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Equivariant homology and cohomology of groups

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Abstract

We provide and study an equivariant theory of group (co)homology of a group G with coefficients in a Γ-equivariant G-module A, when a separate group Γ acts on G and A, generalizing the classical Eilenberg–MacLane (co)homology theory of groups. Relationship with equivariant cohomology of topological spaces is established and application to algebraic K-theory is given.

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0. Introduction

It is well known that the study of groups with operators has many important applications in algebra and topology. The category of groups enriched with an action by automorphisms of a given group provides a suitable setting for the investigation of an extensive list of subjects with recognized mathematical interest. See, for instance, recent results in equivariant stable homotopy theory [6] and articles devoted to equivariant algebraic K-theory [13,29, 24]. The origin of the equivariant investigation in homological algebra, particularly in extension theory of groups, goes back to the article of Whitehead [36]. It should be noted that recently a theory of cohomology of groups with operators was developed [8], motivated by

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the graded categorical groups classification problem which was suggested by Frohlich and Wall [16]. This problem was solved [7,10] by using the third cohomology of groups with operators introduced in [8]. An equivariant version of the classical Brauer–Hasse–Noether result was proved [9] showing that for any Galois finite field extension F/K on which a separate group of operators Γ is acting, there is an isomorphism of equivariant isomorphism classes of finite dimensional central simple K-algebras endowed with a Γ-action and containing F as an equivariant strictly maximal subfield and the second cohomology of groups with operators defined in [8] of the Galois group of the extension. A homology theory of groups with operators corresponding to the cohomology theory of groups with operators [8] has been treated in [11].

In [8] it was stated that the second cohomology group of a Γ-group G with coefficients in a Γ-equivariant G-module A classifies the Γ-equivariant extensions of G by A. From this result arises the natural problem about the cohomological characterization of those Γ-equivariant extensions of G by A which are Γ-splitting. The solution of this problem (see Theorem 20) has motivated an attempt to develop a different equivariant (co)homology theory of groups, which is presented in this paper.

By definition for a Γ-group G its equivariant homology and cohomology groups, $H_n^\Gamma(G,-)$ and $H_\Gamma^n(G,-)$, are defined as relative $\mathrm{Tor}_n^{\mathcal{F}}$ and $\mathrm{Ext}_{\mathcal{F}}^n$, $n \geqslant 0$, functors respectively in the category of Γ-equivariant G-modules (Definition 1). Therefore this (co)homology theory of groups can be considered as a part of the relative homological algebra [12]. We provide equivariant versions of classical homological theorems: (co)chain and cotriple presentations of the homology and cohomology of groups, Hopf formula for the second integral homology, universal coefficient formulas, universal central extensions, cohomological classification of extensions of groups, exact (co)homology sequences, Tate (co)homology of groups and the cup product. Applications in algebraic K-theory (Corollary 24) and the relationship with equivariant cohomology of topological spaces (Theorem 22) are established.

Corollary 24 motivates the following:

Conjecture. *There is an isomorphism $K_3(A) \cong H_3^{\mathrm{St}(\mathbb{Z})}(\mathrm{St}(A))$ for any ring A.*

For its proof equivariant versions of relevant classical homotopy theorems will be probably needed. This is well known when A is a unital ring [17] and in this case $\mathrm{St}(\mathbb{Z})$ acts trivially on $\mathrm{St}(A)$.

Another application will be the construction of an alternative equivariant algebraic K-theory K_*^Γ by using Γ-equivariant commutators (Section 6). Moreover in the near future it is intended to investigated for any ring (not necessarily with unit) the relationship of higher Quillen's algebraic K-groups with the equivariant integral homology of the general linear group under the action of the Steinberg group of the ring of integers. It is also intended to establish the relationship of this alternative equivariant algebraic K-theory with existing equivariant algebraic K-theory and equivariant homotopy theory and to provide higher Hopf formulae for equivariant integral homology of groups.

