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ON DEGREE OF DERIVED FUNCTORS

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Abstract. We show that the degree of derived functors of a group-valued functor defined on some distinguished category is less than or equal to the degree of the initial functor. Some illustrative examples of distinguished categories and applications to our main result are given.

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1. Introduction

Classical homological algebra studies derived functors of additive, or equivalently linear, functors from an abelian category to another abelian category, which are additive functors as well. Homotopical algebra, or non-linear homological algebra, is the generalization of classical homological algebra to arbitrary categories, which results by considering simplicial objects as being generalization of chain complexes (see [11]). One powerful tool of homotopical algebra is the notion of non-abelian derived functors, construction of which is mostly based on the simplicial constructions and techniques.

In [1], the notion of degree of group valued functors is introduced, modifying the classical notion of cross-effects of functors in the non-abelian framework.

The aim of this short note is to investigate the degree of non-abelian derived functors of a group-valued functor. Namely, our main result has the form:

Main Theorem. *Let $\underline{\mathcal{C}}$ be a distinguished category, i.e. belonging to the full subcategory of the category of categories, $\mathfrak{D}$, having enough projective*

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objects. Let $T : \underline{\mathcal{C}} \to \mathfrak{Gr}$ *be a functor with* $T(0) = 0$*, then we have*

$$\deg\big(\mathcal{L}_n T\big) \le \deg\big(T\big), \qquad n \ge 0.$$

(For the definitions of distinguished category and degree of a functor see Subsections 2.2 and 3.1 respectively).

This result generalizes the classical fact that the derived functors of an additive functor are additive as well (see e.g. [3]).

2. Preliminaries

2.1. **Derived functors.** We begin by recalling some well-known notions on (pseudo)simplicial objects in a category and derived functors (see e.g. [7, 13]). Let $\underline{\mathcal{C}}$ be a category and $(d_0, \dots, d_n) : Y \longrightarrow X$ a sequence of morphisms in $\underline{\mathcal{C}}$, then a *simplicial kernel* of $(d_0, \dots, d_n)$ is a sequence $(k_0, \dots, k_{n+1}) : K \longrightarrow Y$ of morphisms in $\underline{\mathcal{C}}$ satisfying $d_i k_j = d_{j-1} k_i$ for $0 \le i, j \le n+1$ and couniversal with respect to this property. An augmented (pseudo)simplicial object (P_*, d_0^0, X) in $\underline{\mathcal{C}}$ is called *projective* if each P_n is a projective object with respect to regular epimorphisms in $\underline{\mathcal{C}}$ and called *simplicially exact* if each natural morphism from P_{n+1} to the simplicial kernel of $(d_0^n, \dots, d_n^n)$ is a regular epimorphism. (P_*, d_0^0, X) is called *projective resolution* of X if it is projective and simplicially exact. Let $\underline{\mathcal{C}}$ be a finitely complete category having enough projective objects, then every object admits a projective resolution.

Let $\mathfrak{Gr}$ denote the categories of groups. Given a finitely complete category $\underline{\mathcal{C}}$ having enough projective objects and a functor $T : \underline{\mathcal{C}} \to \mathfrak{Gr}$, define the n-th *non-abelian left derived functor* $\mathcal{L}_n T : \underline{\mathcal{C}} \to \mathfrak{Gr}$, $n \ge 0$, of T by choosing for each $X \in \underline{\mathcal{C}}$, a projective resolution (P_*, d_0^0, X) and setting

$$\mathcal{L}_n T\big(X\big) = \pi_n\big(T(P_*)\big).$$

2.2. **Distinguished categories.** Now we distinguish a wide class from the category of categories for our purpose. Given a category $\underline{\mathcal{C}}$, denote by $X \vee Y$ the coproduct of two objects $X, Y \in \underline{\mathcal{C}}$.

Definition 1. Let $\underline{\mathcal{C}}$ be a pointed and finitely complete category, having finite coproducts. It will be called *distinguished* if for any simplicially exact augmented (pseudo)simplicial objects (P_*, d_0^0, X) and (Q_*, d_0^0, Y) in $\underline{\mathcal{C}}$, the augmented (pseudo)simplicial object $(P_* \vee Q_*, d_0^0 \vee d_0^0, X \vee Y)$ is simplicially exact.

