)$. Hence, if A is injective, then $Z^1(F_n(G),A)$, $n \geqslant 0$, is q-divisible, thus $H^1(F_n(G),A)$, $n \geqslant 0$, is also q-divisible. Therefore for an injective G-module A the short exact sequence of abelian cosimplicial groups

$$0 \to \mathrm{Tor}(H^1(F_*(G),A),\mathbb{Z}/q) \to H^1(F_*(G),A) \xrightarrow{\times q} H^1(F_*(G),A) \to 0$$

together with the well-known isomorphism $R^n_{\mathscr{F}}H^1(G,A) \cong H^{n+1}(G,A)$, $n \geqslant 0$, imply the equality $R^n_{\mathscr{F}}\,\mathrm{Tor}(H^1(G,A),\mathbb{Z}/q) = 0$, $n \geqslant 0$.

The universal coefficient formula yields a short exact sequence of abelian cosimplicial groups

$$0 \to H^0(F_*(G),A) \otimes \mathbb{Z}/q \to H^1(F_*(G),A;\mathbb{Z}/q) \to \mathrm{Tor}(H^1(F_*(G),A),\mathbb{Z}/q) \to 0,$$

implying the isomorphism $R^n_{\mathscr{F}}H^1(G,A;\mathbb{Z}/q) \cong R^n_{\mathscr{F}}\,\mathrm{Tor}(H^1(G,A),\mathbb{Z}/q)$, $n > 0$. By (8) and the following short exact sequence of abelian cosimplicial groups:

$$0 \to B^1(F_*(G),A;\mathbb{Z}/q) \to Z^1(F_*(G),A;\mathbb{Z}/q) \to H^1(F_*(G),A;\mathbb{Z}/q) \to 0,$$

it is easily seen that $R^n_{\mathscr{F}} Z^1(G,A;\mathbb{Z}/q) \cong R^n_{\mathscr{F}} H^1(G,A;\mathbb{Z}/q)$, $n>0$. Hence, for an injective G-module A we deduce that $R^n_{\mathscr{F}} Z^1(G,A;\mathbb{Z}/q)=0$, $n>0$, and using again (8) one gets $R^n_{\mathscr{F}} B^2(G,A;\mathbb{Z}/q)=0$, $n>0$.

Let us consider the short exact sequence of abelian cosimplicial groups

$$0 \to H^1(F_*(G),A)\otimes \mathbb{Z}/q \to H^2(F_*(G),A;\mathbb{Z}/q) \to \mathrm{Tor}(H^2(F_*(G),A),\mathbb{Z}/q) \to 0$$

induced by the universal coefficient formula, which for an injective G-module A, implies that $H^2(F_n(G),A;\mathbb{Z}/q) \cong \mathrm{Tor}(H^2(F_n(G),A),\mathbb{Z}/q)=0$ for all $n \geqslant 0$.

We also have the following short exact sequence of abelian cosimplicial groups

$$0 \to B^2(F_*(G),A;\mathbb{Z}/q) \to Z^2(F_*(G),A;\mathbb{Z}/q) \to H^2(F_*(G),A;\mathbb{Z}/q) \to 0.$$

Finally, this implies that $R^n_{\mathscr{F}} Z^2(G,A;\mathbb{Z}/q)=0$, $n>0$, if A is an injective G-module.

Now by induction on k, using (7) and (9), one can easily prove that $R^n_{\mathscr{F}} Z^k(G,A;\mathbb{Z}/q)=0$, $n>0$, for an injective G-module A and $k \geqslant 3$.

Clearly, by the universal coefficient formula, one has $H^n(G,A;\mathbb{Z}/q)=0$, $n \geqslant 2$, if A is an injective G-module.

Thus, we have shown that two sequences of functors

(1) $Z^k(G,-;\mathbb{Z}/q)$, $H^{k+1}(G,-;\mathbb{Z}/q)$, $H^{k+2}(G,-;\mathbb{Z}/q),\ldots,$
(2) $Z^k(G,-;\mathbb{Z}/q)$, $R^1_{\mathscr{F}} Z^k(G,-;\mathbb{Z}/q)$, $R^2_{\mathscr{F}} Z^k(G,-;\mathbb{Z}/q),\ldots$

satisfy the following axioms for a connected sequence of additive functors $\{T_n,\theta^n, n \geqslant 0\}$ from the category of G-modules to the category of abelian groups:

(i) $T_0(-)=Z^k(G,-;\mathbb{Z}/q)$;
(ii) for any short exact sequence of G-modules $0 \to A_1 \to A \to A_2 \to 0$ there is a long exact sequence of abelian groups

$$0 \to T_0(A_1) \to T_0(A) \to T_0(A_2) \xrightarrow{\theta^0} T_1(A_1) \to \cdots$$
$$\xrightarrow{\theta^{n-1}} T_n(A_1) \to T_n(A) \to T_n(A_2) \xrightarrow{\theta^n} T_{n+1}(A_1) \to \cdots;$$

(iii) if A is an injective G-module, then $T_n(A)=0$ for all $n \geqslant 1$. □

In particular, Theorem 3.1 allows us to describe the mod q cohomology groups $H^n(G,A;\mathbb{Z}/q)$, $n \geqslant 3$, in terms of the non-abelian derived functors of the functor $Z^2(-,A;\mathbb{Z}/q)$.