1. Definition and (co)cycle description of the equivariant (co)homology of groups

Before defining the equivariant (co)homology of groups we briefly recall the definition of the cohomology of groups with operators introduced in [8].

The n-cochains of a Γ-group G with coefficients in a Γ-equivariant G-module A (see definition below) are the maps

$$f : \bigcup G^{p+1} \times \Gamma^q \to A, \quad p+q = n-1,$$

normalized in the sense that $f(x_1, \dots, x_{p+1}, \sigma_1, \dots, \sigma_q) = 0$ whenever $x_i = 1$ or $\sigma_j = 1$ for some $i = 1, \dots, p+1$ or $j = 1, \dots, q$, and the coboundary operator is introduced in a natural way. One gets a cochain complex whose homology groups are the cohomology groups of the Γ-group G. Note that in this theory the zero cohomology group is trivial and as mentioned above the second cohomology group describes the Γ-equivariant extensions of the Γ-group G by the $G \rtimes \Gamma$-module A.

Now we will give our definition of the equivariant (co)homology of groups. Let G be a Γ-group. A Γ-equivariant G-module A is a G-module equipped with a Γ-module structure and both actions of G and Γ on A satisfy the following condition:

$$^{\sigma}\big({}^{x}a\big) = {}^{\sigma}x\big({}^{\sigma}a\big), \quad x \in G,\ \sigma \in \Gamma,\ a \in A. \tag{1}$$

The category of Γ-equivariant G-modules is equivalent to the category of $G \rtimes \Gamma$-modules, where $G \rtimes \Gamma$ denotes the semidirect product of G and Γ (see [8]). Let B and C be two Γ-equivariant G-modules. Clearly a map $f : B \to C$ is a $G \rtimes \Gamma$-homomorphism if and only if it is compatible with the actions of G and Γ. A $G \rtimes \Gamma$-module free as a G-module with basis a Γ-subset will be called a relatively free $G \rtimes \Gamma$-module. Denote by $\mathcal{P}$ the class of $G \rtimes \Gamma$-modules which are retracts of relatively free $G \rtimes \Gamma$-modules. The elements of $\mathcal{P}$ will be called relatively projective $G \rtimes \Gamma$-modules. A $G \rtimes \Gamma$-homomorphism $f : B \to C$ of $G \rtimes \Gamma$-modules is a $\mathcal{P}$-epimorphism if it is Γ-splitting, that is there is a Γ-map $\gamma : C \to B$ such that $f\gamma = 1_C$. The group ring $Z(G)$ is a relatively free $G \rtimes \Gamma$-module in a natural way with the action of Γ by

$$^{\sigma}\Big(\sum_i m_i g_i\Big) = \sum_i m_i {}^{\sigma}g_i.$$

Let A be a $G \rtimes \Gamma$-module. Denote by $I_{G \rtimes \Gamma}A$ the subgroup of A generated by the elements $^{(g,\sigma)}a - a = {}^{g}({}^{\sigma}a) - a$, $g \in G,\ \sigma \in \Gamma,\ a \in A$, and by $A_{G \rtimes \Gamma}$ the quotient group of A by $I_{G \rtimes \Gamma}A$. Then it is easily checked that one has canonical isomorphisms

$$\mathbb{Z}(G) \otimes_{G \rtimes \Gamma} A \cong A_{\Gamma}, \qquad \mathbb{Z} \otimes_{G \rtimes \Gamma} A \cong A_{G \rtimes \Gamma}, \qquad \mathrm{Hom}_{G \rtimes \Gamma}\big(\mathbb{Z}(G), A\big) \cong A^{\Gamma}.$$

Clearly if Γ acts trivially on A, then $\mathbb{Z}(G) \otimes_{G \rtimes \Gamma} A \cong \mathrm{Hom}_{G \rtimes \Gamma}(\mathbb{Z}(G), A) \cong A$.