Let us denote the category of distinguished categories by $\mathfrak{D}$, which forms a full subcategory of the category of categories.

2.2.1. *Additive category.* A finitely complete additive category belongs to the category $\mathfrak{D}$. In fact, it follows from the facts that products preserve limits and coproducts respect with regular epimorphisms in any category, while finite coproducts coincide with finite products in an additive category.

2.2.2. *The category of groups.* It is well known that in the category $\mathfrak{Gr}$ there exist a zero object (trivial group), finite coproducts and finite limits. The following proposition shows that $\mathfrak{Gr} \in \mathfrak{D}$. It is known as a particular case to [8, Proposition 4.2], but we give our slightly different proof because it uses only purely group-theoretic arguments.

Proposition 2. *Let (F_*, d_0^0, G) and (Q_*, d_0^0, H) be aspherical augmented (pseudo)simplicial groups. Then the augmented (pseudo)simplicial group $(F_* \vee Q_*, d_0^0 \vee d_0^0, G \vee H)$ is aspherical.*

Proof. For any groups G and H there is a canonical homomorphism $\alpha : G \vee H \to G \times H$ and consider the Cartesian subgroup, $G \square H = \mathrm{Ker}(\alpha)$, of $G \vee H$. It is well known [9] that $G \square H$ is a free group generated by the elements $ghg^{-1}h^{-1} \in G \vee H$ with $g \neq 1$ and $h \neq 1$. Hence we have the short exact sequence of augmented (pseudo)simplicial groups

$$0 \longrightarrow \big(F_* \square Q_*, d_0^0 \square d_0^0, G \square H\big) \longrightarrow \big(F_* \vee Q_*, d_0^0 \vee d_0^0, G \vee H\big) \longrightarrow \big(F_* \times Q_*, d_0^0 \times d_0^0, G \times H\big) \longrightarrow 0. \quad (1)$$

It is easy to see that $\big(F_* \times Q_*, d_0^0 \times d_0^0, G \times H\big)$ is aspherical. Then the long exact homotopy sequence, induced by (1), implies the natural isomorphism of groups

$$\pi_i\big(F_* \square Q_*, d_0^0 \square d_0^0, G \square H\big) \cong \pi_i\big(F_* \vee Q_*, d_0^0 \vee d_0^0, G \vee H\big), \quad i \geq -1. \quad (2)$$

Moreover, since (F_*, d_0^0, G) and (Q_*, d_0^0, H) are aspherical augmented (pseudo)simplicial groups, they admit set-theoretic left (right) contractions t_* and t'_*, respectively, preserving the identity elements. Clearly $t_* \times t'_*$ is a set-theoretic left (right) contraction of the augmented (pseudo)simplicial group $\big(F_* \times Q_*, d_0^0 \times d_0^0, G \times H\big)$ preserving the identity elements. Hence, it is easy to see that $\big(F_* \square Q_*, d_0^0 \square d_0^0, G \square H\big)$ is a contractible and consequently aspherical augmented (pseudo)simplicial group. Now the result follows from (2). □

2.2.3. *The category of crossed modules.* We recall the following algebraic concept of Whitehead [14]. A *crossed module* (M, P, μ) is a group homomorphism $\mu : M \to P$ together with an action of P on M, satisfying the following two conditions:

$$\mu({}^p m) = p\mu(m)p^{-1},$$
$${}^{\mu(m)}m' = mm'm^{-1}$$

for $m, m' \in M$, $p \in P$. A morphism $(\varphi, \psi) : (M, P, \mu) \to (N, Q, \nu)$ of crossed modules is a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mu} & P \\ \varphi \downarrow & & \downarrow \psi \\ N & \xrightarrow{\nu} & Q \end{array},$$

with $\varphi({}^pm) = {}^{\psi(p)}\varphi(m)$ for all $m \in M$, $p \in P$. Let us denote the category of crossed modules of groups by $\mathfrak{Xmod}$. It is easy to see that the trivial crossed module $0 : 0 \to 0$ is the zero object in $\mathfrak{Xmod}$. Moreover, for any two crossed modules of groups $\mu : M \to P$ and $\mu' : M' \to P'$ there exists their coproduct in $\mathfrak{Xmod}$, which has the form $\mu \vee \mu' : M \vee M' \to P \vee P'$. Clearly, one can similarly show the existence of finite limits in the category $\mathfrak{Xmod}$ (see also [2]).

