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LOCALISATION AND COLOCALISATION OF TRIANGULATED CATEGORIES AND EQUIVARIANT KK-THEORY

This article is a short review of the results of papers [2] and [3]. Given a thick subcategory of a triangulated category, we define a localisation and a colocalisation as kinds of left Kan extensions. We construct a natural long exact sequence that involves a homological functor and its localisation and colocalisation functors with respect to a thick subcategory [2]. Given a set of prime numbers S, we localize equivariant bivariant Kasparov KK-theory at S and compare this localisation with Kasparov KK-theory by an exact sequence. We study the properties of the resulting variants of Kasparov KK-theory and consequences [3].

1. Left Kan Extension and Localisation and Colocalisation of Functors

The main notions in [2] and [3] are the localisation and colocalisation of functors. In this section, we interpret them as a left Kan extensions. Namely, let $\alpha\colon A \to B$ and $\beta\colon A \to C$ be functors. The *right localisation* of the functor α along the functor β is $\mathbb{R}\alpha = \kappa(\alpha) \cdot \beta$, where the functor $\kappa(\alpha)\colon C \to B$ is the left Kan extension of α along β. Now, let $\gamma\colon X \to Y$ and $\delta\colon Y \to Z$ be functors, too. The *right colocalisation* of the functor δ along the functor γ is the left Kan extension of the composition functor $\delta\gamma$ along the functor γ.

Let $\mathcal{T}$ be a triangulated category, $\mathcal{E}$ a thick subcategory. Let $\varepsilon\colon \mathcal{E} \hookrightarrow \mathcal{T}$ and $\chi\colon \mathcal{T} \to \mathcal{T}/\mathcal{E}$ be the canonical triangulated functors, where $\mathcal{T}/\mathcal{E}$ is the Verdier quotient.

The *cone* of a morphism $f\colon A \to B$ in $\mathcal{T}$ is the object C in an exact triangle $A \xrightarrow{f} B \to C \to A[1]$. A morphism f in $\mathcal{T}$ is an *$\mathcal{E}$-weak equivalence* if its cone belongs to $\mathcal{E}$. Let $\mathrm{we}_{\mathcal{E}}$ be the category of $\mathcal{E}$-weak equivalences. For a fixed object $B \in\in \mathcal{T}$, we consider the category $B \downarrow \mathrm{we}_{\mathcal{E}}$ whose objects are arrows $B \to C$ in $\mathrm{we}_{\mathcal{E}}$.

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Let $\mathcal{C}$ be an Abelian category and $F\colon \mathcal{T} \to \mathcal{C}$ be a functor. Denote by $\mathbb{R}F$ and $\mathbb{R}^{\perp}F$ the *right localisation* and *colocalisation* of F at $\mathcal{E}$, respectively, where localisation and colocalisation are considered along χ and ε, respectively. We get the following interpretation:

$$\mathbb{R}F(B) \simeq \varinjlim_{(s\colon B\to C)\in\in B\downarrow \mathrm{we}_{\mathcal{E}}} F(C).$$

Let $\mathcal{E} \downarrow B$ be the category, whose objects are arrows $f\colon E \to B$ with $E \in\in \mathcal{E}$. The *right colocalisation* at $\mathcal{E}$ is

$$\mathbb{R}^{\perp}F(B) = \varinjlim_{(s\colon C\to B)\in\in \mathcal{E}\downarrow B} F(C).$$

2. The Properties of Localisation and Colocalisation

The above-given definition of localisation and colocalisation does not use the triangulated category structure. However, if $\mathcal{T}$ is a triangulated category, $\mathcal{E}$ a thick subcategory, and $F\colon \mathcal{T} \to \mathfrak{Ab}$ a homological functor, then the right localisation $\mathbb{R}F\colon \mathcal{T} \to \mathfrak{Ab}$ and the right colocalisation $\mathbb{R}^{\perp}F\colon \mathcal{T} \to \mathfrak{Ab}$ are homological [2].