In the category of $G \rtimes \Gamma$-modules there are sufficient relatively projective (free) $G \rtimes \Gamma$-modules. If A is a $G \rtimes \Gamma$-module, take the free G-module $F(A)$ generated by A and define the action of Γ on $F(A)$ by

$$^{\sigma}\big(g|a|\big) = {}^{\sigma}g\big|{}^{\sigma}a\big|, \quad g \in G,\ \sigma \in \Gamma,\ a \in A.$$

Then $F(A)$ becomes a relatively free $G \rtimes \Gamma$-module with basis A being a Γ-subset of $F(A)$ and the canonical map $F(A) \to A$ is a $\mathcal{P}$-epimorphism, since it is Γ-splitting by the map

$$\gamma : A \to F(A), \quad \gamma(a) = |a|, \ a \in A.$$

It is standard to show that one has isomorphisms

$$H_n\big(P_*(A) \otimes_{G \rtimes \Gamma} B\big) \cong H_n\big(A \otimes_{G \rtimes \Gamma} P_*(B)\big), \quad n \geqslant 0,$$

where $P_*(A)$ and $P_*(B)$ are $\mathcal{P}$-projective $G \rtimes \Gamma$-resolutions of A and B, respectively.

Now we are ready to define the equivariant homology $H_*^{\Gamma}(G, A)$ and cohomology $H_{\Gamma}^*(G, A)$ of a Γ-group G with coefficients in a Γ-equivariant G-module A.

Definition 1. $H_n^{\Gamma}(G, A) = \mathrm{Tor}_n^{\mathcal{P}}(\mathbb{Z}, A)$ and $H_{\Gamma}^n(G, A) = \mathrm{Ext}_{\mathcal{P}}^n(\mathbb{Z}, A)$ for $n \geqslant 0$, where G and Γ act trivially on $\mathbb{Z}$.

It is clear that $H_n^{\Gamma}(G, A) \cong H_n(G, A_{\Gamma})$ and $H_{\Gamma}^n(G, A) \cong H^n(G, A^{\Gamma})$ for $n \geqslant 0$, if Γ acts trivially on G and therefore this case is not interesting from the equivariant point of view.

A short exact sequence of $G \rtimes \Gamma$-modules

$$0 \to C_1 \to C \xrightarrow{\beta} C_2 \to 0 \tag{2}$$

will be called proper if β is Γ-splitting, i.e., there is a Γ-map $\gamma : C_2 \to C$ such that $\beta\gamma = 1_{C_2}$.

Let

$$\cdots \to B_n \to \cdots \to B_1 \to B_0 \to \mathbb{Z} \to 0 \tag{3}$$

be the bar resolution of $\mathbb{Z}$, where $B_0 = \mathbb{Z}(G)$, and B_n, $n > 0$, is the free $\mathbb{Z}(G)$-module generated by $[g_1, g_2, \ldots, g_n]$, $g_i \in G$. Define the action of the group Γ on the bar resolution as follows. Γ acts trivially on $\mathbb{Z}$, the action of Γ on B_0 is already defined and if $n > 0$ then ${}^{\sigma}(g[g_1, g_2, \ldots, g_n]) = {}^{\sigma}g[{}^{\sigma}g_1, {}^{\sigma}g_2, \ldots, {}^{\sigma}g_n]$ for the action of Γ on B_n. The well known contraction $\gamma_{-1} : \mathbb{Z} \to B_0$, $\gamma_{-1}(z) = z1$, $\gamma_n : B_n \to B_{n+1}$, $\gamma_n(g[g_1, \ldots, g_n]) = [g, g_1, \ldots, g_n]$, $n \geqslant 0$, is clearly a Γ-map. We deduce that under this action of Γ the bar resolution (3) becomes an exact sequence of $G \rtimes \Gamma$-modules such that each B_n is a relatively free $G \rtimes \Gamma$-module and the sequences

$$0 \to \mathrm{Ker}\,\partial_n \to B_n \to \mathrm{Im}\,\partial_n \to 0, \quad n > 0$$