Corollary 3. *Let* $\big((F_*, Q_*, \mu_*), (d_0^0, d_0^0), (M, P, \mu)\big)$ *and* $\big((F'_*, Q'_*, \mu'_*), (d_0^0, d_0^0), (M', P', \mu)\big)$ *be simplicially exact augmented (pseudo)simplicial crossed modules of groups. Then the augmented (pseudo)simplicial crossed module* $\big((F_* \vee F'_*, Q_* \vee Q'_*, \mu_* \vee \mu'_*), (d_0^0, d_0^0), (M \vee M', P \vee P', \mu \vee \mu')\big)$ *is simplicially exact.*

Proof. Using the fact that regular epimorphisms in the category $\mathfrak{Xmod}$ are just those morphisms $(\varphi, \psi) : (M, P, \mu) \to (N, Q, \nu)$ in $\mathfrak{Xmod}$ such that both φ and ψ are onto maps [2], the assertion directly follows from Proposition 2. □

We have just shown that $\mathfrak{Xmod}$ belongs to the category $\mathfrak{D}$.

2.2.4. *The category of associative algebras.* Now consider the category of associative (not-necessarily unital) algebras over a field k denoted by $\mathfrak{Alg}$. It is well known that $\mathfrak{Alg}$ is finitely complete, having a zero object (zero algebra) and finite coproducts. The following assertion shows that $\mathfrak{Alg}$ belongs to the category $\mathfrak{D}$.

Proposition 4. *Let* (P_*, d_0^0, P_{-1}) *and* (Q_*, d_0^0, Q_{-1}) *be simplicially exact augmented (pseudo) simplicial associative algebras. Then the augmented (pseudo) simplicial associative algebra* $(P_* \vee Q_*, d_0^0 \vee d_0^0, P_{-1} \vee Q_{-1})$ *is simplicially exact.*

Proof. We only have to show that the augmented (pseudo)simplicial associative algebra $(P_* \vee Q_*, d_0^0 \vee d_0^0, P_{-1} \vee Q_{-1})$ is aspherical.

For any $n \geq -1$, consider the following presentation of the coproduct of objects P_n and Q_n in the category $\mathfrak{Alg}$:

$$P_n \vee Q_n = \sum_{i \geq 1} \Big(\mathcal{A}_i \oplus \mathcal{B}_i \Big) ,$$

where k-modules $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined inductively by the formulas

$$\begin{aligned} &\mathcal{A}_i = \mathcal{A}_{i-1} \otimes_k Q_n \quad \text{and} \quad \mathcal{B}_i = \mathcal{B}_{i-1} \otimes_k P_n \quad \text{for even} \quad i, \\ &\mathcal{A}_i = \mathcal{A}_{i-1} \otimes_k P_n \quad \text{and} \quad \mathcal{B}_i = \mathcal{B}_{i-1} \otimes_k Q_n \quad \text{for odd} \quad i > 1, \end{aligned}$$

with $\mathcal{A}_1 = P_n$ and $\mathcal{B}_1 = Q_n$. The multiplication in $P_n \vee Q_n$ is defined in the obvious way.

Using Quillen's spectral sequence argument [12] (cf. [7]), one easily shows that the augmented (pseudo) simplicial modules $\mathcal{A}_i$ and $\mathcal{B}_i$, $i \geq 2$, are homologically trivial.

Now assertion follows from the fact that any direct sum of homologically trivial augmented (pseudo)simplicial k-modules is homologically trivial. □

2.2.5. *The category of commutative algebras.* Let $\mathfrak{ComAlg}$ denote the category of commutative (not-necessarily unital) algebras over a field $\mathbf{k}$. It is also well known that there exist a zero object, finite coproducts and finite limits in the category $\mathfrak{ComAlg}$. Similarly to the (previous) case 2.2.4, one can show that the category $\mathfrak{ComAlg}$ belongs to $\mathfrak{D}$.