Remark 3.2. The assertion similar to Theorem 3.1 has been proved in [15] for the classical (co)homology of groups and associative algebras. Moreover, one can obtain a similar result for mod q homology of groups [16].

4. Vogel cohomology of groups

In this section, we recall the definition of Vogel cohomology and give the proof, due to Vogel [26], that it is a generalisation of Tate–Farrell cohomology.

Recall from Example 1.8 the Hom complex $\mathscr{H}om(C_*, K_*)_*$ in the category $\mathscr{D}_R$. Given C_* and K_* in $\mathscr{D}_R$ the bounded Hom complex $\mathscr{H}om_b(C_*, K_*)_*$ is the subcomplex of $\mathscr{H}om(C_*, K_*)_*$ given by

$$\mathscr{H}om_b(C_*, K_*)_n = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, K_{i+n}).$$

For completeness, we recall also the tensor product complex $(C_* \otimes K_*)_*$ of two complexes C_* and K_* of R-modules (see any book on algebraic homology):

$$(C_* \otimes K_*)_n = \bigoplus_{i\in\mathbb{Z}} C_i \otimes_R K_{n-i}$$

with the differential Δ given by

$$\Delta(x \otimes y) = \mathrm{d}x \otimes y + (-1)^i x \otimes \mathrm{d}y, \quad x \in C_i, \; y \in K_{n-i}.$$

Proposition 4.1. *Let C_i, $i \in \mathbb{Z}$ be a finitely generated R-module. Then there is an isomorphism of complexes*

$$\mathscr{H}om_b(C_*, K_*)_* \cong \mathscr{H}om(C_*, R)_* \otimes K_*.$$

Proof. It is easy to check that for a finitely generated projective R-module C_i there is an isomorphism

$$\mathrm{Hom}_R(C_i, K_{i+n}) \cong \mathrm{Hom}_R(C_i, R) \otimes_R K_{i+n}.$$

Then we have

$$\mathscr{H}om_b(C_*, K_*)_n = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, K_{i+n}) \cong \bigoplus_{i\in\mathbb{Z}} (\mathrm{Hom}_R(C_i, R) \otimes_R K_{i+n})$$

$$= \bigoplus_{i\in\mathbb{Z}} (\mathrm{Hom}_R(C_{-i}, R) \otimes_R K_{n-i}) = (\mathscr{H}om_b(C_*, R)_* \otimes K_*)_n. \quad \square$$

Let $K_* \in \mathscr{D}_R$. Our second example of a functor Φ (see Section 1) associates to $C_* \in \mathscr{D}_R$ the quotient complex

$$\widehat{\mathscr{H}om}(C_*, K_*)_* = \mathscr{H}om(C_*, K_*)_* / \mathscr{H}om_b(C_*, K_*)_*.$$

Then the Φ-cohomology of this complex is written

$$\hat{H}^n(C_*, K_*) := H^n_\Phi(C_*) = H_{-n}(\widehat{\mathscr{H}om}(C_*, K_*)_*).$$

These cohomology groups have the expected property:

Lemma 4.2. *Let $C_*, K_* \in \mathscr{D}_R$. Then the cohomology groups $\hat{H}^n(C_*, K_*)$ depend only on the homotopy classes of C_* and K_*.*

This lemma allows the following

Definition 4.3. Let A and A' be two R-modules. Let L_* be a projective resolution of A and L'_* a projective resolution of A'. Then we set

$$\widehat{\mathrm{Ext}}^n_R(A, A') := \hat{H}^n(L_*, L'_*).$$

Proposition 4.4. *Let $K_* \in \mathscr{D}_R$ and $0 \to C'_* \to C_* \to C''_* \to 0$ be an exact sequence in $\mathscr{D}_R$. Then we have two long exact sequences*

$$\cdots \to \hat{H}^{n-1}(C'_*, K_*) \to \hat{H}^n(C''_*, K_*) \to \hat{H}^n(C_*, K_*) \to \hat{H}^n(C'_*, K_*) \to \hat{H}^{n+1}(C''_*, K_*) \to \cdots$$

and

$$\cdots \to \hat{H}^{n-1}(K_*, C''_*) \to \hat{H}^n(K_*, C'_*) \to \hat{H}^n(K_*, C_*) \to \hat{H}^n(K_*, C''_*) \to \hat{H}^{n+1}(K_*, C'_*) \to \cdots.$$

Proof. The short exact sequence of complexes

$$0 \to \mathscr{H}om(C''_*, K_*)_* \to \mathscr{H}om(C_*, K_*)_* \to \mathscr{H}om(C'_*, K_*)_* \to 0$$

restricts to an exact sequence

$$0 \to \mathscr{H}om_b(C''_*, K_*)_* \to \mathscr{H}om_b(C_*, K_*)_* \to \mathscr{H}om_b(C'_*, K_*)_* \to 0.$$