and

$$0 \to \mathrm{Ker}\,\varepsilon \to B_0 \to \mathbb{Z} \to 0,$$

are proper short exact sequences of $G \rtimes \Gamma$-modules. Therefore (3) is a relatively free $G \rtimes \Gamma$-resolution of $\mathbb{Z}$ which will be called the Γ-equivariant bar resolution of $\mathbb{Z}$. It follows that

$$H_n^{\Gamma}(G, A) \cong H_n(B_* \otimes_{G \rtimes \Gamma} A), \qquad H_{\Gamma}^n(G, A) \cong H_n\big(\mathrm{Hom}_{G \rtimes \Gamma}(B_*, A)\big), \quad n \geqslant 0.$$

These isomorphisms allow us to produce an alternative description by (co)cycles of the equivariant (co)homology of groups. The case of the equivariant homology of groups is clear. To this end consider the Abelian group $C^n_\Gamma(G,A)$ of Γ-maps $f: G^n \to A$, $n > 0$, which will be called the group of nth Γ-cochains. By using the classical cobord operators $\delta^n : C^n_\Gamma(G,A) \to C^{n+1}_\Gamma(G,A)$, $n > 0$, one gets a cochain complex

$$0 \to C^0_\Gamma(G,A) \to C^1_\Gamma(G,A) \to C^2_\Gamma(G,A) \to \cdots \to C^n_\Gamma(G,A) \to \cdots,$$

where $C^0_\Gamma = A^\Gamma$, $\operatorname{Ker}\delta^1 = \operatorname{Der}_\Gamma(G,A)$ is the group of Γ-derivations, and the homology groups of this complex give the Γ-equivariant cohomology groups of G with coefficients in the $G \rtimes \Gamma$-module A.

It is easily checked that any proper short exact sequence (2) of $G \rtimes \Gamma$-modules induces long exact homology and cohomology sequences

$$\begin{aligned}\cdots \to H^\Gamma_{n+1}(G,C_2) \to H^\Gamma_n(G,C_1) \to \cdots \to H^\Gamma_2(G,C_2) \to H^\Gamma_1(G,C_1)\\ \to H^\Gamma_1(G,C) \to H^\Gamma_1(G,C_2) \to H^\Gamma_0(G,C_1) \to H^\Gamma_0(G,C) \to H^\Gamma_0(G,C_2) \to 0,\end{aligned}$$

$$\begin{aligned}0 \to H^0_\Gamma(G,C_1) \to H^0_\Gamma(G,C) \to H^0_\Gamma(G,C_2) \to H^1_\Gamma(G,C_1) \to H^1_\Gamma(G,C)\\ \to H^1_\Gamma(G,C_2) \to H^2_\Gamma(G,C_1) \to \cdots \to H^n_\Gamma(G,C_2) \to H^{n+1}_\Gamma(G,C_1) \to \cdots.\end{aligned}$$

2. Equivariant (co)homology of groups as cotriple (co)homology

To present the equivariant (co)homology of groups as cotriple (co)homology we will use the free cotriple defined in the category $\mathcal{G}_\Gamma$ of Γ-groups given in [21,22] to develop a non-Abelian homology theory of groups. This cotriple corresponds to the tripleability of $\mathcal{G}_\Gamma$ over Γ-sets. The resulting cotriple $\mathcal{F} = (F,\tau,\delta)$ is the free cotriple in the category of groups endowed with the Γ-action defined as follows. For any Γ-group G the action of Γ on the free group $F(G)$ is given by ${}^\sigma|g| = |{}^\sigma g|$, $g \in G$, $\sigma \in \Gamma$. The cotriple thus defined essentially differs from the cotriple introduced in [8] for the cotriple interpretation of the cohomology of groups with operators. Let $\mathcal{P}_\mathcal{F}$ be the projective class induced by the cotriple $\mathcal{F}$ in the category $\mathcal{G}_\Gamma$. It is easy to see that a morphism $f: G \to H$ of Γ-groups is a $\mathcal{P}_\mathcal{F}$-epimorphism if it is surjective and Γ-splitting. Since the category $\mathcal{G}_\Gamma$ has finite limits, any Γ-group G has a $\mathcal{P}_\mathcal{F}$-projective resolution (X_*, ∂^0_0, G) in the category $\mathcal{G}_\Gamma$ in the sense of [34], that is X_* is an augmented pseudo-simplicial Γ-group which is $\mathcal{P}_\mathcal{F}$-exact [34,20] and each X_n, $n \geqslant 0$, belongs to the class $\mathcal{P}_\mathcal{F}$. Many examples of pseudo-simplicial sets which are not simplicial are given in [25,14,15]. A $\mathcal{P}_\mathcal{F}$-epimorphism $f: P \to G$ with P an object of the class $\mathcal{P}_\mathcal{F}$ will be called a projective presentation of the Γ-group G. Any projective presentation of G induces in a natural way a $\mathcal{P}_\mathcal{F}$-projective resolution $P_* \to G$, constructed as follows:

