EXACT SEQUENCES IN HOMOLOGY OF MULTIPLICATIVE LIE RINGS AND A NEW VERSION OF STALLINGS' THEOREM

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Abstract. We show that the non-abelian tensor product of nilpotent, solvable and Engel multiplicative Lie rings is nilpotent, solvable and Engel, respectively. The six term exact sequence in homology of multiplicative Lie rings is obtained. We also prove a new version of Stallings' theorem.

1. Introduction

Multiplicative Lie rings were introduced by Ellis in [12] (see also [1]). This concept generalizes that of groups and Lie rings. Ellis used this structure to investigate an interesting combinatorial problem on group commutators (see also [8]). In [16] Point and Wantiez studied algebraic structural properties of multiplicative Lie rings. In particular, they defined nilpotency and solvability of multiplicative Lie rings and generalized the well-known results for groups and Lie algebras to multiplicative Lie rings. In [1] we investigated further structural properties of multiplicative Lie rings. We introduced two homology theories and studied their relationships to the Eilenberg-MacLane homology of groups and the Chevalley-Eilenberg homology of Lie rings. In the recent paper [7] we introduced a notion of non-abelian tensor product of multiplicative Lie rings and showed that our definition recovers the notions of non-abelian tensor products of groups defined by Brown-Loday [2] and that of Lie rings defined by Ellis [11].

In the present manuscript we are motivated by the desire to establish a unified theory of the non-abelian tensor product and low-dimensional homologies which simultaneously generalizes that for groups and that for Lie rings. With this aim we generalize the results of [10,17,19] to multiplicative Lie rings. In particular, we prove

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that the non-abelian tensor product of nilpotent, solvable and Engel multiplicative Lie rings is nilpotent, solvable and Engel, respectively (see Proposition 3.5 and Proposition 3.8). Using the technique of the non-abelian tensor product we also get the exact sequences relating the homology groups of multiplicative Lie rings in lower dimensions (see Proposition 4.1). As a result we have a combined proof of the six term homology exact sequences for groups and Lie rings. At the end of this article we prove a new version of Stallings' theorem [13, VI. Theorem 9.1] (see Theorem 5.6).

2. Multiplicative Lie rings, homology and non-abelian tensor product

For any group G and elements $x,\, y \in G$, let ${}^{x}y = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$.

A *multiplicative Lie ring* consist of a multiplicative (possibly non-abelian) group $\mathfrak{g}$ together with a binary function $\{\ ,\ \}\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which we shall call Lie product, satisfying the following identities:

$$\{x, x\} = 1, \tag{2.1}$$

$$\{x, yy'\} = \{x, y\}\, {}^{y}\{x, y'\}, \tag{2.2}$$

$$\{xx', y\} = {}^{x}\{x', y\}\{x, y\}, \tag{2.3}$$

$$\{\{x, y\}, {}^{y}z\}\{\{y, z\}, {}^{z}x\}\{\{z, x\}, {}^{x}y\} = 1, \tag{2.4}$$

$${}^{z}\{x, y\} = \{{}^{z}x, {}^{z}y\} \tag{2.5}$$

for all $x, x', y, y', z \in \mathfrak{g}$.

Examples 2.1.

(a) Any group G can also be given the structure of a multiplicative Lie ring by defining $\{x, y\} = 1$ for all $x, y \in G$. It is called the abelian multiplicative Lie ring of G and is sometimes denoted by $G_{\bullet}$.

(b) Any group G is a multiplicative Lie ring under $\{x, y\} = xyx^{-1}y^{-1}$ for all $x, y \in G$. It is denoted by $G_{[\,,\,]}$.

(c) Any Lie ring $\mathscr{g}$ is a multiplicative Lie ring under $\{x, y\}$ the Lie product for all $x, y \in \mathscr{g}$. Moreover, if $\mathfrak{g}$ is a multiplicative Lie ring whose underlying group is abelian then $\mathfrak{g}$ is an ordinary Lie ring.

(d) Given a ring R with identity, there is defined the Steinberg multiplicative Lie ring $St^{mlr}(R)$ by $St^{mlr}(R) = \varinjlim_{n} St_n^{mlr}(R)$, $n \geq 3$, where $St_n^{mlr}(R)$ is the multiplicative Lie ring defined by generators $x_{ij}(s)$, $s \in R$, $1 \leq i \neq j \leq n$, subject to the relations

$$\begin{aligned} & x_{ij}(s) + x_{ij}(t) = x_{ij}(s + t), & \\ & \{x_{ij}(s), x_{kl}(t)\} = 1, & \text{if } i \neq l, j \neq k, \\ & \{x_{ij}(s), x_{kl}(t)\} = x_{il}(st), & \text{if } i \neq l, j = k. \end{aligned}$$

A *morphism* $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ of multiplicative Lie rings is a group homomorphism such that $\phi\{x, y\} = \{\phi x, \phi y\}$ for all $x, y \in \mathfrak{g}$. Denote the category of multiplicative Lie rings by $\mathcal{LM}$.

A subgroup $\mathfrak{n}$ of a multiplicative Lie ring $\mathfrak{g}$ will be a *subring* of $\mathfrak{g}$ if $\{x, y\} \in \mathfrak{n}$ for all $x, y \in \mathfrak{n}$. It will be an *ideal* of $\mathfrak{g}$ if it is a normal subgroup and if $\{x, y\} \in \mathfrak{n}$ for all $x \in \mathfrak{n}$ and $y \in \mathfrak{g}$.

Let $\mathfrak{n}$ and $\mathfrak{h}$ be subgroups of $\mathfrak{g}$. The subgroup of $\mathfrak{g}$ generated by all elements $\{x, y\}$, $x \in \mathfrak{n}$, $y \in \mathfrak{h}$ is denoted by $\{\mathfrak{n}, \mathfrak{h}\}$. The subgroup $\{\mathfrak{g}, \mathfrak{g}\}$ is an ideal and is called the *Lie commutator* of the multiplicative Lie ring $\mathfrak{g}$.

A multiplicative Lie ring $\mathfrak{g}$ is called *perfect* if $\mathfrak{g} = \{\mathfrak{g}, \mathfrak{g}\}$.

2.2. **Homology theories of multiplicative Lie rings.** Now we briefly recall the construction of two homology theories of multiplicative Lie rings studied in [1].

Let *Sets* denote the category of sets and $\mathfrak{Gr}$ denote the category of groups. Let $\mathcal{U}\colon \mathcal{LM} \to Sets$ and $\mathcal{V}\colon \mathcal{LM} \to \mathfrak{Gr}$ be the natural forgetful functors. The functors $\mathcal{U}$ and $\mathcal{V}$ admit left adjoints $\mathfrak{F}\colon Sets \to \mathcal{LM}$ and $\mathfrak{L}\colon \mathfrak{Gr} \to \mathcal{LM}$, respectively (see [1] for detailed description of $\mathfrak{F}$ and $\mathfrak{L}$). It is well known that every adjoint pair of functors induces a cotriple (see, for example, [14, Chapter 2]). Let $(\mathfrak{F}\mathcal{U}, \tau, \delta)$ (resp. $(\mathfrak{L}\mathcal{V}, \tau', \delta'))$ denote the cotriple in $\mathcal{LM}$ defined by the adjoint pair $(\mathfrak{F}, \mathcal{U})$ (resp. $(\mathfrak{L}, \mathcal{V})$). Let $\mathcal{P}$ and $\mathcal{Q}$ denote the projective classes in the category $\mathcal{LM}$ induced by the cotriples $(\mathfrak{F}\mathcal{U}, \tau, \delta)$ and $(\mathfrak{L}\mathcal{V}, \tau', \delta')$, respectively. It is easy to see that the category $\mathcal{LM}$ has finite limits. This implies, cf. [14, Chapter 2], the existence of non-abelian left derived functors $\mathcal{L}_n^{\mathcal{P}}\mathfrak{Ab}\colon \mathcal{LM} \to \mathfrak{Gr}$, $n \geq 0$, and $\mathcal{L}_n^{\mathcal{Q}}\mathfrak{Ab}\colon \mathcal{LM} \to \mathfrak{Gr}$, $n \geq 0$, of the abelianization functor $\mathfrak{Ab}\colon \mathcal{LM} \to \mathfrak{Gr}$, $\mathfrak{g} \mapsto \mathfrak{Ab}(\mathfrak{g}) = \frac{\mathfrak{g}}{\{\mathfrak{g},\mathfrak{g}\}}$, relative to the projective classes $\mathcal{P}$ and $\mathcal{Q}$, respectively. It is known (see again [14, Chapter 2]) that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple.