3. Cross-effects of Functors

3.1. **Degree of functors.** We recall the notions of the cross-effect and degree of a group-valued functor from [1, 10], generalizing the classical notions of MacLane. Let $\underline{\mathcal{C}}$ be a category with a zero object and finite coproducts. Let $T : \underline{\mathcal{C}} \longrightarrow \mathfrak{Gr}$ be a functor with $T(0) = 0$ and $q \geq 2$. Then, for objects $X_1, \ldots, X_q \in \underline{\mathcal{C}}$, the *q-th cross-effect* of T is denoted by $T(X_1|\ldots|X_q)$ and defined by the equality

$$T\big(X_1|\cdots|X_q\big) = \operatorname{Ker}\Big(T\big(X_1 \vee \cdots \vee X_q\big) \xrightarrow{\big(T(s_1),\ldots,T(s_q)\big)} \prod_{j=1}^{q} T\big(X_1 \vee \cdots \vee \hat{X}_j \vee \cdots \vee X_q\big)\Big)\ ,$$

where the retractions s_j are given by

$$s_j = \big(1, \ldots, 1, \underset{j}{0}, 1, \ldots, 1\big) : X_1 \vee \cdots \vee X_q \to X_1 \vee \cdots \vee \hat{X}_j \vee \cdots \vee X_q$$

for $1 \leq j \leq q$. Moreover, T is called a functor *of degree* q if the $(q+1)$-st cross-effect $T(X_1|\cdots|X_{q+1})$ vanishes for all $X_1, \ldots, X_{q+1} \in \underline{\mathcal{C}}$ and the q-th cross-effect $T(Y_1|\cdots|Y_q)$ is non-zero for some $Y_1, \ldots, Y_q \in \underline{\mathcal{C}}$. In this case we write $\deg(T) = q$.

Lemma 5. *Let $T : \underline{\mathcal{C}} \longrightarrow \mathfrak{Gr}$ be a functor with $T(0) = 0$, $q \geq 2$ and $X_1, \ldots, X_q \in \underline{\mathcal{C}}$. Then the q-th cross-effect $T(X_1|\cdots|X_q)$ is described as follows:*

$$T\big(X_1|\cdots|X_q\big) = \operatorname{Ker}\Big(T\big(X_1|\cdots|X_{q-2}|X_{q-1} \vee X_q\big) \longrightarrow T\big(X_1|\cdots|X_{q-2}|X_{q-1}\big) \times T\big(X_1|\cdots|X_{q-2}|X_q\big)\Big).$$

Proof. By definition

$$T(X_1|\cdots|X_{q-2}|X_{q-1}\vee X_q)=\operatorname{Ker}\Big(T(X_1\vee\cdots\vee X_{q-2}\vee X_{q-1}\vee X_q)\longrightarrow \prod_{j=1}^{q-2}T(X_1\vee\cdots\vee\hat{X}_j\vee\cdots\vee X_{q-2}\vee X_{q-1}\vee X_q)\times T(X_1\vee\cdots\vee X_{q-2})\Big),$$

which clearly implies the result. □

3.2. **Relation between Cross-efects and homotopy.** Let $\underline{\mathcal{SC}}$ denote the category of (pseudo) simplicial objects in $\underline{\mathcal{C}}$. It is easy to see that $\underline{\mathcal{SC}}$ has a zero object, which is the constant (pseudo)simplicial object whose value is 0. Moreover, $\underline{\mathcal{SC}}$ has finite coproducts, which are dimension-wise finite coproducts of (pseudo)simplicial objects in $\underline{\mathcal{C}}$. Given a functor $T:\underline{\mathcal{C}}\to\mathfrak{Gr}$ with $T(0)=0$ and $i\geq 0$, we define the functor

$$T_i:\underline{\mathcal{SC}}\longrightarrow\mathfrak{Gr}\quad\text{by}\quad T_i(X_*)=\pi_i\big(T(X_*)\big).$$

Clearly $T_i(0)=0$. On the other hand, for any (pseudo)simplicial objects $X_*^1,\dots,X_*^q\in\underline{\mathcal{SC}}$, let $T(X_*^1|\cdots|X_*^q)$ denote the (pseudo)simplicial group obtained by applying the functor $T(\overbrace{-|\cdots|-}^{q-times})$ dimension-wise to the (pseudo)simplicial n-tuple $(X_*^1,\dots,X_*^q)$.