By diagram chasing these two exact sequences induce a third one

$$0 \to \widehat{\mathscr{H}om}(C''_*, K_*)_* \to \widehat{\mathscr{H}om}(C_*, K_*)_* \to \widehat{\mathscr{H}om}(C'_*, K_*)_* \to 0$$

implying the first long exact sequence. The second long exact sequence is obtained in the same way. □

Corollary 4.5. *Let M be an R-module and $0 \to A' \to A \to A'' \to 0$ an exact sequence of R-modules. Then we have two long exact sequences*

$$\cdots \to \widehat{\mathrm{Ext}}^{n-1}(A', M) \to \widehat{\mathrm{Ext}}^{n}(A'', M) \to \widehat{\mathrm{Ext}}^{n}(A, M) \to \widehat{\mathrm{Ext}}^{n}(A', M) \to \cdots$$

and

$$\cdots \to \widehat{\mathrm{Ext}}^{n-1}(M, A'') \to \widehat{\mathrm{Ext}}^{n}(M, A') \to \widehat{\mathrm{Ext}}^{n}(M, A) \to \widehat{\mathrm{Ext}}^{n}(M, A'') \to \cdots.$$

Vogel's Ext functors have applications outside group theory [19,27], but, to keep to our subject, we just relate them, when R is a group ring, with Farrell cohomology theory (see [10] or e.g. [4]). From now on the ring R is $\mathbb{Z}[G]$ with G a group and we give the definition of Vogel cohomology of groups.

Definition 4.6. Let G be a group and A a G-module. Then Vogel cohomology groups are given, for $n \in \mathbb{Z}$, by

$$\hat{H}^n(G, A) := \widehat{\mathrm{Ext}}^{n}_{G}(\mathbb{Z}, A).$$

Before giving the proof, due to Vogel, that his cohomology theory is a generalisation of Farrell cohomology we recall the definition of Farrell cohomology [10].

Definition 4.7. A complete resolution for a group G is a pair (F_*, F''_*) of complexes of G-modules such that

(i) F_* is acyclic,
(ii) F''_* is a resolution of the G-module $\mathbb{Z}$,
(iii) F_* and F''_* coincide in higher dimensions.

In the sequel, we will always suppose a complete resolution to be projective, i.e. F_* and F''_* are complexes of projective G-modules. We shall say that a group G satisfies condition (CR) if there exists a complete resolution (F_*, F''_*) for G, if such a complete resolution is unique up to homotopy and if there exists a surjective morphism $F_* \to F''_*$ which is the identity in higher dimensions. We shall say that G satisfies condition (CR_f) if, furthermore, there exists a complete resolution with each F_i and F''_i finitely generated, $i \in \mathbb{Z}$.

Remark 4.8. The existence of the morphism $F_* \to F''_*$ is a consequence of the construction of the complete resolution [4, Proposition X 2.3]. Furthermore, this morphism can be made surjective by a change of F_*.

Definition 4.9. Let G be a group satisfying condition (CR), A a G-module and (F_*, F''_*) a complete resolution for G. Then Farrell cohomology groups with coefficients in A are the groups $\hat{H}^n_{Fa}(G,A) = H^n(F_*, A)$.

Tate cohomology is Farrell cohomology for finite groups. Tate first build complete resolutions in this case by splicing a resolution and a coresolution [4]. Then Farrell checked the condition (CR) for groups with finite virtual cohomological dimension (vcd) [10]. Finally, Ikenaga introducing a generalised cohomological dimension proved that condition (CR) is valid for a wider class of groups [13].

Theorem 4.10 (Vogel [26]). *Let G be a group satisfying condition* (CR_f). *We suppose that, given an acyclic projective complex F_*, the complex $\mathscr{H}om(F_*, \mathbb{Z}[G])_*$ is acyclic. Then the Farrell cohomology of G and the Vogel cohomology of G coincide.*

Proof. Let A be a G-module and L_* a projective G-resolution of A. By condition (CR_f) there exists a complete projective resolution (F_*, F''_*) for G with each F_i and F''_i finitely generated, $i \in \mathbb{Z}$.

Let F'_* be the kernel of the canonical epimorphism $F_* \to F''_*$. We have an exact sequence of complexes $0 \to F'_* \to F_* \to F''_* \to 0$, thus an exact sequence

$$0 \to \widehat{\mathscr{H}om}(F''_*, L_*)_* \to \widehat{\mathscr{H}om}(F_*, L_*)_* \to \widehat{\mathscr{H}om}(F'_*, L_*)_* \to 0. \tag{10}$$

As L_* is bounded beneath and F'_* is bounded overhead we have

$$\mathscr{H}om(F'_*, L_*)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_G(F'_i, L_{n+i}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_G(F'_i, L_{n+i}) = \mathscr{H}om_b(F'_*, L_*)_n.$$

Thus, $\mathscr{H}om(F'_*, L_*)_* = \mathscr{H}om_b(F'_*, L_*)_*$ and $\widehat{\mathscr{H}om}(F'_*, L_*)_* = 0$. Sequence (10) gives an isomorphism $\widehat{\mathscr{H}om}(F''_*, L_*)_* \cong \widehat{\mathscr{H}om}(F_*, L_*)_*$.