Let $\mathfrak{g}$ denote a multiplicative Lie ring and $n \geq 1$. Define the n-th *strong homology* group of $\mathfrak{g}$ by
$$HS_n^{mlr}(\mathfrak{g}) = \mathcal{L}_{n-1}^{\mathcal{P}}\mathfrak{Ab}(\mathfrak{g}).$$
Define the n-th *weak homology* group of $\mathfrak{g}$ by
$$HW_n^{mlr}(\mathfrak{g}) = \mathcal{L}_{n-1}^{\mathcal{Q}}\mathfrak{Ab}(\mathfrak{g}).$$

It is easy to check that $\mathfrak{Ab}$ is a right exact functor. Hence by [14],
$$HS_1^{mlr}(\mathfrak{g}) = \mathcal{L}_0^{\mathcal{P}}\mathfrak{Ab}(\mathfrak{g}) \cong \mathfrak{Ab}(\mathfrak{g}) = \frac{\mathfrak{g}}{\{\mathfrak{g},\mathfrak{g}\}} \cong \mathcal{L}_0^{\mathcal{Q}}\mathfrak{Ab}(\mathfrak{g}) = HW_1^{mlr}(\mathfrak{g}).$$

Moreover, we have the following theorem (see [1, Theorem 3.6]).

Theorem 2.3 ([1]). *For any multiplicative Lie ring $\mathfrak{g}$ there is a natural isomorphism of groups*
$$HS_2^{mlr}(\mathfrak{g}) \cong HW_2^{mlr}(\mathfrak{g}).$$

In general HS_n^{mlr} and HW_n^{mlr} are different for $n \geq 3$ (see [1]).

2.4. **Action of multiplicative Lie rings.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings. By an action of $\mathfrak{g}$ on $\mathfrak{h}$ we mean an underlying group action of $\mathfrak{g}$ on $\mathfrak{h}$, given by a group homomorphism $\Phi\colon \mathfrak{g} \to \operatorname{Aut}\mathfrak{h}$, together with a map $\mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, $(x, y) \mapsto \langle x, y\rangle$, satisfying the following conditions:
$$\begin{aligned}
&\langle x, yy'\rangle = \langle x, y\rangle \ \langle {}^{y}x, {}^{y}y'\rangle,\\
&\langle xx', y\rangle = \langle {}^{x}x', {}^{x}y\rangle \ \langle x, y\rangle\,,\\
&\left\langle \{x, x'\}, {}^{x'}y\right\rangle \left\langle {}^{y}x, \langle x', y\rangle\right\rangle^{-1} \left\langle {}^{x}x', \langle x, y\rangle^{-1}\right\rangle^{-1} = 1,\\
&\left\langle {}^{y'}x, \{y, y'\}\right\rangle \{{}^{y}y', \langle x, y\rangle\} \{{}^{x}y, \langle x, y'\rangle^{-1}\} = 1,
\end{aligned}$$
where $x, x' \in \mathfrak{g}$, $y, y' \in \mathfrak{h}$, ${}^{x}y = \Phi(x)(y)$, ${}^{x}x' = xx'x^{-1}$, ${}^{y}y' = yy'y^{-1}$.

Now we give natural examples of actions of multiplicative Lie rings.

Examples 2.5.

(a) Let $\mathfrak{g}$ and $\mathfrak{h}$ be two ideals in a multiplicative Lie ring. Then the action of underlying group $\mathcal{V}(\mathfrak{g})$ on $\mathcal{V}(\mathfrak{h})$ defined by conjugation together with Lie multiplication is an action of multiplicative Lie ring $\mathfrak{g}$ on $\mathfrak{h}$.

(b) Let G and H be two groups. Any group action of G on H with trivial angle bracket gives an action of the multiplicative Lie ring $G_\bullet$ on $H_\bullet$.

(c) Let G and H be groups again. Any group action of G on H with angle bracket defined by $\langle g, h\rangle = {}^{g}hh^{-1}$, for all $g \in G$ and $h \in H$, is an action of the multiplicative Lie ring $G_{[,]}$ on $H_{[,]}$.

(d) Any Lie ring L acts on other Lie ring M via trivial group action and Lie ring action.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other. The actions are said to be compatible if

$$\begin{aligned} {}^{{}^{x}y}x' &= {}^{xyx^{-1}}x', & {}^{{}^{y}x}y' &= {}^{yxy^{-1}}y', \\ \big\langle \langle y, x\rangle^{-1}, y' \big\rangle &= \{\langle x, y\rangle, y'\}, & \big\langle \langle x, y\rangle^{-1}, x' \big\rangle &= \{\langle y, x\rangle, x'\}, \\ {}^{\langle x,y\rangle\langle y,x\rangle}y' &= y', & {}^{\langle x,y\rangle\langle y,x\rangle}x' &= x', \end{aligned}$$

for all $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$.

Remark 2.6. The first two conditions say that the underlying groups of $\mathfrak{g}$ and $\mathfrak{h}$ act on each other compatibly. If the underlying groups of $\mathfrak{g}$ and $\mathfrak{h}$ are abelian, then we have a compatible actions of Lie rings in the sense of [11].

Examples 2.7.

(a) Let $\mathfrak{g}$ and $\mathfrak{h}$ be two ideals in a multiplicative Lie ring acting on each other as in Example 2.5(a). Then, the mutual actions of $\mathfrak{g}$ and $\mathfrak{h}$ are compatible.

(b) Let G and H be two groups acting on each other compatibly and let $G_{[,]}$ and $H_{[,]}$ be the corresponding multiplicative Lie rings acting on each other as in Example 2.5(c). Then, the mutual actions of $G_{[,]}$ on $H_{[,]}$ are compatible.

2.8. **Definition of non-abelian tensor product.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other. Then the non-abelian tensor product $\mathfrak{g} \otimes \mathfrak{h}$ is the multiplicative Lie ring generated by the symbols $x \otimes y$ (for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$) subject to the following relations:

$$x \otimes (yy') = (x \otimes y)({}^{y}x \otimes {}^{y}y'), \tag{2.6}$$

$$(xx') \otimes y = ({}^{x}x' \otimes {}^{x}y)(x \otimes y), \tag{2.7}$$

$$(\{x, x'\} \otimes {}^{x'}y)({}^{y}x \otimes \langle x', y\rangle)^{-1}({}^{x}x' \otimes \langle x, y\rangle^{-1})^{-1} = 1, \tag{2.8}$$

$$({}^{y'}x \otimes \{y, y'\})(\langle y, x\rangle^{-1} \otimes {}^{y}y')^{-1}(\langle y', x\rangle \otimes {}^{x}y)^{-1} = 1, \tag{2.9}$$

$$\{x \otimes y, x' \otimes y'\} = \langle y, x\rangle^{-1} \otimes \langle x', y'\rangle\,, \tag{2.10}$$

for all $x, x' \in \mathfrak{g}$, $y, y' \in \mathfrak{h}$.

The following two propositions proved in [7] show that our definition of non-abelian tensor product of multiplicative Lie rings generalizes that of groups [2] and Lie rings [11].

Proposition 2.9. *Let G and H be groups acting on each other compatibly. Then there is a natural isomorphism of multiplicative Lie rings*

$$G_{[\,,]} \otimes H_{[\,,]} \cong \big(G \otimes H\big)_{[\,,]},$$

where the symbol $\otimes$ on the right side denotes the non-abelian tensor product of groups.

Proposition 2.10. *Let $\mathscr{g}$ and $\mathscr{h}$ be Lie rings acting on each other by Lie action. Then there is an isomorphism of Lie rings*

$$\mathscr{g} \otimes \mathscr{h} \cong \mathscr{g} \widetilde{\otimes} \mathscr{h}.$$

where $\widetilde{\otimes}$ denotes the non-abelian tensor product of Lie rings defined by Ellis.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two ideals in a multiplicative Lie ring acting on each other as in Example 2.5(a). Then, the non-abelian exterior product of $\mathfrak{g}$ and $\mathfrak{h}$ is defined by

$$\mathfrak{g} \wedge \mathfrak{h} = \mathfrak{g} \otimes \mathfrak{h} / \{ \text{ ideal generated by } x \otimes x,\ x \in \mathfrak{g} \cap \mathfrak{h} \}.$$

There is a well-defined homomorphism $\mathfrak{g} \wedge \mathfrak{h} \to \{\mathfrak{g}, \mathfrak{h}\}$, $x \wedge y \mapsto \{x, y\}$. We denote this homomorphism by $\theta_{\mathfrak{g},\mathfrak{h}}$.

Given an extension of multiplicative Lie rings $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$, we have the following exact sequence of multiplicative Lie rings:

$$\mathfrak{g} \wedge \mathfrak{n} \to \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h} \wedge \mathfrak{h} \to 1. \tag{2.11}$$

We will use the following theorem proved in [7].

Theorem 2.11. *Let $\mathfrak{g}$ be a multiplicative Lie ring. Then there is a natural isomorphism of groups*

$$HS_2^{mlr}(\mathfrak{g}) \cong \mathsf{Ker}\big(\mathfrak{g} \wedge \mathfrak{g} \xrightarrow{\theta_{\mathfrak{g},\mathfrak{g}}} \{\mathfrak{g}, \mathfrak{g}\}\big).$$

3. Non-abelian tensor product of nilpotent, solvable and Engel multiplicative Lie rings

In this section we generalize the results of [17, 19] to multiplicative Lie rings. In particular, we show that if the actions are compatible, then the non-abelian tensor product of nilpotent, solvable and Engel multiplicative Lie rings is nilpotent, solvable and Engel, respectively.