Proposition 6. *Let $T:\underline{\mathcal{C}}\longrightarrow\mathfrak{Gr}$ be a functor with $T(0)=0$, $q\geq 2$ and $X_*^1,\dots,X_*^q\in\underline{\mathcal{SC}}$. Then there is a natural isomorphism*

$$T_i\big(X_*^1|\cdots|X_*^q\big)\cong\pi_i\big(T(X_*^1|\cdots|X_*^q)\big),\qquad i\geq 0.$$

Proof. We use induction on q. Let $q=2$ and $X_*^1,X_*^2\in\underline{\mathcal{SC}}$, then there is a short exact sequence of (pseudo)simplicial groups

$$0\longrightarrow T\big(X_*^1|X_*^2\big)\longrightarrow T\big(X_*^1\vee X_*^2\big)\xrightarrow{\big(T(s_{1*}),T(s_{2*})\big)} T\big(X_*^1\big)\times T\big(X_*^2\big)\longrightarrow 0.$$

It is easy to see that the (pseudo)simplicial group homomorphism $\big(T(s_{1*}),T(s_{2*})\big)$ has a set-theoretic (pseudo)simplicial section $T(\iota_{1*})\times T(\iota_{2*})$, where $T(\iota_{1*})$ and $T(\iota_{2*})$ are (pseudo)simplicial group homomorphisms induced by the structural simplicial morphisms ι_{1*} and ι_{2*}, respectively. Hence the induced long exact homotopy sequence implies the required isomorphism

$$\pi_i\big(T(X_*^1|X_*^2)\big)\cong\operatorname{Ker}\Big(T_i\big(X_*^1\vee X_*^2\big)\xrightarrow{\big(T_i(s_{1*}),T_i(s_{2*})\big)} T_i\big(X_*^1\big)\times T_i\big(X_*^2\big)\Big),\quad i\geq 0.$$

Proceeding by induction, we suppose that the assertion is true for $q-1$ and we will prove it for q.

Let $X^1_*, \dots, X^q_* \in \underline{\mathcal{SC}}$, then by Lemma 5 we have a short exact sequence of (pseudo)simplicial groups

$$0 \longrightarrow T\big(X^1_*|\cdots|X^{q-1}_*|X^q_*\big) \longrightarrow T\big(X^1_*|\cdots|X^{q-2}_*|X^{q-1}_* \vee X^q_*\big) \xrightarrow{\big(T(s'_{1*}),T(s'_{2*})\big)} T\big(X^1_*|\cdots|X^{q-2}_*|X^{q-1}_*\big) \times T\big(X^1_*|\cdots|X^{q-2}_*|X^q_*\big) \longrightarrow 0\,.$$

Similarly as above, the (pseudo)simplicial group homomorphism $\big(T(s'_{1*}),T(s'_{2*})\big)$ has a set-theoretic (pseudo)simplicial section $T(\iota'_{1*})\times T(\iota'_{2*})$, where $T(\iota'_{1*})$ and $T(\iota'_{2*})$ are (pseudo)simplicial group homomorphisms. Hence, using the inductive hypothesis, the induced long exact homotopy sequence implies the isomorphism

$$\pi_i\big(T(X^1_*|\cdots|X^{q-1}_*|X^q_*)\big) \cong \operatorname{Ker}\Big(T_i(X^1_*|\cdots|X^{q-2}_*|X^{q-1}_* \vee X^q_*) \to \to T_i(X^1_*|\cdots|X^{q-2}_*|X^{q-1}_*) \times T_i(X^1_*|\cdots|X^{q-2}_*|X^q_*)\Big),\ i \geq 0.$$

Now the result follows from Lemma 5. □

4. Proof of Main Theorem and Applications

4.1. **Proof of main theorem.** There is noting to prove if $\deg(T) = \infty$, so assume that $\deg(T) = q$ with $1 \leq q < \infty$. Hence we have to show that $\mathcal{L}_i T(X_1|\cdots|X_{q+1}) = 0$ for all $X_1, \dots, X_{q+1} \in \underline{\mathcal{C}}$.