As the complex $\mathscr{H}om(F_*, \mathbb{Z}[G])_*$ is acyclic the complex $\mathscr{H}om(F_*, \mathbb{Z}[G])_* \otimes L_*$ is acyclic and, by Proposition 4.1, $\mathscr{H}om_b(F_*, L_*)_*$ is acyclic too. Thus, the canonical morphism $\mathscr{H}om(F_*, L_*)_* \to \widehat{\mathscr{H}om}(F''_*, L_*)_*$ is an homology equivalence. The complexes $\mathscr{H}om(F_*, L_*)_*$ and $\mathscr{H}om(F_*, A[0])_*$, where A is in degree 0, are homotopy equivalent, since L_* is a resolution of A, see Example 1.8. Finally, the complexes $\widehat{\mathscr{H}om}(F''_*, L_*)_*$ and $\mathscr{H}om(F_*, A[0])_*$ have the same homology, that is Vogel and Farrell cohomology coincide. □

Remark 4.11. Let (K_*, K''_*) be an other complete projective resolution for G. Plainly, as, by condition (CR), the complexes F_* and K_* are homotopy equivalent the complexes $\mathscr{H}om_b(F_*, L_*)_*$ and $\mathscr{H}om_b(K_*, L_*)_*$ are equivalent. Whence $\mathscr{H}om_b(K_*, L_*)_*$ is acyclic even if the groups K_i are not finitely generated.

Finite groups G satisfy condition (CR_f) and, given an acyclic projective complex F_*, the complex $\mathscr{H}om(F_*, \mathbb{Z}[G])_*$ is acyclic [4]. Thus, we have

Corollary 4.12. *For a finite group G Tate and Vogel cohomology of G coincide.*

Condition (CR) is true for a group G with $\mathrm{vcd}(G)$ finite but condition (CR_f) is not always true. Nevertheless, Remark 4.11 allows to extend the corollary in this case.

Corollary 4.13. *Let G be a group satisfying condition* (CR). *We suppose that, for any G-module A, $\hat{H}^n_{Fa}(G, A) = 0$ if, for any finite subgroup H of G, $\hat{H}^n_{Fa}(H, A) = 0$. Then Farrell and Vogel cohomology of G coincide.*

Proof. Let (F_*, F''_*) be a complete resolution for G. It is as well a complete resolution for any (finite) subgroup H and a complex L_* of projective G-modules is a complex of projective H-modules. By definition $\hat{H}^n_{Fa}(G, A) = 0$ (resp $\hat{H}^n_{Fa}(H, A) = 0$) means that the complex $\mathscr{H}om_G(C_*, A)_*$ (resp $\mathscr{H}om_H(C_*, A)_*$) is acyclic. Either by hand calculation or by use of a spectral sequence associated to the bicomplex $\mathrm{Hom}_G(C_i, L'_j)$ we see that, for any bounded G-complex L'_*, the complex $\mathscr{H}om(C_*, L'_*)_*$ is acyclic if and only if, for each $i \in \mathbb{Z}$, the complex $\mathscr{H}om(C_*, L'_i)_*$ is acyclic. Thus the hypothesis is equivalent to the same hypothesis where the G-module A is replaced by a bounded complex.

Let A be a G-module and L_* a projective G-resolution of A. Then, for any subgroup H of G and bounded subcomplex $L'_* \in \mathscr{D}_R$ of L_* the complex $\mathscr{H}om_H(F_*, L'_*)$ is acyclic by Proposition 4.1, the proof of Theorem 4.10 and Remark 4.11.Thus, the complex $\mathscr{H}om_G(F_*, L'_*)$ is acyclic. Whence, as a colimit, the complex $\mathscr{H}om(F_*, L_*)$ is acyclic and we apply the end of the proof of Theorem 4.10. □

The hypothesis in Corollary 4.13 is true for groups with finite vcd [4, Lemma X5.1]. It does not work for all groups considered by Ikenaga but he exhibited among them a large class of groups, called C_∞, containing the class of groups with finite vcd but larger and for which this hypothesis is true, see [13] for details. Thus we have

Corollary 4.14. *If the group G belongs to the class C_∞ of Ikenaga, in particular if G has a finite* vcd, *Farrell and Vogel cohomology of G coincide.*

Remark 4.15. (i) Vogel introduced also a cohomology theory in which he replaces the complex $\mathscr{H}om_b(C_*, K_*)_*$ by the subcomplex $\mathscr{H}om_f(C_*, K_*)_*$ of morphisms which factor through bounded complexes of finitely generated projective R-modules. This gives the same theory for a finite group but not in general.