Let $\mathfrak{g}$ be a multiplicative Lie ring. The *descending central series* of $\mathfrak{g}$ is defined by

$$\Gamma_1(\mathfrak{g}) \supseteq \Gamma_2(\mathfrak{g}) \supseteq \Gamma_3(\mathfrak{g}) \supseteq \cdots,$$

where $\Gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\Gamma_{i+1}(\mathfrak{g}) = \{\Gamma_i(\mathfrak{g}), \mathfrak{g}\}$. We say that $\mathfrak{g}$ is *nilpotent*, if $\Gamma_i(\mathfrak{g}) = 1$ for some $i \geq 1$.

The *derived series* of $\mathfrak{g}$ is defined by

$$\Gamma^{(1)}(\mathfrak{g}) \supseteq \Gamma^{(2)}(\mathfrak{g}) \supseteq \Gamma^{(3)}(\mathfrak{g}) \supseteq \cdots,$$

where $\Gamma^{(1)}(\mathfrak{g}) = \mathfrak{g}$ and $\Gamma^{(i+1)}(\mathfrak{g}) = \{\Gamma^{(i)}(\mathfrak{g}), \Gamma^{(i)}(\mathfrak{g})\}$. We say that $\mathfrak{g}$ is *solvable*, if $\Gamma^{(i)}(\mathfrak{g}) = 1$ for some $i \geq 1$.

An extension of multiplicative Lie rings $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$ is said to be *central*, if $\mathfrak{n}$ is contained in the center of $\mathfrak{g}$, i.e, $\{\mathfrak{n}, \mathfrak{g}\} = 1$. We also call it a *central extension* of $\mathfrak{h}$.

Remark 3.1. The definition immediately implies that a central extension of a nilpotent (solvable) multiplicative Lie ring is nilpotent (solvable). It is also clear that every subring of a nilpotent (solvable) multiplicative Lie ring is nilpotent (solvable).

Lemma 3.2. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other. If the actions of the underlying groups are compatible, then*

$$\langle x,y\rangle\langle g,h\rangle\langle x,y\rangle^{-1} = \langle {}^{[x,y]}g,\ {}^{[x,y]}h\rangle,\quad \langle y,x\rangle\langle h,g\rangle\langle y,x\rangle^{-1} = \langle {}^{[y,x]}h,\ {}^{[y,x]}g\rangle,$$

for all $x, g \in \mathfrak{g}$ and $y, h \in \mathfrak{h}$.

Proof. Since both equalities are essentially the same, we will check only the first one. For all $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$, we have:

$$\begin{aligned}\langle xx', yy'\rangle &= \langle {}^{x}x',\ {}^{x}y\,{}^{x}y'\rangle\langle x, yy'\rangle = \langle {}^{x}x',\ {}^{x}y\rangle\langle {}^{xyx}x',\ {}^{xyx}y'\rangle\langle x,y\rangle\langle {}^{y}x,\ {}^{y}y'\rangle\\ &= \langle {}^{x}x',\ {}^{x}y\rangle\langle {}^{xy}x',\ {}^{xy}y'\rangle\langle x,y\rangle\langle {}^{y}x,\ {}^{y}y'\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle xx', yy'\rangle &= \langle xx', y\rangle\langle {}^{y}x\,{}^{y}x',\ {}^{y}y'\rangle = \langle {}^{x}x',\ {}^{x}y\rangle\langle x,y\rangle\langle {}^{yxy}x',\ {}^{yxy}y'\rangle\langle {}^{y}x,\ {}^{y}y'\rangle\\ &= \langle {}^{x}x',\ {}^{x}y\rangle\langle x,y\rangle\langle {}^{yx}x',\ {}^{yx}y'\rangle\langle {}^{y}x,\ {}^{y}y'\rangle.\end{aligned}$$

Thus,

$$\langle x,y\rangle\langle {}^{yx}x',\ {}^{yx}y'\rangle\langle x,y\rangle^{-1} = \langle {}^{xy}x',\ {}^{xy}y'\rangle.$$

Set $g = {}^{yx}x'$ and $h = {}^{yx}y'$. Then, we get the desired result. □

Lemma 3.3. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other compatibly. Then,*

$$\Big\{x\otimes y, \prod_{i=1}^{n}(x_i\otimes y_i)^{\varepsilon_i}\Big\} = \langle y,x\rangle^{-1}\otimes\prod_{i=1}^{n}\langle x_i,y_i\rangle^{\varepsilon_i},$$

where $x, x_1, \dots, x_n \in \mathfrak{g}$, $y, y_1, \dots, y_n \in \mathfrak{h}$, and $\varepsilon_i = 1$ or $\varepsilon_i = -1$ for all $i = 1, 2, \dots, n$.

Proof. We will show this identity by induction with respect to n. If $n = 1$ and $\varepsilon_1 = 1$, then it is true. If $n = 1$ and $\varepsilon_1 = -1$, then taking into account the compatibility of actions we will have:

$$\begin{aligned}&\{x\otimes y, (x_1\otimes y_1)^{-1}\} = {}^{(x_1\otimes y_1)^{-1}}\{x\otimes y, x_1\otimes y_1\}^{-1}\\ &= \{{}^{[x_1,y_1]^{-1}}x\otimes {}^{[x_1,y_1]^{-1}}y,\ {}^{[x_1,y_1]^{-1}}x_1\otimes {}^{[x_1,y_1]^{-1}}y_1\}^{-1}\\ &= \big(\langle {}^{[y_1,x_1]}y,\ {}^{[y_1,x_1]}x\rangle^{-1}\otimes\langle {}^{[x_1,y_1]^{-1}}x_1,\ {}^{[x_1,y_1]^{-1}}y_1\rangle\big)^{-1}\\ &= \big({}^{\langle y_1,x_1\rangle}\langle y,x\rangle^{-1}\otimes {}^{\langle x_1,y_1\rangle^{-1}}\langle x_1,y_1\rangle\big)^{-1}\\ &= \big({}^{\langle x_1,y_1\rangle^{-1}}\langle y,x\rangle^{-1}\otimes {}^{\langle x_1,y_1\rangle^{-1}}\langle x_1,y_1\rangle\big)^{-1} = \langle y,x\rangle^{-1}\otimes\langle x_1,y_1\rangle^{-1}.\end{aligned}$$

Now, suppose that the lemma is true for $n-1$. Then, using the previous lemma we will have:

$$\{x\otimes y, \prod_{i=1}^{n}(x_i\otimes y_i)^{\varepsilon_i}\} = \{x\otimes y, (x_1\otimes y_1)^{\varepsilon_1}\}^{(x_1\otimes y_1)^{\varepsilon_1}}\{x\otimes y, \prod_{i=2}^{n}(x_i\otimes y_i)^{\varepsilon_i}\}$$

$$= \{x\otimes y, (x_1\otimes y_1)^{\varepsilon_1}\}\{{}^{[x_1,y_1]^{\varepsilon_1}}x\otimes {}^{[x_1,y_1]^{\varepsilon_1}}y, \prod_{i=2}^{n}\big({}^{[x_1,y_1]^{\varepsilon_1}}x_i\otimes {}^{[x_1,y_1]^{\varepsilon_1}}y_i\big)^{\varepsilon_i}\}$$

(by the induction hypothesis)

$$= \big(\langle y,x\rangle^{-1}\otimes\langle x_1,y_1\rangle^{\varepsilon_1}\big)\big(\langle{}^{[x_1,y_1]^{\varepsilon_1}}y, {}^{[x_1,y_1]^{\varepsilon_1}}x\rangle^{-1}\otimes\prod_{i=2}^{n}\langle{}^{[x_1,y_1]^{\varepsilon_1}}x_i, {}^{[x_1,y_1]^{\varepsilon_1}}y_i\rangle^{\varepsilon_i}\big)$$

$$= \big(\langle y,x\rangle^{-1}\otimes\langle x_1,y_1\rangle^{\varepsilon_1}\big)\big(\langle{}^{[y_1,x_1]^{-\varepsilon_1}}y, {}^{[y_1,x_1]^{-\varepsilon_1}}x\rangle^{-1}\otimes\prod_{i=2}^{n}\langle{}^{[x_1,y_1]^{\varepsilon_1}}x_i, {}^{[x_1,y_1]^{\varepsilon_1}}y_i\rangle^{\varepsilon_i}\big)$$

$$= \big(\langle y,x\rangle^{-1}\otimes\langle x_1,y_1\rangle^{\varepsilon_1}\big)\big({}^{\langle y_1,x_1\rangle^{-\varepsilon_1}}\langle y,x\rangle^{-1}\otimes\prod_{i=2}^{n}{}^{\langle x_1,y_1\rangle^{\varepsilon_1}}\langle x_i,y_i\rangle^{\varepsilon_i}\big)$$

$$= \big(\langle y,x\rangle^{-1}\otimes\langle x_1,y_1\rangle^{\varepsilon_1}\big)\big({}^{\langle x_1,y_1\rangle^{\varepsilon_1}}\langle y,x\rangle^{-1}\otimes {}^{\langle x_1,y_1\rangle^{\varepsilon_1}}\prod_{i=2}^{n}\langle x_i,y_i\rangle^{\varepsilon_i}\big)$$

$$= \langle y,x\rangle^{-1}\otimes\prod_{i=1}^{n}\langle x_i,y_i\rangle^{\varepsilon_i}. \quad \square$$