$$\cdots \;\begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\; F(L_2G) \xrightarrow{\tau_{L_2G}} L_2G \;\underset{l^2_0}{\overset{l^2_2}{\begin{smallmatrix}\longrightarrow\\ \vdots\\ \longrightarrow\end{smallmatrix}}}\; F(L_1G) \xrightarrow{\tau_{L_1G}} L_1G \;\underset{l^1_0}{\overset{l^1_1}{\rightrightarrows}}\; P \xrightarrow{f} G,$$

where (L_1G, l^1_0, l^1_1) is the simplicial kernel of the morphism f, $(L_2G, l^2_0, l^2_1, l^2_2)$ the simplicial kernel of the pair of morphisms $(l^1_0\tau_{L_1G}, l^1_1\tau_{L_1G})$ and if $(L_nG, l^n_0, \ldots, l^n_n)$ has been constructed, then $(L_{n+1}G, l_0^{n+1}, \ldots, l_{n+1}^{n+1})$ is the simplicial kernel of the sequence of morphisms $(l_0^n \tau_{L_n G}, \ldots, l_n^n \tau_{L_n G})$. Simplicial kernels are defined in [34,20]. Denote $P_0 = P$, $P_n = F(L_n(G))$ and $\partial_i^n = l_i^n \tau_{L_n(G)}$ for $n > 0$.

Let T be a functor from the category $\mathcal{G}_\Gamma$ to the category $\mathcal{G}$ of groups. Then by definition the left cotriple derived functors $L_n^{\mathcal{F}} T(G)$ are equal to $\pi_n(TF_*(G))$, $n \geqslant 0$, where $\tau : F_*(G) \to G$ is the free cotriple resolution of G with $F_0 = F(G)$, $F_n = F(F^n(G))$, $n \geqslant 1$, $\partial_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$ (see [33]) and the left derived functors $L_n^{\mathcal{P}_\mathcal{F}} T(G)$ with respect to the projective class $\mathcal{P}_\mathcal{F}$ are equal to $\pi_n(T(X_*))$, $n \geqslant 0$ (see [19,20]). If the functor T is a contravariant functor with values in the category of Abelian groups, one has also its right derived functors.

Proposition 2. *The left cotriple derived functors of a functor $T : \mathcal{G}_\Gamma \to \mathcal{G}$ are isomorphic to its left $\mathcal{P}_\mathcal{F}$-derived functors.*