(ii) Proposition 4.1 is still valid if, instead of C_i finitely generated, we suppose the K_i, $i \in \mathbb{Z}$, finitely generated. Hence, if the G-module A admits a projective resolution by finitely generated projective G-modules, Vogel and Ikenaga cohomology with coefficients in A coincide if G satisfies condition (CR).

5. Mod q Vogel cohomology of groups

In this section, we extend Vogel's definition to get mod q cohomology. Then we investigate its properties among which we generalise some classical properties of Tate–Farrell cohomology.

Lemmas 1.7 and 4.2 allow the following

Definition 5.1. Let G be a group, q a and A a G-module. Let L_* (resp K_*) a projective G-resolution of $\mathbb{Z}$ (resp A). Then mod q Vogel cohomology groups are given by

$$\hat{H}^n(G, A; \mathbb{Z}/q) := H_{-n+1}(\widehat{\mathscr{H}om}(L_*, K_*)_*; \mathbb{Z}/q).$$

Now, as an immediate consequence of Lemma 1.9 the mapping cones of L_* and of $\widehat{\mathscr{H}om}(L_*, K_*)_*$ are related.

Lemma 5.2. *Let $C_*, K_* \in \mathscr{D}_R$. For all $n \in \mathbb{Z}$ we have a canonical isomorphism*

$$\widehat{\mathscr{H}om}(\mathrm{Mc}(C_*, q)_*, K_*)_n \cong \mathrm{Mc}(\widehat{\mathscr{H}om}(C_*, K_*)_*, q)_{n+1}.$$

Hence we have the following

Proposition 5.3. *Let G be a group and A a G-module. Then, for all $n \in \mathbb{Z}$, we have an isomorphism*

$$\hat{H}^n(G, A; \mathbb{Z}/q) = \widehat{\mathrm{Ext}}^n_G(\mathbb{Z}/q, A).$$

Proof. If L_* is a projective G-resolution of $\mathbb{Z}$ we claim $\mathrm{Mc}(L_*,q)_*$ is a projective G-resolution of $\mathbb{Z}/q$. It is always true but for our purpose it is enough to consider the standard bar resolution see Section 2. Whence Lemma 5.2 gives the result. □

Remark 5.4. The Farrell mod q cohomology of a group of finite virtual cohomological dimension can be defined in the same way Farrell defined his cohomology by taking a complete resolution of $\mathbb{Z}/q$ instead of $\mathbb{Z}$. For instance, the pair of mapping cones $(\mathrm{Mc}(F_*,q)_*, \mathrm{Mc}(F''_*,q)_*)$, where (F_*,F''_*) is a complete resolution of $\mathbb{Z}$, is a complete resolution of $\mathbb{Z}/q$. Then the proofs of Theorem 4.10 and Corollary 4.13 work to show that Farrell mod q cohomology coincides with Vogel mod q cohomology.

In this context Corollary 1.3 becomes

Proposition 5.5 (Universal Coefficient Formula). *Let G be a group and A a G-module. Then, for all $n\in\mathbb{Z}$, there is a short exact sequence of abelian groups*

$$0 \to \hat{H}^{n-1}(G,A)\otimes\mathbb{Z}/q \to \hat{H}^n(G,A;\mathbb{Z}/q) \to \mathrm{Tor}(\hat{H}^n(G,A),\mathbb{Z}/q) \to 0.$$

Corollary 5.6. *Let G be a group with $\mathrm{vcd}(G)<\infty$ and A a G-module. Then the canonical map $H^n(G,A;\mathbb{Z}/q) \to \hat{H}^n(G,A;\mathbb{Z}/q)$ induces an isomorphism for $n \geqslant \mathrm{vcd}(G)+2$ and a surjection for $n=\mathrm{vcd}(G)+1$.*

Proof. We have the following commutative diagram of groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}(G,A)\otimes\mathbb{Z}/q & \longrightarrow & H^n(G,A;\mathbb{Z}/q) & \longrightarrow & \mathrm{Tor}(H^n(G,A),\mathbb{Z}/q) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{H}^{n-1}(G,A)\otimes\mathbb{Z}/q & \longrightarrow & \hat{H}^n(G,A;\mathbb{Z}/q) & \longrightarrow & \mathrm{Tor}(\hat{H}^n(G,A),\mathbb{Z}/q) & \longrightarrow & 0 \end{array}$$

with exact rows; the vertical homomorphisms are the canonical maps. By K.S. Brown [4] the first vertical map is surjective and the third vertical map is an isomorphism for $n-1 \geqslant \mathrm{vcd}(G)$; furthermore, the first vertical map is an isomorphism still for $n-1 > \mathrm{vcd}(G)$. Whence the result by the five lemma. □