□

Lemma 3.4. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other compatibly and $\phi\colon \mathfrak{g}\otimes\mathfrak{h}\to\mathfrak{h}$ be a map defined by $x\otimes y\mapsto\langle x,y\rangle$, for $x\in\mathfrak{g}$, $y\in\mathfrak{h}$. Then $\phi\colon\mathfrak{g}\otimes\mathfrak{h}\to\mathfrak{h}$ is a well-defined homomorphism of multiplicative Lie rings.*

Proof. We have to show the following identities:

$$\phi\big(x\otimes(yy')\big)=\phi(x\otimes y)\ \phi({}^{y}x\otimes{}^{y}y'),$$
$$\phi\big((xx')\otimes y\big)=\phi({}^{x}x'\otimes{}^{x}y)\ \phi(x\otimes y),$$
$$\phi(\{x,x'\}\otimes{}^{x'}y)\ \phi\big({}^{y}x\otimes\langle x',y\rangle\big)^{-1}\ \phi\big({}^{x}x'\otimes\langle x,y\rangle^{-1}\big)^{-1}=1,$$
$$\phi({}^{y'}x\otimes\{y,y'\})\ \phi(\langle y,x\rangle^{-1}\otimes{}^{y}y')^{-1}\ \phi(\langle y',x\rangle\otimes{}^{x}y)^{-1}=1,$$
$$\phi(\{x\otimes y,x'\otimes y'\})=\phi(\langle y,x\rangle^{-1}\otimes\langle x',y'\rangle).$$

The first three identities are true even without compatibility conditions. We will check the last two identities.

$$
\begin{aligned}
&\phi({}^{y'}x \otimes \{y,y'\})\ \phi(\langle y,x\rangle^{-1} \otimes {}^{y}y')^{-1}\ \phi(\langle y',x\rangle \otimes {}^{x}y)^{-1}\\
&= \langle {}^{y'}x, \{y,y'\}\rangle\ \big\langle \langle y,x\rangle^{-1}, {}^{y}y'\big\rangle^{-1}\ \big\langle \langle y',x\rangle, {}^{x}y\big\rangle^{-1}\\
&= \langle {}^{y'}x, \{y,y'\}\rangle\ \{\langle x,y\rangle, {}^{y}y'\}^{-1}\ \big\langle \langle y',x\rangle, {}^{x}y\big\rangle^{-1}\\
&= \langle {}^{y'}x, \{y,y'\}\rangle\ \{{}^{y}y', \langle x,y\rangle\}\ \big\langle \langle y',x\rangle^{-1}, {}^{\langle y',x\rangle x}y\big\rangle\\
&= \langle {}^{y'}x, \{y,y'\}\rangle\ \{{}^{y}y', \langle x,y\rangle\}\ \{\langle x,y'\rangle, {}^{\langle y',x\rangle x}y\}\\
&= \langle {}^{y'}x, \{y,y'\}\rangle\ \{{}^{y}y', \langle x,y\rangle\}\ \{\langle x,y'\rangle^{-1}, {}^{\langle x,y'\rangle\langle y',x\rangle x}y\}^{-1}\\
&= \langle {}^{y'}x, \{y,y'\}\rangle\ \{{}^{y}y', \langle x,y\rangle\}\ \{{}^{x}y, \langle x,y'\rangle^{-1}\} = 1.
\end{aligned}
$$

$$
\begin{aligned}
&\phi(\{x\otimes y, x'\otimes y'\}) = \{\langle x,y\rangle, \langle x',y'\rangle\}\\
&= \big\langle \langle y,x\rangle^{-1}, \langle x',y'\rangle\big\rangle = \phi(\langle y,x\rangle^{-1} \otimes \langle x',y'\rangle).
\end{aligned}
$$

□

Proposition 3.5. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other compatibly. If $\mathfrak{h}$ is nilpotent (solvable), then $\mathfrak{g}\otimes\mathfrak{h}$ is also nilpotent (solvable).*

Proof. We have a short exact sequence of multiplicative Lie rings

$$1 \to \mathsf{Ker}\,\phi \to \mathfrak{g}\otimes\mathfrak{h} \to \phi(\mathfrak{g}\otimes\mathfrak{h}) \to 1,$$

where $\phi\colon \mathfrak{g}\otimes\mathfrak{h} \to \mathfrak{h}$ is a homomorphism of multiplicative Lie rings defined as in Lemma 3.4. If $\mathfrak{h}$ is a nilpotent (solvable) multiplicative Lie ring, then so is $\phi(\mathfrak{g}\otimes\mathfrak{h})$. Therefore, it suffices to show that $\mathsf{Ker}\,\phi$ is contained in the center of $\mathfrak{g}\otimes\mathfrak{h}$, i.e. $\{\omega,\omega'\} = 1$, for each $\omega\in\mathfrak{g}\otimes\mathfrak{h}$ and $\omega'\in\mathsf{Ker}\,\phi$. Clearly, it is enough to consider the case $\omega = x\otimes y$, for each $x\in\mathfrak{g}$ and $y\in\mathfrak{h}$. Let $(x_1\otimes y_1)^{\varepsilon_1}(x_2\otimes y_2)^{\varepsilon_2}\cdots(x_n\otimes y_n)^{\varepsilon_n}\in \mathsf{Ker}\,\phi$, where $x_1,\dots,x_n\in\mathfrak{g}$, $y_1,\dots,y_n\in\mathfrak{h}$ and $\varepsilon_1,\dots,\varepsilon_n\in\{1,-1\}$. By Lemma 3.3 we have:

$$
\begin{aligned}
&\big\{x\otimes y, \prod_{i=1}^{n}(x_i\otimes y_i)^{\varepsilon_i}\big\} = \langle y,x\rangle^{-1}\otimes\prod_{i=1}^{n}\langle x_i,y_i\rangle^{\varepsilon_i}\\
&= \langle y,x\rangle^{-1}\otimes\phi\big(\prod_{i=1}^{n}\langle x_i,y_i\rangle^{\varepsilon_i}\big) = \langle y,x\rangle^{-1}\otimes 1 = 1.
\end{aligned}
$$

□

Corollary 3.6 ([19])**.** *Let G and H be two groups acting on each other compatibly. If H is nilpotent (solvable), then $G\otimes H$ is also nilpotent (solvable).*

Proof. This follows from Proposition 3.5 and Proposition 2.9. □

Corollary 3.7 ([17])**.** *Let $\mathcal{g}$ and $\mathcal{h}$ be two Lie rings acting on each other compatibly. If $\mathcal{h}$ is nilpotent (solvable), then $\mathcal{g}\otimes\mathcal{h}$ is also nilpotent (solvable).*

Proof. This follows from Proposition 3.5 and Proposition 2.10. □

Let $\mathfrak{g}$ be a multiplicative Lie ring. For each $x,y\in\mathfrak{g}$, set $\{x,_1 y\} = \{x,y\}$ and $\{x,_{n+1} y\} = \{\{x,_n y\},y\}$ for $n\geq 1$. We say that $\mathfrak{g}$ is an Engel multiplicative Lie ring, if for each $x,y\in\mathfrak{g}$ there exists $n = n(x,y)$ such that $\{x,_n y\} = 1$. Clearly, every subring of an Engel multiplicative Lie ring is Engel. Moreover, it is easy to see that every central extension of an Engel multiplicative Lie ring is Engel.

Proposition 3.8. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two multiplicative Lie rings acting on each other compatibly. If $\mathfrak{h}$ is an Engel multiplicative Lie ring, then $\mathfrak{g} \otimes \mathfrak{h}$ is also Engel.*

Proof. As in Proposition 3.5 we have a short exact sequence of multiplicative Lie rings

$$1 \to \mathsf{Ker}\,\phi \to \mathfrak{g} \otimes \mathfrak{h} \to \phi(\mathfrak{g} \otimes \mathfrak{h}) \to 1,$$

where $\phi\colon \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$ is a homomorphism of multiplicative Lie rings defined as in Lemma 3.4. This is a central extension of $\phi(\mathfrak{g} \otimes \mathfrak{h})$ which itself is a subring of an Engel multiplicative Lie ring $\mathfrak{h}$. Therefore, $\mathfrak{g} \otimes \mathfrak{h}$ is Engel. □

This proposition immediately implies the following results.

Corollary 3.9. *Let G and H be two groups acting on each other compatibly. If H is an Engel group, then $G \otimes H$ is also Engel.*

Corollary 3.10. *Let $\mathpzc{g}$ and $\mathpzc{h}$ be two Lie rings acting on each other compatibly. If $\mathpzc{h}$ is an Engel Lie ring, then $\mathpzc{g} \otimes \mathpzc{h}$ is also Engel.*

4. Exact sequences in homology

In this section we generalize the well-know five term exact sequence in homology of groups [13, Chapter 6] to multiplicative Lie rings.

Proposition 4.1. *Let $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$ be an extension of multiplicative Lie rings. Then, we have the following exact sequence of groups:*

$$\begin{aligned}&\mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}}\colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{h}) \to \mathfrak{n}/\{\mathfrak{n},\mathfrak{g}\} \to HS_1^{mlr}(\mathfrak{g})\\ &\to HS_1^{mlr}(\mathfrak{h}) \to 1,\end{aligned}$$

where $\theta_{\mathfrak{n},\mathfrak{g}}$ is defined as in Section 2.