Proof. Let (P_*, ∂_0^0, G) be the standard $\mathcal{P}_\mathcal{F}$-resolution of G. Then it is easy to see that this resolution is left contractible in the category of Γ-sets. Thus the augmented pseudosimplicial Γ-groups $(F_i(P_*), F_i(\partial_0^0), F_i(G))$ are left contractible for $i \geqslant 0$ in the category of Γ-groups (for the categorical definition of left contractibility see [33,20]). On the other hand the augmented simplicial Γ-groups $(F_*(P_j), \tau_j, P_j)$ are also left contractible for $j \geqslant 0$. It follows that the homotopy groups π_n, $n > 0$, of the pseudosimplicial groups $TF_i(P_*)$ and $TF_*(P_j)$ for $i, j \geqslant 0$ are trivial and the homotopy groups of $\pi_0(TF_i(P_*))$ and $\pi_0(TF_*(P_j))$ give the left projective and cotriple derived functors respectively of the functor T. Consider now the bipseudosimplicial group $G_{**}(G)$ by putting $G_{pq}(G) = TF_p(P_q(G))$ and apply the Quillen spectral sequences [30,19,20] for a bipseudosimplicial group. It follows that the nth homotopy groups, $n \geqslant 0$, of $TP_*(G)$ and $TF_*(G)$ are both isomorphic to the nth homotopy group of the diagonal pseudosimplicial group ΔG_{**}. It remains to apply Theorems 1.2 and 2.1 of [19] showing that the definition of the left projective derived functors are independent of the projective resolution of G. $\square$

Note that this proposition is known for the left derived functors of functors (right derived functors of contravariant functors) with values in the category of Abelian groups [34].

Let A be a fixed Γ-module and ${}_A\mathcal{G}_\Gamma$ the category of Γ-groups acting on A such that the condition (1) holds. Consider the following functors from the category ${}_A\mathcal{G}_\Gamma$ to the category of Abelian groups: $I(-) \otimes_{G \rtimes \Gamma} A$ and $\mathrm{Der}_\Gamma(-, A)$, where $I(G)$ is the kernel of the canonical homomorphism $\varepsilon : Z(G) \to Z$ of $G \rtimes \Gamma$-modules and $\mathrm{Der}_\Gamma(G, A)$ is the group of Γ-derivations from G to A consisting of derivations $f : G \to A$ such that $f({}^\sigma g) = {}^\sigma f(g)$, $g \in G$, $\sigma \in \Gamma$ [8].

Theorem 3. *There are isomorphisms*

$$H_n^\Gamma(G, A) \cong L_{n-1}^{\mathcal{F}}\big(I(G) \otimes_{G \rtimes \Gamma} A\big),$$

$$H_\Gamma^n(G, A) \cong R_{\mathcal{F}}^{n-1} \mathrm{Der}_\Gamma(G, A), \quad n \geqslant 2.$$

Proof. Apply the functor $I(-)$ to the free cotriple resolution $\tau : F_* \to G$ of the Γ-group G. One gets an augmented simplicial $G \rtimes \Gamma$-module $I(F_*) \to I(G)$. We introduce the notations:

$$IF_n(G) \cong \sum_{y \in F^n(G)} \mathbb{Z}\big(F^n(G)\big)(y-e) = D_n(G),$$
$$\sum_{y \in F^n(G)} \mathbb{Z}(G)(y-e) = E_n(G)$$

for $n \geqslant 0$, $F^0(G) = G$.

There are natural homomorphisms

$$\alpha_n : D_n(G) \to E_n(G), \quad n \geqslant 1,$$

induced by the homomorphism $\tau \partial_0^1 \partial_0^2 \cdots \partial_0^{n-2} \partial_0^{n-1} : F^n(G) \to G$ such that we obtain a morphism of augmented simplicial $G \rtimes \Gamma$-modules

$$\big(D_*(G) \to I(G)\big) \to \big(E_*(G) \to I(G)\big).$$

The left Γ-contractibility of the cotriple resolution $F_*(G) \to G$ implies the Γ-contractibility of the corresponding induced Abelian chain complexes

$$\cdots \to D_n(G) \to \cdots \to D_2(G) \to D_1(G) \to I(G) \to 0,$$
$$\cdots \to E_n(G) \xrightarrow{\varepsilon_n} \cdots \to E_2(G) \xrightarrow{\varepsilon_2} E_1(G) \xrightarrow{\varepsilon_1} I(G) \to 0,$$

where $\varepsilon_n = \sum_i (-1)^i \varepsilon_i^n$, $n \geqslant 1$. In effect the canonical Γ-injections $\{f, f_n,\ n \geqslant 1\}$,