Corollary 5.7. *Let G be a finite group and A a G-module. Then*

$$\hat{H}^n(G,A;\mathbb{Z}/q) = H^n(G,A;\mathbb{Z}/q), \quad n \geqslant 2,$$

$$\hat{H}^{-n}(G,A;\mathbb{Z}/q) = H_n(G,A;\mathbb{Z}/q), \quad n \geqslant 2,$$

furthermore the groups $\hat{H}^{-1}(G,A;\mathbb{Z}/q)$, $\hat{H}^0(G,A;\mathbb{Z}/q)$ and $\hat{H}^1(G,A;\mathbb{Z}/q)$ are new and enter into short exact sequences

$$0 \to H_1(G,A)\otimes\mathbb{Z}/q \to \hat{H}^{-1}(G,A;\mathbb{Z}/q) \to \mathrm{Tor}(\hat{H}^{-1}(G,A),\mathbb{Z}/q) \to 0,$$

$$0 \to \hat{H}^{-1}(G,A)\otimes\mathbb{Z}/q \to \hat{H}^0(G,A;\mathbb{Z}/q) \to \mathrm{Tor}(\hat{H}^0(G,A),\mathbb{Z}/q) \to 0,$$

$$0 \to \hat{H}^0(G,A)\otimes\mathbb{Z}/q \to \hat{H}^1(G,A;\mathbb{Z}/q) \to \mathrm{Tor}(H^1(G,A),\mathbb{Z}/q) \to 0.$$

Using again the universal coefficient formula for a group G of order k, we see that, for $n \in \mathbb{Z}$ and $x \in \hat{H}^n(G,A;\mathbb{Z}/q)$ we have $k^2 x = 0$. Whence the groups $\hat{H}^n(G,A;\mathbb{Z}/q)$ are finite when G is finite and A is a finitely generated G-module.

It is easy to check that Shapiro's lemma holds for mod q Vogel cohomology of groups which states that, if H is a subgroup of finite index in a group G and A is an H-module, one has an isomorphism

$$\hat{H}^*(H,A;\mathbb{Z}/q) \cong \hat{H}^*(G,\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A;\mathbb{Z}/q).$$

The proof is similar to the case of Vogel homology [11, Lemma 4.4].

We have a cup product, actually a composition product [4], on Vogel cohomology $\hat{H}^*(G,-)$ [11]. For $\mathrm{vcd}(G) < \infty$, a fortiori for G finite, we recover the usual cup product. We shall extend this cup product to mod q cohomology for groups with finite virtual cohomology dimensions.

A group G is said to have periodic cohomology if there exists an integer $d \neq 0$ such that, for any $n \in \mathbb{Z}$, the functors $\hat{H}^n(G,-)$ and $\hat{H}^{n+d}(G,-)$ are isomorphic. In case $\mathrm{vcd}(G) \leqslant \infty$ it is equivalent to the existence of an element $u \in \hat{H}^d(G,\mathbb{Z})$ which is invertible in the ring $\hat{H}^*(G,\mathbb{Z})$. Then [4] the cup product with u gives, for any $n \in \mathbb{Z}$ and any G-module A, a periodicity isomorphism

$$u \cup - : \hat{H}^n(G,A) \cong \hat{H}^{n+d}(G,A).$$

Note that, at least for $\mathrm{vcd}(G) < \infty$, if G has periodic cohomology, the period d is even.

Theorem 5.8. *Let G be a finite group, L_* a complete resolution of $\mathbb{Z}$ for G, A and B two G-modules. Then*

(i) *the cochain product $\cup$ of Tate cohomology induces a cup product*

$$\hat{H}^p(G,A) \otimes \hat{H}^n(G,B;\mathbb{Z}/q) \xrightarrow{\cup} \hat{H}^{p+n}(G,A \otimes B;\mathbb{Z}/q)$$

given by

$$f \cdot (g,h) = (f \cdot g, (-1)^p f \cdot h),$$

where $f \in \mathrm{Hom}_G(L_*,A)_p$ and $(g,h) \in \mathrm{Hom}_G(\mathrm{Mc}(L_*,q)_*,B)_n$;

(ii) *for G with periodic cohomology of period d the cup product with $u \in \hat{H}^d(G,\mathbb{Z})$ induces an isomorphism*

$$\hat{H}^n(G,B;\mathbb{Z}/q) \cong \hat{H}^{n+d}(G,B;\mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$ and any G-module B.

Proof. (i) It is easily checked that one has the equality

$$\delta(f(g,h)) = \delta f \cdot (g,h) + (-1)^p f \cdot \delta(g,h)$$

implying the correctness of the cup product.

(ii) To prove the periodicity the defining properties of the Tate cohomology cup product are used [1, Theorem 7.1]. One gets the following commutative diagram of groups:

$$\begin{array}{ccccccccc} \cdots \xrightarrow{\times q} & \hat{H}^{n-1}(G,B) & \longrightarrow & \hat{H}^{n}(G,B;\mathbb{Z}/q) & \longrightarrow & \hat{H}^{n}(G,B) & \xrightarrow{\times q} \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \xrightarrow{\times q} & \hat{H}^{n-1+d}(G,B) & \longrightarrow & \hat{H}^{n+d}(G,B;\mathbb{Z}/q) & \longrightarrow & \hat{H}^{n+d}(G,B) & \xrightarrow{\times q} \cdots \end{array}$$

with exact rows and the vertical homomorphisms are induced by the cup product given in (i). Since the periodicity holds for the Tate cohomology [1,4], it remains to apply the five lemma. □

Remark 5.9. By the same way the periodicity theorem can be proved for groups with $\operatorname{vcd} G < \infty$ having periodic cohomology.