Proof. By (2.11) we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & \mathfrak{n} \wedge \mathfrak{g} \arrow[r] \arrow[d, "\theta_{\mathfrak{n},\mathfrak{g}}"'] & \mathfrak{g} \wedge \mathfrak{g} \arrow[r] \arrow[d, "\theta_{\mathfrak{g},\mathfrak{g}}"'] & \mathfrak{h} \wedge \mathfrak{h} \arrow[r] \arrow[d, "\theta_{\mathfrak{h},\mathfrak{h}}"'] & 1 \\
1 \arrow[r] & \mathfrak{n} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{h} \arrow[r] & 1.
\end{tikzcd}$$

Now the snake lemma and Theorem 2.11 imply the proposition. □

Remark 4.2.

(i) Let $1 \to R \to G \to H \to 1$ be an extension of groups. Suppose that $R_{[,]}, G_{[,]}$ and $H_{[,]}$ are multiplicative Lie rings defined as in Example 2.1(b). By [1, Proposition 3.9] $HS_2^{mlr}(G_{[,]})$ and $HS_2^{mlr}(H_{[,]})$ are isomorphic to the Eilenberg-MacLane homologies $H_2(G)$ and $H_2(H)$, respectively. Moreover, by Proposition 2.9 $(R \wedge G)_{[,]} \cong R_{[,]} \otimes G_{[,]}$. Thus, Proposition 4.1 yields the well-known six term exact sequence in homology of groups.

(ii) Let $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$ be an extension of Lie rings. By definition the weak homologies of a Lie ring are isomorphic to the Chevalley-Eilenberg homologies of the same ring (see [1]). Hence, by Theorem 2.3 $HS_2^{mlr}(\mathfrak{g})$ and $HS_2^{mlr}(\mathfrak{h})$ are isomorphic to the second Chevalley-Eilenberg homologies of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Thus, Proposition 4.1 yields the six term exact sequence in homology of Lie rings.

Proposition 4.3. *Let $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$ be an extension of multiplicative Lie rings with $\mathfrak{n}$ being perfect. Then, we have the following exact sequence of homology groups:*

$$HS_2^{mlr}(\mathfrak{n}) \to HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{h}) \to 1.$$

Proof. First we will prove that

$$\mathsf{Im}\big(\mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}\big) = \mathsf{Im}\big(\mathfrak{n} \wedge \mathfrak{n} \to \mathfrak{g} \wedge \mathfrak{g}\big). \tag{4.1}$$

It is enough to show that $n \wedge g \in \mathsf{Im}\big(\mathfrak{n} \wedge \mathfrak{n} \to \mathfrak{g} \wedge \mathfrak{g}\big)$ for each $n \in \mathfrak{n}$ and $g \in \mathfrak{g}$. Since $\mathfrak{n}$ is perfect, every element n in $\mathfrak{n}$ can be written as a finite product $\{x_1, x_1'\}\{x_2, x_2'\} \cdots \{x_t, x_t'\}$, where $x_i, x_i' \in \mathfrak{n}$ for each $i \in \{1, \dots, t\}$. Therefore, by (2.7) it suffices to show that $\{x, x'\} \wedge g \in \mathsf{Im}\big(\mathfrak{n} \wedge \mathfrak{n} \to \mathfrak{g} \wedge \mathfrak{g}\big)$ for each $x, x' \in \mathfrak{n}$ and $g \in \mathfrak{g}$. By (2.8) we have:

$$\{x, x'\} \wedge g = ({}^{x}x' \wedge \{{}^{x'^{-1}}g, x\})({}^{x'^{-1}gx'}x \wedge \{x',\ {}^{x'^{-1}}g\}) \in \mathsf{Im}\big(\mathfrak{n} \wedge \mathfrak{n} \to \mathfrak{g} \wedge \mathfrak{g}\big).$$

Using (4.1) we can modify the diagram in Proposition 4.1 as follows:

$$\begin{tikzcd}
 & \mathfrak{n} \wedge \mathfrak{n} \arrow[r] \arrow[d, "\theta_{\mathfrak{n},\mathfrak{n}}"'] & \mathfrak{g} \wedge \mathfrak{g} \arrow[r] \arrow[d, "\theta_{\mathfrak{g},\mathfrak{g}}"'] & \mathfrak{h} \wedge \mathfrak{h} \arrow[r] \arrow[d, "\theta_{\mathfrak{h},\mathfrak{h}}"'] & 1 \\
1 \arrow[r] & \mathfrak{n} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{h} \arrow[r] & 1.
\end{tikzcd}$$

The snake lemma and Theorem 2.11 finish the proposition. □

Corollary 4.4. *Let $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$ be a central extension of multiplicative Lie rings with $\mathfrak{g}$ being perfect. Then we have the following exact sequence of groups:*

$$1 \to HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{h}) \to \mathfrak{n} \to 1.$$

Proof. By Proposition 4.1 it is enough to prove that $\mathfrak{n} \wedge \mathfrak{g} = 1$. Since $\mathfrak{g}$ is perfect, it suffices to show that $n \wedge \{g, g'\} = 1$ for each $n \in \mathfrak{n}$ and $g, g' \in \mathfrak{g}$. By (2.9) we have:

$$n \wedge \{g, g'\} = \big(\{g',\ {}^{g'^{-1}}n\} \wedge\ {}^{g'^{-1}ng'}g\big)\ \big(\{{}^{g'^{-1}}n, g\} \wedge\ {}^{g}g'\big) = 1,$$

because $\mathfrak{n}$ is contained in the center of $\mathfrak{g}$. □

Let R be a unital associative algebra over $\mathbb{Z}$. Denote by $\mathfrak{gl}_n(R)$, $n \geq 2$, the Lie ring of $n \times n$ matrices with entries in R and denote by $\mathfrak{sl}_n(R)$, $n \geq 2$, the Lie subring of $\mathfrak{gl}_n(R)$ generated by all strictly upper and lower triangular matrices. Let $[R, R]$ be an additive subgroup of R generated by $ab - ba$ for all $a, b \in R$. There is a homomorphism of Lie rings $\mathtt{tr}\colon \mathfrak{gl}_n(R) \to R/[R, R]$ (here $R/[R, R]$ is a Lie ring with trivial Lie bracket) given by $\alpha \mapsto \mathtt{tr}(\alpha) + [R, R]$, where $\mathtt{tr}(\alpha)$ denotes the trace of a matrix $\alpha \in \mathfrak{gl}_n(R)$, $n \geq 2$. If $n \geq 3$, then $\mathfrak{sl}_n(R)$ is a perfect Lie ring. Moreover, we have the following extension of Lie rings:

$$0 \to \mathfrak{sl}_n(R) \to \mathfrak{gl}_n(R) \xrightarrow{\mathtt{tr}} R/[R, R] \to 0.$$

Corollary 4.5. *Let R be an associative algebra over $\mathbb{Z}$. Then we have the following exact sequence of groups*

$$HC_1(R) \to H_2\big(\mathfrak{gl}_n(R)\big) \to \frac{R}{[R, R]} \wedge \frac{R}{[R, R]} \to 1, \quad n \geq 5,$$

where $HC_1(R)$ denotes the first cyclic homology group of R and $H_2\big(\mathfrak{gl}_n(R)\big)$ denotes the second Chevalley-Eilenberg homology of $\mathfrak{gl}_n(R)$.

Proof. We know that $HS_2^{mlr}(\mathfrak{gl}_n(R)) \cong H_2(\mathfrak{gl}_n(R))$ (see [1]). Therefore, by Proposition 4.3 we have the following exact sequence:

$$H_2\big(\mathfrak{sl}_n(R)\big) \to H_2\big(\mathfrak{gl}_n(R)\big) \to H_2\big(R/[R,R]\big) \to 1, \quad n \geq 3.$$

Since $R/[R,R]$ has the trivial Lie bracket, $H_2\big(R/[R,R]\big) = \frac{R}{[R,R]} \wedge \frac{R}{[R,R]}$. On the other hand, by [15] $H_2\big(\mathfrak{sl}_n(R)\big) = HC_1(R)$ for $n \geq 5$. □

An extension of multiplicative Lie rings $1 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 1$ is said to be a free presentation of $\mathfrak{g}$, if $\mathfrak{f}$ is a free multiplicative Lie ring over a set. In [7] we have posed the following question:

Question. Let $\mathfrak{g}$ be a multiplicative Lie ring and $1 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 1$ be a free presentation of $\mathfrak{g}$. Is there an isomorphism between $HS_3^{mlr}(\mathfrak{g})$ and $\mathsf{Ker}\,\theta_{\mathfrak{r},\mathfrak{f}}$?

We do not know the answer to this question yet. In [7] we proved the following:

Proposition 4.6. *Let $\mathfrak{g}$ be a multiplicative Lie ring and $1 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 1$ be a free presentation of $\mathfrak{g}$. Suppose that for any free presentation $1 \to \mathfrak{r}' \to \mathfrak{f}' \to \mathfrak{f} \to 1$ of the free multiplicative Lie ring $\mathfrak{f}$, $\mathsf{Ker}\,\theta_{\mathfrak{r}',\mathfrak{f}'} = 1$. Then, there is an isomorphism between $HS_3^{mlr}(\mathfrak{g})$ and $\mathsf{Ker}\,\theta_{\mathfrak{r},\mathfrak{f}}$.*

Proposition 4.6 says that if the answer to the aforementioned question is positive for free multiplicative Lie rings, then it will be the same for arbitrary multiplicative Lie rings. Based on the hypothesis in Proposition 4.6 we can prove the eight term exact sequence in the homology of multiplicative Lie rings.