In fact, let (P^i_*, d^0_0, X_i) be a projective resolution of X_i for $1 \leq i \leq q+1$ in the category $\underline{\mathcal{C}}$. Since finite coproducts of projective objects are again projective objects in $\underline{\mathcal{C}}$, the augmented (pseudo)simplicial object $(P^1_* \vee \cdots \vee P^{q+1}_*, d^0_0 \vee \cdots \vee d^0_0, X_1 \vee \cdots \vee X_{q+1})$ is projective. Since $\underline{\mathcal{C}} \in \mathfrak{D}$, the augmented (pseudo)simplicial object $(P^1_* \vee \cdots \vee P^{q+1}_*, d^0_0 \vee \cdots \vee d^0_0, X_1 \vee \cdots \vee X_{q+1})$ is simplicially exact.

Hence, evaluating the non-abelian derived functors of T for $X_1 \vee \cdots \vee X_{q+1}$, we have to calculate the homotopy groups of the (pseudo)simplicial group $T(P^1_* \vee \cdots \vee P^{q+1}_*)$. Therefore, by Proposition 6, we have

$$\mathcal{L}_n T\big(X_1|\cdots|X_{q+1}\big) = T_n\big(P^1_*|\cdots|P^{q+1}_*\big) \cong \pi_n\big(T(P^1_*|\cdots|P^{q+1}_*)\big), \quad n \geq 0. \tag{3}$$

Since $\deg(T) = q$, the (pseudo)simplicial group $T(P^1_*|\cdots|P^{q+1}_*)$ vanishes. Clearly the result follows from (3). □

4.2. **Some applications.** As an illustration of our result we calculate the degrees of the non-abelian derived functors of some known functors, in particular, the 'nilization of degree k' functors studied in [5] and CCG-homology of crossed modules of groups introduced and investigated in [2].

Given a category $\underline{\mathcal{C}}$, denote the category of its abelian group objects by $\mathfrak{Ab}\underline{\mathcal{C}}$. It is often the case that the obvious forgetful functor $\mathfrak{Ab}\underline{\mathcal{C}} \to \underline{\mathcal{C}}$ admits a left adjoint $\mathrm{Ab} : \underline{\mathcal{C}} \to \mathfrak{Ab}\underline{\mathcal{C}}$, called the *abelianization functor*, and that $\mathfrak{Ab}\underline{\mathcal{C}}$ is an abelian category. Hence the abelianization carries finite coproducts to products, i.e. it has degree one (see more general notions of the cross-effect and degree of a functor in [6]). Then using Main Theorem, the n-th (Quillen) homology functor from $\underline{\mathcal{C}} \in \mathfrak{D}$ to $\mathfrak{Ab}\underline{\mathcal{C}}$, defined as the n-th derived functor of Ab [11, 12] (see also [4]), has degree one as well. From this discussion and that of given in 2.2.3 we have the following assertion as a particular case.

Proposition 7. *Let* $\mathcal{H}_n^{\mathrm{CCG}} : \mathfrak{Xmod} \longrightarrow \mathfrak{Ab}\,\mathfrak{Xmod} \subset \mathfrak{Xmod}$, $n \geq 1$, *denote* n*-th CCG-homology functor defined in* [2] *by the formula*

$$\mathcal{H}_n^{\mathrm{CCG}}(M,P,\mu) = \pi_{n-1}\Big(\,\mathrm{Ab}\,\big((M,P,\mu)_*\big)\Big),$$

where $(M,P,\mu)_*$ *is a projective resolution of* (M,P,μ) *in* $\mathfrak{Xmod}$, $\mathrm{Ab}(M,P,\mu) = (M/[P,M], P/[P,P], \mu)$ *and* $[P,M]$ *is a subgroup of* M *generated by the elements* ${}^{p}mm^{-1}$, $m \in M$, $p \in P$. *Then*