Notice too that there is a cup product action of Tate cohomology on the right:

$$\hat{H}^n(G,B;\mathbb{Z}/q) \otimes \hat{H}^p(G,A) \xrightarrow{\cup} \hat{H}^{n+p}(G,B\otimes A;\mathbb{Z}/q)$$

given by $(g,h)\cdot f = (g\cdot f, h\cdot f)$. In this case

$$\delta((g,h)\cdot f) = \delta(g,h)\cdot f + (-1)^n (g,h)\cdot \delta f,$$

where $(g,h)\in \operatorname{Hom}_G(\operatorname{Mc}(L_*,q)_*,B)_n$, $f\in \operatorname{Hom}_G(L_*,B)_p$, and the mod q Tate cohomology $\hat{H}^*(G,B;\mathbb{Z}/q)$ becomes an $\hat{H}^*(G,\mathbb{Z})$-bimodule for any G-module B.

From Theorem 5.8, we deduce that one has periodicity of the mod q Tate cohomology for finite cyclic groups having periodic cohomology of period 2 and for finite subgroups of the multiplicative group of the quaternion algebra having periodic cohomology of period 4. Moreover one has

Corollary 5.10. *Let C_m be the cyclic group of order m, t a generator of C_m. Then for any C_m-module A one gets*

$$\hat{H}^{2n}(C_m,A;\mathbb{Z}/q) = \{(a,a') \mid Na + qa' = 0,\ ta' = a'\}/\tilde{D}(A\oplus A), \quad n\in\mathbb{Z},$$

$$\hat{H}^{2n+1}(C_m,A;\mathbb{Z}/q) = \{(a,a') \mid Da + qa' = 0,\ Na' = 0\}/\tilde{N}(A\oplus A), \quad n\in\mathbb{Z}$$

with $N = 1 + t + \cdots + t^{m-1}$, $D = t - 1 \in \mathbb{Z}[G]$ and where the homomorphisms $\tilde{D}: A\oplus A \to A\oplus A$ and $\tilde{N}: A\oplus A \to A\oplus A$ are defined by $\tilde{D}(a,a') = (Da + qa', -Na')$ and $\tilde{N}(a,a') = (Na + qa', -Da')$.

Proof. Follows from Theorem 5.8(ii) and [16, Proposition 3.10]. □

Remark 5.11. The question of periodic cohomology for a wider class of groups has been considered in classical cohomology in the context of "periodicity after k steps" [23,24].

Theorem 5.12. *Let G be a p-group whose order $|G| = p^m$ divides q and A a G-module. Then the following conditions are equivalent*:

(i) $\hat{H}^n(G,A;\mathbb{Z}/q) = 0$ *for some* $n \in \mathbb{Z}$.
(ii) *A is cohomologically trivial.*

If in addition A is p-torsion-free, then (i) *and* (ii) *are equivalent to*

(iii) *A/pA is free over* $(\mathbb{Z}/p)[G]$.

Proof. First suppose that A is p-torsion-free. According to [1, Theorem 9.2] it suffices to show the equivalence of the following two conditions:

(i) $\hat{H}^n(G,A;\mathbb{Z}/q) = 0$ for some $n \in \mathbb{Z}$.
(iv) $\hat{H}^n(G,A) = 0$ for two consecutive integers n.

(iv) $\Rightarrow$ (i) if $\hat{H}^n(G,A) = \hat{H}^{n+1}(G,A) = 0$, then by Theorem 5.5 $\hat{H}^{n+1}(G,A;\mathbb{Z}/q) = 0$.

(i) $\Rightarrow$ (iv) if $\hat{H}^n(G,A;\mathbb{Z}/q) = 0$, the homomorphism $\hat{H}^{n-1}(G,A) \xrightarrow{\times q} \hat{H}^{n-1}(G,A)$ is surjective and the homomorphism $\hat{H}^n(G,A) \xrightarrow{\times q} \hat{H}^n(G,A)$ is injective. Thus, for $x \in \hat{H}^{n-1}(G,A)$ there is an element $y \in \hat{H}^{n-1}(G,A)$ with $qy = x$. On the other hand, one has $p^m y = 0$ whence $qy = 0$. If $x \in \hat{H}^n(G,A)$, the equality $p^m x = 0$ implies $qx = 0$ and therefore $x = 0$.

The equivalence of (i) and (ii) for any G-module A is reduced to the previous case by use of dimension-shifting. Take a short exact sequence of G-modules

$$0 \to A' \to F \to A \to 0$$

with F free over $\mathbb{Z}[G]$. Then one has the isomorphisms

$$\hat{H}^n(G,A) \cong \hat{H}^{n+1}(G,A') \quad \text{and} \quad \hat{H}^n(G,A;\mathbb{Z}/q) \cong \hat{H}^{n+1}(G,A';\mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$ with A' torsion-free. □

We end with a last example of extension of a classical property to Vogel cohomology.