Proposition 4.7. *Let $\mathfrak{g}$ be a multiplicative Lie ring and $1 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 1$ be a free presentation of $\mathfrak{g}$. Suppose that for any free presentation $1 \to \mathfrak{r}' \to \mathfrak{f}' \to \mathfrak{f} \to 1$ of $\mathfrak{f}$, $\mathsf{Ker}\,\theta_{\mathfrak{r}',\mathfrak{f}'} = 1$. Then, for any extension of multiplicative Lie rings $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$, we have the following exact sequence of groups:*

$$HS_3^{mlr}(\mathfrak{g}) \to HS_3^{mlr}(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}} \colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{h})$$
$$\to \mathfrak{n}/\{\mathfrak{n},\mathfrak{g}\} \to HS_1^{mlr}(\mathfrak{g}) \to HS_1^{mlr}(\mathfrak{h}) \to 1.$$

Proof. Let $\mathfrak{s}$ be an ideal of $\mathfrak{f}$ such that $\mathfrak{f}/\mathfrak{s} = \mathfrak{h}$. By Proposition 4.6 we have:

$$HS_3^{mlr}(\mathfrak{g}) \cong \mathsf{Ker}\,\theta_{\mathfrak{r},\mathfrak{f}}, \qquad HS_3^{mlr}(\mathfrak{h}) \cong \mathsf{Ker}\,\theta_{\mathfrak{s},\mathfrak{f}}. \tag{4.2}$$

The extensions $1 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 1$ and $1 \to \mathfrak{r} \to \mathfrak{s} \to \mathfrak{n} \to 1$ imply the following exact sequence:

$$\mathfrak{s} \otimes \mathfrak{r} \times \mathfrak{r} \otimes \mathfrak{f} \to \mathfrak{s} \otimes \mathfrak{f} \to \mathfrak{n} \otimes \mathfrak{g} \to 1.$$

Using this sequence we get the following exact sequence:

$$\mathfrak{r} \wedge \mathfrak{f} \to \mathfrak{s} \wedge \mathfrak{f} \to \mathfrak{n} \wedge \mathfrak{g} \to 1.$$

Thus, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & \mathfrak{r} \wedge \mathfrak{f} \arrow[r] \arrow[d, "\theta_{\mathfrak{r},\mathfrak{f}}"'] & \mathfrak{s} \wedge \mathfrak{f} \arrow[r] \arrow[d, "\theta_{\mathfrak{s},\mathfrak{f}}"'] & \mathfrak{n} \wedge \mathfrak{g} \arrow[r] \arrow[d, "\theta_{\mathfrak{n},\mathfrak{g}}"'] & 1 \\
1 \arrow[r] & \mathfrak{r} \arrow[r] & \mathfrak{s} \arrow[r] & \mathfrak{n} \arrow[r] & 1.
\end{tikzcd}$$

The snake lemma and (4.2) imply the following exact sequence:

$$HS_3^{mlr}(\mathfrak{g}) \to HS_3^{mlr}(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}} \colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to \mathfrak{r}/\{\mathfrak{r},\mathfrak{f}\} \to \mathfrak{s}/\{\mathfrak{s},\mathfrak{f}\}.$$

It is easy to see that

$$\mathsf{Im}\left(\mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}} \colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to \mathfrak{r}/\{\mathfrak{r},\mathfrak{f}\}\right) \subseteq \frac{\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{f}\}}.$$

Therefore, we have an exact sequence of groups:

$$HS_3^{mlr}(\mathfrak{g}) \to HS_3^{mlr}(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}} \colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to \frac{\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{f}\}} \to \frac{\mathfrak{s} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{s},\mathfrak{f}\}}.$$

Using the Hopf formula [1] we get an exact sequence of groups:

$$HS_3^{mlr}(\mathfrak{g}) \to HS_3^{mlr}(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}} \colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{h}).$$

The rest of the proof follows from Proposition 4.1. □

Proposition 4.8. *Let $\mathfrak{g}$ be a multiplicative Lie ring and $1 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 1$ be an extension with $\mathfrak{f}$ being a free multiplicative Lie ring over a group. Suppose that for any extension $1 \to \mathfrak{r}' \to \mathfrak{f}' \to \mathfrak{f} \to 1$ of $\mathfrak{f}$, where $\mathfrak{f}'$ is a free multiplicative Lie ring over a group, $\mathsf{Ker}\,\theta_{\mathfrak{r}',\mathfrak{f}'} = 1$. Then, for any extension of multiplicative Lie rings $1 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{h} \to 1$, we have the following exact sequence of groups:*

$$HW_3^{mlr}(\mathfrak{g}) \to HW_3^{mlr}(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{n},\mathfrak{g}} \colon \mathfrak{n} \wedge \mathfrak{g} \to \mathfrak{n}\right) \to HW_2^{mlr}(\mathfrak{g}) \to HW_2^{mlr}(\mathfrak{h})$$

$$\to \mathfrak{r}/\{\mathfrak{r},\mathfrak{g}\} \to HW_1^{mlr}(\mathfrak{g}) \to HW_1^{mlr}(\mathfrak{h}) \to 1.$$

Proof. We have an analogue of Proposition 4.6 for the third weak homology group of a multiplicative Lie ring (see [7]). Therefore, the proof of Proposition 4.8 can be carried out as that of Proposition 4.7. □

In the next theorem we will use our method to reprove the eight term exact sequence in the homology of Lie algebras which is implicitly given in [11].

Theorem 4.9. *Let k be a commutative ring with identity and $1 \to \mathscr{n} \to \mathscr{g} \to \mathscr{h} \to 1$ be an extension of Lie algebras over k. Then we have the following exact sequence of groups:*

$$\mathsf{H}_3(\mathscr{g}) \to \mathsf{H}_3(\mathscr{h}) \to \mathsf{Ker}\left(\theta_{\mathscr{n},\mathscr{g}} \colon \mathscr{n} \wedge \mathscr{g} \to \mathscr{n}\right) \to \mathsf{H}_2(\mathscr{g}) \to \mathsf{H}_2(\mathscr{h})$$

$$\to \mathscr{n}/[\mathscr{n},\mathscr{g}] \to \mathsf{H}_1(\mathscr{g}) \to \mathsf{H}_1(\mathscr{h}) \to 1,$$

where H_i denotes the i-th Chevalley-Eilenberg homology for $i = 1, 2, 3$.

Proof. Let Lie denote the category of Lie algebras over k. Let $\mathsf{U}\colon \mathsf{Lie} \to Sets$ be the natural forgetful functor. The functor U admits a left adjoint functor $\mathsf{F}\colon Sets \to \mathsf{Lie}$. We know that the Chevalley-Eilenberg homology of a Lie algebra $\mathscr{g}$ is isomorphic to the left derived functor of the abelianization functor Ab, $\mathsf{Ab}(\mathscr{g}) = \frac{\mathscr{g}}{[\mathscr{g},\mathscr{g}]}$, $\mathsf{H}_n(\mathscr{g}) \cong \mathcal{L}_{n-1}^{\mathsf{P}}\mathsf{Ab}(\mathscr{g})$, $n \geq 1$, where P is a projective class defined by the adjoint pair (F, U). The method used in the proof of Proposition 4.7 shows that it suffices to prove $\mathsf{Ker}\,\theta_{\mathscr{r},\mathscr{f}'} = 1$, for all extension of Lie algebras $1 \to \mathscr{r} \to \mathscr{f}' \xrightarrow{p} \mathscr{f} \to 1$ with free Lie algebras f and f'. Let $\mu\colon r \wedge f' \to f' \wedge f'$ be a natural homomorphism. We have the following commutative diagram:

$$\begin{tikzcd} r \wedge f' \arrow[r, "\mu"] \arrow[d, "\theta_{r,f'}"'] & f' \wedge f' \arrow[d, "\theta_{f',f'}"] \\ r \arrow[r] & f'. \end{tikzcd}$$

Since f' is a free Lie algebra, $\mathsf{Ker}\,\theta_{f',f'} = 1$. Therefore, it is enough to show that $\mathsf{Ker}\,\mu = 1$. Since f is a free Lie algebra, there is a homomorphism of Lie algebras $i\colon f \to f'$ such that $pi = 1_f$. Now [11, Proposition 13] implies that μ is injective. □

5. A new version of Stallings' theorem

Given a multiplicative Lie ring $\mathfrak{g}$, let $\Gamma_n(\mathfrak{g})$ (resp. $\Gamma^{(n)}(\mathfrak{g})$), $n \geq 1$, denote the lower central series (resp. the derived series) of $\mathfrak{g}$ defined in Section 3. Using Proposition 4.1 we are able to prove Stallings' theorem (see [13, VI. Theorem 9.1] or [18]) and Dwyer's theorem (see [9, Theorem 1.1]) in the multiplicative Lie ring framework.