$$f : G \to F(G), \qquad f_n : F^n(G) \to F^{n+1}(G) = \big(F\big(F^n(G)\big)\big), \quad n \geqslant 1,$$

yield the left Γ-contractibility of $F_*(G) \to G$ in the category of Γ-sets (see [33, Lemma 1.2]). Therefore we obtain Γ-homomorphisms

$$\mathbb{Z}(f) : \mathbb{Z}(G) \to \mathbb{Z}\big(F(G)\big),$$
$$\mathbb{Z}(f_n) : \mathbb{Z}\big(F^n(G)\big) \to \mathbb{Z}\big(F^{n+1}(G)\big), \quad n \geqslant 1,$$

of free Abelian Γ-groups induced by $\{f, f_n,\ n \geqslant 1\}$, where the action of Γ on $\mathbb{Z}(F^n(G))$, $n \geqslant 0$, $F^0(G) = G$, is induced by the above defined action of Γ on $F^n(G)$. The action of Γ on $IF_n(G) \cong \sum_{y \in F^n(G)} \mathbb{Z}(F^n(G))(y-e)$, $n \geqslant 0$, is induced by the action of Γ on $\mathbb{Z}(F^n(G))$, namely

$${}^{\sigma}\big(x(y-e)\big) = {}^{\sigma}x\big({}^{\sigma}y - e\big), \quad x \in \mathbb{Z}\big(F^n(G)\big),\ y \in F^n(G).$$

The Γ-homomorphisms $\{\mathbb{Z}(f), \mathbb{Z}(f_n),\ n \geqslant 0\}$ induce Γ-homomorphisms

$$IG \to IF_0(G), \qquad IF^n(G) \to IF^{n+1}(G), \quad n \geqslant 0,$$

and yield the required Γ-contraction in $IF_*(G)$.

Thus each short exact sequence

$$0 \to \operatorname{Ker} \varepsilon_n \to E_n(G) \to \operatorname{Im} \varepsilon_n \to 0, \quad n \geqslant 1,$$

is Γ-splitting and it follows that $(E_*(G) \to I(G))$ is a relatively free resolution of the $G \rtimes \Gamma$-module $I(G)$.

It is obvious that the homomorphisms $\alpha_n,\ n \geqslant 1$, induce isomorphisms

$$D_n(G) \otimes_{F^n(G)\rtimes\Gamma} A \cong E_n(G) \otimes_{G\rtimes\Gamma} A,$$
$$\mathrm{Hom}_{F^n(G)\rtimes\Gamma}\big(D_n(G), A\big) \cong \mathrm{Hom}_{G\rtimes\Gamma}\big(E_n(G), A\big). \tag{4}$$

Whence we deduce from (4) that $L_n^{\mathcal{F}}(I(G) \otimes A) \cong \mathrm{Tor}_n^{\mathcal{P}}(I(G), A)$, $n \geqslant 0$. It is easily checked that the well-known isomorphism

$$\mathrm{Hom}_{F^n(G)}\big(D_n(G), A\big) \cong \mathrm{Der}\big(F^n(G), A\big)$$

is compatible with the action of Γ, whence its restriction on the subgroup $\mathrm{Hom}_{F^n(G)\rtimes\Gamma}(D_n(G), A)$ gives an isomorphism with $\mathrm{Der}_\Gamma(F^n(G), A)$. Thus from (4) one gets $R_{\mathcal{F}}^n \mathrm{Der}_\Gamma(G, A) \cong \mathrm{Ext}_{\mathcal{P}}^n(I(G), A),\ n \geqslant 0$.