Let $F\colon \mathcal{T} \to \mathfrak{Ab}$ be a homological functor. The following assertions are equivalent:

(1) the natural transformation $F \Rightarrow \mathbb{R}F$ is invertible;
(2) $F(E) \cong 0$ for all $E \in\in \mathcal{E}$;
(3) $F(s)$ is invertible for all $s \in \mathrm{we}_{\mathcal{E}}$;
(4) F factors through a homological functor $\mathcal{T}/\mathcal{E} \to \mathfrak{Ab}$.

Furthermore, $\mathbb{R}F$ always satisfies these equivalent conditions.

A homological functor with the above equivalent properties is called *local*. Condition (4) means that local homological functors $\mathcal{T} \to \mathfrak{Ab}$ are equivalent to homological functors $\mathcal{T}/\mathcal{E} \to \mathfrak{Ab}$. The localisation $\mathbb{R}F$ is the universal local homological functor on $\mathcal{T}$ equipped with a natural transformation $F \Rightarrow \mathbb{R}F$: if G is any local homological functor on $\mathcal{T}$, then there is a natural bijection between natural transformations $F \Rightarrow G$ and natural transformations $\mathbb{R}F \Rightarrow G$. This universal property characterizes $\mathbb{R}F$ uniquely up to natural isomorphism [2].

We call a homological functor $F\colon \mathcal{T} \to \mathfrak{Ab}$ *colocal* if the natural transformation $\mathbb{R}^{\perp}F \to F$ is invertible. Let $F\colon \mathcal{E} \to \mathfrak{Ab}$ be a homological functor. Then there is a unique colocal homological functor $\bar{F}\colon \mathcal{T} \to \mathfrak{Ab}$ that extends F. Thus, colocal homological functors $\mathcal{T} \to \mathfrak{Ab}$ are essentially equivalent to homological functors $\mathcal{E} \to \mathfrak{Ab}$. Furthermore, $\mathbb{R}^{\perp}G$ is colocal for any homological functor $G\colon \mathcal{T} \to \mathfrak{Ab}$. The natural transformation $\mathbb{R}^{\perp}G \Rightarrow G$ is universal among natural transformations from colocal functors to G.

Theorem 2.1. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{E}$ a thick subcategory. Let $F\colon \mathcal{T} \to \mathfrak{Ab}$ be a homological functor to the category of Abelian groups. Then there is a natural exact sequence*

$$\cdots \to \mathbb{R}^{\perp}F_1(B) \to F_1(B) \to \mathbb{R}F_1(B) \to \mathbb{R}^{\perp}F_0(B) \to F_0(B) \to \mathbb{R}F_0(B) \to \cdots$$

This is the main exact sequence promised in the introduction. This is Theorem 4.2, the main result in [2]. In addition to the filtered categories $B \downarrow \mathrm{we}_{\mathcal{E}}$ and $\mathcal{E} \downarrow B$, to relate the localisation and colocalisation of $\mathcal{T}$ at $\mathcal{E}$, we introduce a third filtered category $\triangle_{\mathcal{E}} B$ that combines $B \downarrow \mathrm{we}_{\mathcal{E}}$ and $\mathcal{E} \downarrow B$ for an object B of $\mathcal{T}$. Objects of $\triangle_{\mathcal{E}} B$ are exact triangles of the form

$$E \to B \xrightarrow{s} C \to E[1]$$

with $E \in\in \mathcal{E}$ or, equivalently, $s \in \mathrm{we}_{\mathcal{E}}$; arrows in $\triangle_{\mathcal{E}} B$ are morphisms of triangles of the form

$$\begin{array}{ccccccc} E & \longrightarrow & B & \xrightarrow{s} & C & \longrightarrow & E[1] \\ \downarrow & & \| & & \downarrow & & \downarrow \\ E' & \longrightarrow & B & \xrightarrow{s'} & C' & \longrightarrow & E'[1]. \end{array}$$

There are obvious forgetful functors from $\triangle_{\mathcal{E}} B$ to $B \downarrow \mathrm{we}_{\mathcal{E}}$ and $\mathcal{E} \downarrow B$ that extract the map $B \to C$ or the map $E \to B$, respectively. Since $s \in \mathrm{we}_{\mathcal{E}}$ if and only if $E \in\in \mathcal{E}$, any object of $B \downarrow \mathrm{we}_{\mathcal{E}}$ or $\mathcal{E} \downarrow B$ is in the range of this forgetful functor [2].