$$\deg\big(\mathcal{H}_n^{\mathrm{CCG}}\big) = 1.$$

Now consider the 'nilization of degree k' functor $Z_k : \mathfrak{Gr} \to \mathfrak{Gr}$, $k \geq 2$, which is defined as follows: for a group G, let $Z_k(G) = G/\gamma_k(G)$; for a group homomorphism α, let $Z_k(\alpha)$ be the natural homomorphism induced by α. Here $\gamma_k = \gamma_k(G)$ is the (lower) central series for the group G,

$$G = \gamma_1 \supseteq \gamma_2 \supseteq \cdots \supseteq \gamma_k \supseteq \cdots$$

defined inductively by $\gamma_k = [G, \gamma_{k-1}]$. The non-abelian derived functors of Z_k functors are studied in [5]. We have the following.

Proposition 8. *There holds the inequality*

$$\deg\big(\mathcal{L}_n Z_k\big) \leq k-1 \quad \textit{for} \quad k \geq 2 \quad \textit{and} \quad n \geq 0.$$

Proof. We use the same arguments as in the proof of [10, 3.3 Lemma]. In fact, for any group G there is an epimorphism

$$(G^{ab})^{\otimes k} \longrightarrow \gamma_k(G)/\gamma_{k+1}(G).$$

Since the group abelianization is a degree one functor, it is easy to check that $\deg((-^{ab})^{\otimes k}) \leq k$ and consequently $\deg(\gamma_k/\gamma_{k+1}) \leq k$. Hence, using the short exact sequences

$$1 \longrightarrow \gamma_k(G)/\gamma_{k+1}(G) \longrightarrow G/\gamma_{k+1}(G) \longrightarrow G/\gamma_k(G) \longrightarrow 1, \quad k \geq 2,$$

we show inductively that $\deg\big(Z_k\big) \leq k-1$, $k \geq 2$. Now Main Theorem gives the desired result. □

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References

1. H. J. Baues and T. Pirashvili, Quadratic endofunctors of the category of groups. *Adv. Math.* **141** (1999), No. 1, 167–206.
2. P. Carrasco, A. M. Cegarra and A. R. Grandjeán, (Co)homology of crossed modules. Category theory 1999 (Coimbra). *J. Pure Appl. Algebra* **168** (2002), No. 2–3, 147–176.
3. H. Cartan and S. Eilenberg, Homological algebra. *Princeton University Press, Princeton, New Jersey*, 1956,
4. J. M. Casas, G. Ellis, M. Ladra and T. Pirashvili, Derived functors and homology of n-types. *J. Algebra* **256** (2002), No. 2, 583–598.
5. G. Donadze, N. Inassaridze and T. Porter, N-fold Čech derived functors and generalised hopf type formulas. *K-Theory* **35** (2005), No. 3–4, 341–373.
6. M. Hartl and B. Loiseau, Internal object actions in homological categories. *arXiv:1003.0096*, 2010.
7. H. Inassaridze, Non-abelian homological algebra and its applications. *Mathematics and its Applications*, 421, *Kluwer Academic Publishers, Dordrecht*, 1997.
8. D. M. Kan and W. P. Thurston, Every connected space has the homology of a $K(\pi, 1)$. *Topology* **15** (1976), No. 3, 253–258.
9. F. W. Levi, The commutator group of a free product. *J. Indian Math. Soc.* (*N. S.*), **4** (1940), 136–144.
10. T. Pirashvili, Simplicial degrees of functors. *Math. Proc. Cambridge Philos. Soc.* **126** (1999), No. 1, 45–62.
11. D. Quillen, Homotopical Algebra. *Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York*, **43** (1967).
12. D. Quillen, Spectral sequences of a double semi-simplicial group. *Topology* **5** (1966), 155–157.
13. M. Tierney and W. Vogel, Simplicial resolutions and derived functors. *Math. Z.* **111** (1969), 1–14.
14. J. H. C. Whitehead, Combinatorial homotopy II. *Bull. Amer. Math. Soc.* **55** (1949), 453–496.

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