Theorem 5.1. *Let $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of multiplicative Lie rings. If the induced homomorphism $\phi_*\colon HS_1^{mlr}(\mathfrak{g}) \to HS_1^{mlr}(\mathfrak{g}')$ is an isomorphism and $\phi_*\colon HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{g}')$ is an epimorphism, then ϕ induces isomorphisms $\phi_*\colon \mathfrak{g}/\Gamma_n(\mathfrak{g}) \to \mathfrak{g}'/\Gamma_n(\mathfrak{g}')$, for each $n \geq 1$. Consequently, if $\mathfrak{g}$ and $\mathfrak{g}'$ are nilpotent, then ϕ is an isomorphism.*

Proof. The proof follows [13, VI.Theorem 9.1] mutatis mutandis. □

Given a multiplicative Lie ring $\mathfrak{g}$, let $\Phi_k(\mathfrak{g})$, $k \geq 2$, denote the kernel of the natural map $HS_2^{mlr}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{g}/\Gamma_{k-1}(\mathfrak{g}))$.

Theorem 5.2. *Let $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of multiplicative Lie rings which induces an isomorphism $\phi_*\colon HS_1^{mlr}(\mathfrak{g}) \to HS_1^{mlr}(\mathfrak{g}')$. Then the following three conditions are equivalent:*

(i) ϕ induces an epimorphism $HS_2^{mlr}(\mathfrak{g})/\Phi_k(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{g}')/\Phi_k(\mathfrak{g}')$.

(ii) ϕ induces an isomorphism $\mathfrak{g}/\Gamma_k(\mathfrak{g}) \to \mathfrak{g}'/\Gamma_k(\mathfrak{g}')$.

(iii) ϕ induces an isomorphism $HS_2^{mlr}(\mathfrak{g})/\Phi_k(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{g}')/\Phi_k(\mathfrak{g}')$ and an injection $HS_2^{mlr}(\mathfrak{g})/\Phi_{k+1}(\mathfrak{g}) \to HS_2^{mlr}(\mathfrak{g}')/\Phi_{k+1}(\mathfrak{g}')$.

Proof. The proof follows [9, Theorem 1.1] mutatis mutandis. □

We do not have such a nice relationship between the derived series and homology. For instance, if D_k is the dihedral group of order $2k$, then the natural projection $D_k \to \mathbb{Z}/2\mathbb{Z}$ satisfies the hypothesis of Theorem 5.1 for each odd number $k \geq 3$. But if $n \geq 3$, we do not have an isomorphism between $D_k/\Gamma^{(n)}(D_k)$ and $\mathbb{Z}/2\mathbb{Z}$.

Various analogous of Stallings' theorem and Dwyer's theorem showing the relationships between the derived series and homology are given in [3–6]. In this section our aim is to show that if ϕ satisfies the hypothesis of Theorem 5.1, then ϕ induces an isomorphism $\mathfrak{g}/\big(\{\mathsf{Ker}\,\phi, \mathfrak{g}\}\,\Gamma^{(n)}(\mathfrak{g})\big) \to \mathfrak{g}'/\Gamma^{(n)}(\mathfrak{g}')$ for all $n \geq 1$. This version of "Stallings' theorem" differs from those of [3–6] and we refer it as the new version of Stallings' theorem.

Lemma 5.3. *Let* $1 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\alpha} \mathfrak{g} \to 1$ *and* $1 \to \mathfrak{n} \to \mathfrak{g} \xrightarrow{\beta} \mathfrak{q} \to 1$ *be extensions of multiplicative Lie rings. Then, there is an exact sequence:*

$$\frac{\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{r}\}} \to \frac{\mathfrak{s} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{s},\mathfrak{s}\}} \to \frac{\mathfrak{n}}{\{\mathfrak{n},\mathfrak{n}\}} \to \frac{\mathfrak{g}}{\{\mathfrak{g},\mathfrak{g}\}} \to \frac{\mathfrak{q}}{\{\mathfrak{q},\mathfrak{q}\}} \to 1,$$

where $\mathfrak{s} = \mathsf{Ker}(\beta \circ \alpha)$.

Proof. We have the following exact sequence:

$$1 \to \frac{\mathfrak{n} \cap \{\mathfrak{g},\mathfrak{g}\}}{\{\mathfrak{n},\mathfrak{n}\}} \to \frac{\mathfrak{n}}{\{\mathfrak{n},\mathfrak{n}\}} \to \frac{\mathfrak{g}}{\{\mathfrak{g},\mathfrak{g}\}} \to \frac{\mathfrak{q}}{\{\mathfrak{q},\mathfrak{q}\}} \to 1. \tag{5.1}$$

Since $\mathfrak{s} = \alpha^{-1}(\mathfrak{n})$ and $\mathfrak{r} = \mathsf{Ker}\,\big(\alpha \colon \mathfrak{s} \to \mathfrak{n}\big)$, the following sequence is also exact:

$$\frac{\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{r}\}} \to \frac{\mathfrak{s} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{s},\mathfrak{s}\}} \to \frac{\mathfrak{n} \cap \{\mathfrak{g},\mathfrak{g}\}}{\{\mathfrak{n},\mathfrak{n}\}} \to 1. \tag{5.2}$$

By (5.1) and (5.2) we get the required results. $\square$

Lemma 5.3 immediately implies the following results:

Corollary 5.4. *Let* $1 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\alpha} \mathfrak{g} \to 1$ *and* $1 \to \mathfrak{n} \to \mathfrak{g} \xrightarrow{\beta} \mathfrak{q} \to 1$ *be extensions of multiplicative Lie rings. Suppose that* $\mathfrak{s} = \mathsf{Ker}(\beta \circ \alpha)$ *and that* $\bar{\mathfrak{s}}$ *is an ideal of* $\mathfrak{s} \cap \{\mathfrak{f},\mathfrak{f}\}$. *Then, we have the following exact sequence:*

$$\frac{\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{r}\}} \to \frac{\mathfrak{s} \cap \{\mathfrak{f},\mathfrak{f}\}}{\bar{\mathfrak{s}}\{\mathfrak{s},\mathfrak{s}\}} \to \frac{\mathfrak{n}}{\alpha(\bar{\mathfrak{s}})\{\mathfrak{n},\mathfrak{n}\}} \to \frac{\mathfrak{g}}{\{\mathfrak{g},\mathfrak{g}\}} \to \frac{\mathfrak{q}}{\{\mathfrak{q},\mathfrak{q}\}} \to 1.$$

Corollary 5.5. *Let* $1 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\alpha} \mathfrak{g} \to 1$ *and* $1 \to \mathfrak{n} \to \mathfrak{g} \xrightarrow{\beta} \mathfrak{q} \to 1$ *be extensions of multiplicative Lie rings. Suppose that* $\bar{\mathfrak{r}}$ *is an ideal of* $\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}$. *Then, we have the following exact sequence:*

$$\frac{\mathfrak{r} \cap \{\mathfrak{f},\mathfrak{f}\}}{\bar{\mathfrak{r}}\{\mathfrak{r},\mathfrak{r}\}} \to \frac{\mathfrak{s} \cap \{\mathfrak{f},\mathfrak{f}\}}{\bar{\mathfrak{r}}\{\mathfrak{s},\mathfrak{s}\}} \to \frac{\mathfrak{n}}{\{\mathfrak{n},\mathfrak{n}\}} \to \frac{\mathfrak{g}}{\{\mathfrak{g},\mathfrak{g}\}} \to \frac{\mathfrak{q}}{\{\mathfrak{q},\mathfrak{q}\}} \to 1,$$

where $\mathfrak{s} = \mathsf{Ker}(\beta \circ \alpha)$.

Now we are ready to prove the new version of Stallings' theorem.

Theorem 5.6. *Let* $\phi \colon \mathfrak{g} \to \mathfrak{g}'$ *be a homomorphism of multiplicative Lie rings. Suppose that* $HS_1^{mlr}(\phi)$ *is an isomorphism and that* $HS_2^{mlr}(\phi)$ *is an epimorphism. Then*

(i) ϕ *induces an isomorphism*

$$\frac{\mathfrak{g}}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i)}(\mathfrak{g})} \xrightarrow{\cong} \frac{\mathfrak{g}'}{\Gamma^{(i)}(\mathfrak{g}')}, \qquad i \geq 1;$$

(ii) if both $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are solvable multiplicative Lie rings, then* ϕ *induces an isomorphism*

$$\frac{\mathfrak{g}}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}} \xrightarrow{\cong} \mathfrak{g}'.$$

Moreover, $\{\mathsf{Ker}\,\phi,\mathfrak{g}\} = \{\{\mathsf{Ker}\,\phi,\mathfrak{g}\},\mathfrak{g}\} = \{\{\{\mathsf{Ker}\,\phi,\mathfrak{g}\},\mathfrak{g}\},\mathfrak{g}\} = \cdots$.

Proof.