3. Central Localisation and Colocalisation

Let R be a commutative unital ring and let S be a multiplicatively closed subset of R. Let $S^{-1}R$ denote the localisation of R at S. This is a unital ring equipped with a natural unital ring homomorphism $i_S \colon R \to S^{-1}R$. Let $\mathcal{T}$ be an R-linear triangulated category, that is, each morphism space in $\mathcal{T}$ is an R-module and composition of morphisms is R-linear. Let $S^{-1}\mathcal{T}$ be an $S^{-1}R$-linear additive category with morphism spaces

$$S^{-1}\mathcal{T}(A, B) = \mathcal{T}(A, B) \otimes_R S^{-1}R$$

and the obvious composition. The natural map $i_S \colon R \to S^{-1}R$ induces an R-linear functor $\mathcal{T} \to S^{-1}\mathcal{T}$.

An object A of $\mathcal{T}$ is called *S-finite* if $s \cdot \mathrm{id}_A = 0$ for some $s \in S$.

The category $S^{-1}\mathcal{T}$ together with the functor $\mathcal{T} \to S^{-1}\mathcal{T}$ is the localisation of $\mathcal{T}$ at the thick subcategory $\mathcal{N}_S$ of finite objects. Let $F \colon \mathcal{T} \to \mathfrak{Ab}$ be a homological functor. The functor

$$S^{-1}F \colon \mathcal{T} \to \mathfrak{Ab}, \qquad S^{-1}F(A) = F(A) \otimes_R S^{-1}R.$$

is the localisation of F with respect to the thick subcategory of S-finite objects [3].

The groups $\mathcal{T}(A, B; S^{-1}R/R)$ behave like the morphism spaces in a triangulated category, except that they lack unit morphisms.

If $F\colon \mathcal{T} \to \mathfrak{Ab}$ is a homological functor, then the map $F \to S^{-1}F$ embeds in an exact sequence

$$\begin{aligned} \cdots \to F_1(A) \to S^{-1}F_1(A) \to F_1(A; S^{-1}R/R) \\ \to F_0(A) \to S^{-1}F_0(A) \to F_0(A; S^{-1}R/R) \\ \to F_{-1}(A) \to S^{-1}F_{-1}(A) \to \cdots \end{aligned} \tag{3.1}$$

with $F_{n+1}(A; S^{-1}R/R) = \mathbb{R}^{\perp}_n F(A)$ in the notation of [3].

The definition in [2] is not useful to actually compute $\mathcal{T}(A, B; S^{-1}R/R)$. To address this problem, recall that

$$S^{-1}R/R \cong \varinjlim x^{-1}R/R = \varinjlim R/(x),$$

where $(x) = x \cdot R \cong R$ is the principal ideal generated by x. Hence we expect that $\mathcal{T}(A, B; S^{-1}R/R)$ is a colimit of theories $\mathcal{T}(A, B; s)$ "with finite coefficients." Namely, for an object A of $\mathcal{T}$ and a homological functor $F\colon \mathcal{T} \to \mathfrak{Ab}$, we have $\operatorname{Tor}^R_n(F_*(A); S^{-1}R/R) = 0$ for all $n \geq 2$, and there is a natural group extension

$$\operatorname{Tor}^R_0(F_0(A); S^{-1}R/R) \rightarrowtail F_0(A; S^{-1}R/R) \twoheadrightarrow \operatorname{Tor}^R_1(F_{-1}(A); S^{-1}R/R).$$

which is a direct limit, when $s \in S$, of

$$\operatorname{coker}\bigl(s\colon F_0(A) \to F_0(A)\bigr) \rightarrowtail F_0(A; s) \twoheadrightarrow \ker\bigl(s\colon F_{-1}(A) \to F_{-1}(A)\bigr). \tag{3.2}$$