Proposition 5.13. *Let G be a group and A a projective G-module. Then*

$$\hat{H}^*(G,A;\mathbb{Z}/q) = 0.$$

Proof. We take $L_0 = A$ and $L_n = 0$ for $n \neq 0$ as a projective resolution of A. □

Acknowledgements

The authors would like to thank the referee for useful comments and suggestions. The authors are grateful to Professor Daniel Guin for stating the problem solved in Section 3 and Professor Donadze for his remarks on Proposition 4.1. The second and third authors would like to thank the Université Rennes 1 for its hospitality during the work on this paper. The authors were partially supported by INTAS Georgia Grant No. 213. The second and the third authors were also supported by INTAS Grant No. 00 566 and FNRS Grant No. 7GEPJ065513.01. The first author is a member of the GDRE 1110.

References

[1] M.F. Atiyah, C.T.C. Wall, Cohomology of groups, in: J.W.S. Cassels, A. Frohlich (Eds.), Algebraic Number Theory, Academic Press, London, New York, 1967.

[2] M. Barr, J. Beck, Acyclic modules and triples, Proceedings of the Conference on Categorical Algebra, La Jolla 1965, Springer, Berlin, New York, 1966, pp. 336–343.

[3] W. Browder, Algebraic K-theory with coefficients $\mathbb{Z}/p$, in: M.G. Barratt, M.E. Mahowald (Eds.), Geometric Applications of Homotopy Theory I, Lecture Notes, Vol. 657, Springer, Berlin, 1978, pp. 40–84.

[4] K.S. Brown, Cohomology of groups, Graduated Texts in Mathematics, Springer, Berlin, 1982.

[5] R. Brown, Cohomology with chains as coefficients, Proc. London Math. Soc. 14 (3) (1964) 545–565.

[6] R. Brown, q-perfect groups and universal q-central extensions, Publ. Mat. 34 (1990) 291–297.

[7] R. Brown, J.-L. Loday, Van Kampen theorems for diagrams of spaces, Topology 26 (1987) 311–335.

[8] D. Conduché, C. Rodriguez-Fernandez, Non-abelian tensor and exterior products modulo q and universal q-central relative extension, J. Pure Applied Algebra 78 (1992) 139–160.

[9] G.J. Ellis, Tensor products and q-crossed modules, J. London Math. Soc. 51 (2) (1995) 243–258.

[10] F.T. Farrell, An extension of Tate cohomology to a class of groups, J. Pure Appl. Algebra 10 (1977) 153–161.

[11] F. Goichot, Homologie de Tate–Vogel équivariante, J. Pure Appl. Algebra 82 (1992) 39–64.

[12] P.J. Hilton, U. Stammbach, A course in homological algebra, Graduate Text in Mathematics, Springer, Berlin, 1996.

[13] B. Ikenaga, Homological dimension and Farrell cohomology, J. Algebra 87 (1984) 422–457.

[14] H. Inassaridze, Non-abelian Homological Algebra and its Applications, Kluwer Academic Publishers, Amsterdam, 1997.

[15] H. Inassaridze, More about (co)homology of groups and associative algebras, The Mathematics Preprint Server: http://www.mathpreprints.com/math/Preprint/hvedri/20030427/1.

[16] N. Inassaridze, On nonabelian tensor product modulo q, Comm. Algebra 29 (6) (2001) 2657–2687.

[17] M. Karoubi, Th. Lambre, Quelques classes caractéristiques en théorie des nombres, J. Reine Angew. Math. 543 (2000) 169–186.

[18] S. MacLane, Homology, Springer, Berlin, 1963.

[19] A. Martsinkovsky, New homological invariants for modules over local rings, I, J. Pure Appl. Algebra 110 (1996) 1–8.

[20] J. Neisendorfer, Primary homotopy theory, Mem. Amer. Math. Soc. 25 (232) (1980) 1–67.

[21] J.-P. Serre, Cohomologie Galoisienne, in: Lecture Notes in Mathematics, Vol. 5, Springer, Berlin, 1964.

[22] A. Suslin, V. Voevodsky, Bloch–Kato conjecture and motivic cohomology with finite coefficients, The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), NATO Science Series C Mathematics and Physical Science, Vol. 548, Kluwer Academic Publ., Dordrecht, 2000, pp. 117–189.

[23] O. Talelli, On cohomological periodicity for infinite groups, Comment. Math. Helv. 55 (1980) 178–192.

[24] O. Talelli, On groups with periodic cohomology after 1 step, J. Pure Appl. Algebra 30 (1983).

[25] V. Voevodsky, The Milnor conjecture, K-theory preprint archive no. 170, 1996.

[26] P. Vogel, Talks given at Paris and Rennes, 1983–1984.

[27] Y. Yoshino, Tate–Vogel completion of half-exact functors, Algebras Representation Theory 4 (2001) 171–200.