(i) We will proceed by the induction with respect to i. If $i = 1$, then the proposition is true. Assume that it is true for i. Let $1 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\alpha} \mathfrak{g} \to 1$ be a free presentation of $\mathfrak{g}$. Set $\mathfrak{n} = \{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i)}(\mathfrak{g})$, $\mathfrak{r}' = \mathsf{Ker}(\phi \circ \alpha)$ and $\mathfrak{s} = \mathsf{Ker}(\beta \circ \alpha)$, where β is defined as the natural projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$. By the assumption we have an isomorphism $\mathfrak{g}/\big(\{\mathsf{Ker}\,\phi, \mathfrak{g}\}\,\Gamma^{(i)}(\mathfrak{g})\big) \xrightarrow{\cong} \mathfrak{g}'/\Gamma^{(i)}(\mathfrak{g}')$ induced by ϕ. Therefore, $\mathfrak{r}' \subseteq \mathfrak{s}$. Consider the following commutative diagram:

$$\begin{tikzcd}
\frac{\mathfrak{r}\cap\{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{r}\}} \arrow[r] \arrow[d, "\gamma_1"] & \frac{\mathfrak{s}\cap\{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r}',\mathfrak{f}\}\{\mathfrak{s},\mathfrak{s}\}} \arrow[r] \arrow[d, "\gamma_2"] & \frac{\mathfrak{n}}{\alpha(\{\mathfrak{r}',\mathfrak{f}\})\{\mathfrak{n},\mathfrak{n}\}} \arrow[r] \arrow[d, "\gamma_3"] & HS_1^{mlr}(\mathfrak{g}) \arrow[r] \arrow[d, "\gamma_4"] & HS_1^{mlr}\left(\frac{\mathfrak{g}}{\mathfrak{n}}\right) \arrow[d, "\gamma_5"] \\
\frac{\mathfrak{r}'\cap\{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r}',\mathfrak{f}\}} \arrow[r] & \frac{\mathfrak{s}\cap\{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r}',\mathfrak{f}\}\{\mathfrak{s},\mathfrak{s}\}} \arrow[r] & \frac{\Gamma^{(i)}(\mathfrak{g}')}{\{\Gamma^{(i)}(\mathfrak{g}'),\Gamma^{(i)}(\mathfrak{g}')\}} \arrow[r] & HS_1^{mlr}(\mathfrak{g}') \arrow[r] & HS_1^{mlr}\left(\frac{\mathfrak{g}'}{\Gamma^{(i)}(\mathfrak{g}')}\right),
\end{tikzcd}$$

where $\gamma_1, \dots, \gamma_5$ are defined in a natural way. By Corollary 5.4 and 5.5, the upper and lower rows of this diagram are exact. By the Hopf formula we have:

$$HS_2^{mlr}(\mathfrak{g}) = \frac{\mathfrak{r}\cap\{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r},\mathfrak{f}\}}, \qquad HS_2^{mlr}(\mathfrak{g}') = \frac{\mathfrak{r}'\cap\{\mathfrak{f},\mathfrak{f}\}}{\{\mathfrak{r}',\mathfrak{f}\}}.$$

Therefore, γ_1 is an epimorphism. Moreover, since γ_2, γ_4 and γ_5 are isomorphisms, the above diagram implies that γ_3 is also an isomorphism. But $\alpha(\{\mathfrak{r}',\mathfrak{f}\}) = \{\alpha(\mathfrak{r}'),\alpha(\mathfrak{f})\} = \{\mathsf{Ker}\,\phi,\mathfrak{g}\}$. Thus, we get that

$$\frac{\mathfrak{n}}{\alpha(\{\mathfrak{r}',\mathfrak{f}\})\{\mathfrak{n},\mathfrak{n}\}} = \frac{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i)}(\mathfrak{g})}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\{\Gamma^{(i)}(\mathfrak{g}),\Gamma^{(i)}(\mathfrak{g})\}},$$

and ϕ induces an isomorphism

$$\frac{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\Gamma^{(i)}(\mathfrak{g})}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\{\Gamma^{(i)}(\mathfrak{g}),\Gamma^{(i)}(\mathfrak{g})\}} \xrightarrow[\alpha_3]{\cong} \frac{\Gamma^{(i)}(\mathfrak{g}')}{\{\Gamma^{(i)}(\mathfrak{g}'),\Gamma^{(i)}(\mathfrak{g}')\}}.$$

Now, the following commutative diagram completes the proof:

$$\begin{tikzcd}
1 \arrow[r] & \frac{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i)}(\mathfrak{g})}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i+1)}(\mathfrak{g})} \arrow[r] \arrow[d, "\alpha_3", "\cong"'] & \frac{\mathfrak{g}}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i+1)}(\mathfrak{g})} \arrow[r] \arrow[d] & \frac{\mathfrak{g}}{\{\mathsf{Ker}\,\phi,\mathfrak{g}\}\,\Gamma^{(i)}(\mathfrak{g})} \arrow[r] \arrow[d, "\cong"'] & 1 \\
1 \arrow[r] & \frac{\Gamma^{(i)}(\mathfrak{g}')}{\Gamma^{(i+1)}(\mathfrak{g}')} \arrow[r] & \frac{\mathfrak{g}'}{\Gamma^{(i+1)}(\mathfrak{g}')} \arrow[r] & \frac{\mathfrak{g}'}{\Gamma^{(i)}(\mathfrak{g}')} \arrow[r] & 1.
\end{tikzcd}$$

(ii) Using (i) we obtain an isomorphism $\mathfrak{g}/\{\mathsf{Ker}\,\phi,\mathfrak{g}\} \xrightarrow{\cong} \mathfrak{g}'$ induced by ϕ. Let $\phi_1\colon \mathfrak{g} \to \mathfrak{g}/\{\mathsf{Ker}\,\phi,\mathfrak{g}\}$ be the natural projection. Then $HS_1^{mlr}(\phi_1)$ is an isomorphism and $HS_2^{mlr}(\phi_1)$ is an epimorphism. Therefore, ϕ_1 induces an isomorphism $\mathfrak{g}/\{\mathsf{Ker}\,\phi_1,\mathfrak{g}\} \xrightarrow{\cong} \mathfrak{g}/\{\mathsf{Ker}\,\phi,\mathfrak{g}\}$. This implies that $\{\mathsf{Ker}\,\phi,\mathfrak{g}\} = \{\mathsf{Ker}\,\phi_1,\mathfrak{g}\} = \{\{\mathsf{Ker}\,\phi,\mathfrak{g}\},\mathfrak{g}\}$. Similarly using the natural projection $\phi_2\colon \mathfrak{g} \to \mathfrak{g}/\{\{\mathsf{Ker}\,\phi,\mathfrak{g}\},\mathfrak{g}\}$ one can prove that $\{\{\mathsf{Ker}\,\phi,\mathfrak{g}\},\mathfrak{g}\} = \{\{\{\mathsf{Ker}\,\phi,\mathfrak{g}\},\mathfrak{g}\},\mathfrak{g}\}$, and so on. □

In the next corollary H_1 and H_2 denote the first and second Eilenberg-MacLane homology functors, respectively.

Corollary 5.7. *Let $\phi\colon G \to G'$ be a homomorphism of groups. Suppose that $H_1(\phi)$ is an isomorphism and that $H_2(\phi)$ is an epimorphism. Then*

(i) ϕ induces an isomorphism

$$\frac{G}{[\mathsf{Ker}\,\phi, G]\,\Gamma^{(i)}(G)} \xrightarrow{\cong} \frac{G'}{\Gamma^{(i)}(G')}, \quad i \geq 1;$$

(ii) if both G and G' are solvable groups, then ϕ induces an isomorphism

$$\frac{G}{[\mathsf{Ker}\,\phi, G]} \xrightarrow{\cong} G'.$$

Moreover, $[\mathsf{Ker}\,\phi, G] = [[\mathsf{Ker}\,\phi, G], G] = [[[\mathsf{Ker}\,\phi, G], G], G] = \cdots$.

Proof. This follows from Theorem 5.6 and [1, Proposition 3.9]. □

In the next corollary H_1 and H_2 denote the first and second Chevalley-Eilenberg homology functors, respectively.

Corollary 5.8. *Let* $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ *be a homomorphism of Lie rings. Suppose that* $\mathsf{H}_1(\phi)$ *is an isomorphism and that* $\mathsf{H}_2(\phi)$ *is an epimorphism. Then*

(i) ϕ *induces an isomorphism*

$$\frac{\mathfrak{g}}{[\mathsf{Ker}\,\phi, \mathfrak{g}]\,\Gamma^{(i)}(\mathfrak{g})} \xrightarrow{\cong} \frac{\mathfrak{g}'}{\Gamma^{(i)}(\mathfrak{g}')}, \quad i \geq 1;$$

(ii) if both $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are solvable Lie rings, then* ϕ *induces an isomorphism*

$$\frac{\mathfrak{g}}{[\mathsf{Ker}\,\phi, \mathfrak{g}]} \xrightarrow{\cong} \mathfrak{g}'.$$

Moreover, $[\mathsf{Ker}\,\phi, \mathfrak{g}] = [[\mathsf{Ker}\,\phi, \mathfrak{g}], \mathfrak{g}] = [[[\mathsf{Ker}\,\phi, \mathfrak{g}], \mathfrak{g}], \mathfrak{g}] = \cdots$.

Proof. This follows from Theorem 5.6 because $\mathsf{H}_2(\mathfrak{g}) \cong HS_2^{mlr}(\mathfrak{g})$ (see [1]). □

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