4. Application to Kasparov KK^G-theory

4.1. **Rational KK^G-theory.** Now we apply the general theory developed above to equivariant Kasparov KK^G-theory, viewed as a triangulated category. We only consider central localisations where R is the ring $\mathbb{Z}$ of integers. Finer information may be obtained by considering the larger ring $\operatorname{Rep}(G)$ instead, but we leave this to future investigation.

Let us first consider the *rational* KK^G*-theory*. Here $S = \mathbb{Z} \setminus \{0\}$ and $S^{-1}\mathbb{Z} = \mathbb{Q}$. Following the definitions above, we let

$$\mathrm{KK}^G_n(A, B; \mathbb{Q}) = \mathrm{KK}^G_n(A, B) \otimes \mathbb{Q}, \tag{4.1}$$

where A and B are G-C^*-algebras.

This differs from the definition in Exercise 23.15.6 of [1], where rational KK^G-theory for complex C^*-algebras is defined as $\mathrm{KK}^G_n(A, B \otimes D_{\mathbb{Q}})$ for an C^*-algebra $D_{\mathbb{Q}}$ in the bootstrap class with $\mathrm{K}_0(D_{\mathbb{Q}}) = \mathbb{Q}$ and $\mathrm{K}_1(D_{\mathbb{Q}}) = 0$. The definition of $\mathrm{KK}^G(A, B; \mathbb{Q})$ above yields again a triangulated category. This is crucial to apply methods from the stable homotopy theory and homological algebra.

For A in the bootstrap class, the Universal Coefficient Theorem yields

$$\mathrm{KK}_0(A, B \otimes D_{\mathbb{Q}}) \cong \operatorname{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B) \otimes \mathbb{Q}) \cong \operatorname{Hom}_{\mathbb{Q}}(\mathrm{K}_*(A) \otimes \mathbb{Q}, \mathrm{K}_*(B) \otimes \mathbb{Q})$$

because Abelian groups of the form $\mathrm{K}_*(B) \otimes \mathbb{Q}$ are injective. Hence the bootstrap class with these morphisms is equivalent to the category of countable $\mathbb{Q}$-vector spaces. This category is triangulated and Abelian at the same time. And we may also view it as the localisation of $\mathcal{KK}$ at the class of C^*-algebras with vanishing rational K-theory $\mathrm{K}_*(\llcorner) \otimes \mathbb{Q}$. But this observation depends on an explicit computation of the category.

4.2. **Localisation at multiplicatively closed subsets of $\mathbb{Z}$.** If S is any multiplicatively closed subset of $\mathbb{Z}$, then we define *S-rational* KK^G*-theory*

$$\mathrm{KK}^G_*(A, B; S^{-1}\mathbb{Z}) = S^{-1}\mathrm{KK}^G_*(A, B) = \mathrm{KK}^G_*(A, B) \otimes S^{-1}\mathbb{Z}.$$

By our general theory, these groups form the morphism spaces of an $S^{-1}\mathbb{Z}$-linear triangulated category. It is the localisation of $\mathcal{KK}^G$ at the class of S-finite G-C^*-algebras. Here A is S-finite if and only if there is $s \in S$ with $s \cdot \mathrm{id}_A = 0$.

The colocalisation also produces an *S-torsion* KK^G*-theory* $\mathrm{KK}^G_*(A, B; S^{-1}\mathbb{Z}/\mathbb{Z})$ that fits to a natural long exact sequence

$$\begin{aligned}\cdots \to \mathrm{KK}^G_{n+1}(A, B) \to \mathrm{KK}^G_{n+1}(A, B; S^{-1}\mathbb{Z}) \to \mathrm{KK}^G_{n+1}(A, B; S^{-1}\mathbb{Z}/\mathbb{Z})\\ \to \mathrm{KK}^G_n(A, B) \to \mathrm{KK}^G_n(A, B; S^{-1}\mathbb{Z})\\ \to \mathrm{KK}^G_n(A, B; S^{-1}\mathbb{Z}/\mathbb{Z}) \to \cdots .\end{aligned} \tag{4.2}$$

This includes the rational KK^G-theory

$$\mathrm{KK}^G_n(A, B; \mathbb{Q}) = \mathrm{KK}^G_n(A, B) \otimes \mathbb{Q}$$

and a torsion theory $\mathrm{KK}^G_*(A, B; \mathbb{Q}/\mathbb{Z})$ as special cases.

The S-rational and S-torsion KK^G-theories inherit basic properties like homotopy invariance, C^*-stability, excision and Bott periodicity from KK^G. All this is contained in the statement that they are bifunctors on $\mathcal{KK}^G$, homological in the first and cohomological in the second variable. Furthermore, the maps in (4.2) are natural transformations. Since the S-rational KK^G-theory is again a triangulated category, we get an associative product

$$\mathrm{KK}^G_n(A, B; S^{-1}\mathbb{Z}) \otimes_{S^{-1}\mathbb{Z}} \mathrm{KK}^G_m(B, C; S^{-1}\mathbb{Z}) \to \mathrm{KK}^G_{n+m}(A, C; S^{-1}\mathbb{Z}).$$

4.3. **Real versus complex Kasparov KK-theory.** To illustrate the usefulness of localisation, we reformulate some well-known results about the relationship between real and complex Kasparov KK-theory and K-theory. Roughly speaking, these two theories become equivalent when we localize at 2, that is, work with $\mathbb{Z}[\frac{1}{2}]$-coefficients. The results in this subsection are due to Max Karoubi and Thomas Schick [4, 5].

Thomas Schick related the KK-theories of two real C^*-algebras A and B and their complexifications $A_\mathbb{C}$ and $B_\mathbb{C}$ by an exact sequence

$$\begin{aligned}\cdots \to \mathrm{KKO}^\Gamma_{n-1}(A, B) \xrightarrow{\chi} \mathrm{KKO}^\Gamma_n(A, B) \xrightarrow{c} \mathrm{KK}^\Gamma_n(A_\mathbb{C}, B_\mathbb{C})\\ \xrightarrow{\delta} \mathrm{KKO}^\Gamma_{n-2}(A, B) \xrightarrow{\chi} \mathrm{KKO}^\Gamma_{n-1}(A, B) \xrightarrow{c} \mathrm{KK}^\Gamma_{n-1}(A_\mathbb{C}, B_\mathbb{C}) \to \cdots .\end{aligned} \tag{4.3}$$

In this paper, Γ is assumed to be a discrete group, but the same arguments work if Γ is replaced by a locally compact group or even groupoid; A and B are separable real Γ-C^*-algebras; χ is given by Kasparov product with the generator of $\mathrm{KKO}_1^\Gamma(\mathbb{R},\mathbb{R}) = \mathbb{Z}/2$; c is the complexification functor; and δ is the composition of the inverse of the complex Bott periodicity isomorphism with the functor that forgets the complex structure. More generally, the same argument yields:

Theorem 4.1 ([3]). *Let G be a second countable locally compact group, let A and B be separable real G-C^*-algebras. There is a natural isomorphism*

$$\mathrm{KK}_n^\Gamma(A_\mathbb{C}, B_\mathbb{C}; H) \cong \mathrm{KKO}_n^\Gamma(A, B; H) \oplus \mathrm{KKO}_{n-2}^\Gamma(A, B; H)$$

for the following coefficients:

(1) *$H = S^{-1}\mathbb{Z}$ with $2 \in S$ (localisation);*
(2) *$H = \mathbb{Z}/s\mathbb{Z}$ with odd s (finite coefficients);*
(3) *$H = S^{-1}\mathbb{Z}/\mathbb{Z}$ if S contains only odd numbers (colocalisation).*

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