The proper short exact sequence of $G\rtimes\Gamma$-modules

$$0 \to I(G) \to Z(G) \to Z \to 0 \tag{5}$$

yields long exact sequences of the relative derived functors of the functors $- \otimes_{G\rtimes\Gamma} A$ and $\mathrm{Hom}_{G\rtimes\Gamma}(-, A)$ implying the isomorphisms

$$H_{n+1}^\Gamma(G, A) \cong \mathrm{Tor}_n^{\mathcal{P}}\big(I(G), A\big) \quad \text{and}$$
$$H_\Gamma^{n+1}(G, A) \cong \mathrm{Ext}_{\mathcal{P}}^n\big(I(G), A\big), \quad n \geqslant 1,$$

which give the required isomorphisms. □

It is clear that $L_0^{\mathcal{F}}(I(G) \otimes_{G\rtimes\Gamma} A) \cong I(G) \otimes_{G\rtimes\Gamma} A$ and $R_{\mathcal{F}}^0 \mathrm{Der}_\Gamma(G, A) \cong \mathrm{Der}_\Gamma(G, A)$.

Definition 4. A Γ-group G will be called Γ-free, if it is a free group with basis a Γ-subset.

Corollary 5. *If G is a retract of a Γ-free group, then $H_n^\Gamma(G, A) = 0$ and $H_\Gamma^n(G, A) = 0$ for $n > 1$ and any $G \rtimes \Gamma$-module A.*

Proof. The augmented simplicial group $F_*(G) \to G$ is left contractible [33] implying the triviality of the homotopy groups $\pi_n(IF_*(G) \otimes_{G\rtimes\Gamma} A)$ and $\pi_n \mathrm{Der}_\Gamma(F_*(G), A)$ for $n \geqslant 1$. □

Corollary 6. *If G and Γ act trivially on $\mathbb{Z}$, then $H_1^\Gamma(G, \mathbb{Z}) \cong I(G) \otimes_{G\rtimes\Gamma} \mathbb{Z}$.*

Proof. The proof follows immediately from the long exact sequence of the functors $\mathrm{Tor}_n^{\mathcal{P}}(-, \mathbb{Z})$ induced by (5), since $\mathbb{Z}(G) \otimes_{G\rtimes\Gamma} \mathbb{Z} \cong \mathbb{Z}$. □

Proposition 7. *The cotriple derived functors $L_n^{\mathcal{F}} H_1^\Gamma(-, A)$ are isomorphic to $H_{n+1}^\Gamma(-, A)$, $n > 0$.*

Proof. The long exact sequence of the functors $\mathrm{Tor}_n^{\mathcal{F}}(-, A)$ for the sequence (5) yields the exact sequence

$$0 \to H_1^\Gamma(G, A) \to I(G) \otimes_{G\rtimes\Gamma} A \to \mathbb{Z}(G) \otimes_{G\rtimes\Gamma} A \to \mathbb{Z} \otimes_{G\rtimes\Gamma} A \to 0.$$

It follows that there is a functorial short exact sequence

$$0 \to H_1^{\Gamma}(G, A) \to I(G) \otimes_{G \rtimes \Gamma} A \to I_{G \rtimes \Gamma} A / I_{\Gamma} A \to 0$$

inducing a short exact sequence of Abelian simplicial groups

$$0 \to H_1^{\Gamma}\big(F_*(G), A\big) \to I\big(F_*(G)\big) \otimes_{G \rtimes \Gamma} A \to I_{F_*(G) \rtimes \Gamma} A / I_{\Gamma} A \to 0.$$

It remains to apply the corresponding long exact homotopy sequence and to see that $I_{F_*(G) \rtimes \Gamma} A / I_{\Gamma} A$ is a constant Abelian simplicial group. □

Denote by $[G, G]_{\Gamma}$ the subgroup of the Γ-group G generated by $[G, G]$ and by the elements of the form ${}^{\sigma}g \cdot g^{-1},\ g \in G,\ \sigma \in \Gamma$. This subgroup will be called the Γ-commutator subgroup of G. It is obvious that $[G, G]_{\Gamma}$ is a normal Γ-subgroup of G and Γ acts trivially on the Abelian group $G/[G, G]_{\Gamma}$. If H is a normal Γ-subgroup of G, we denote by $[G, H]_{\Gamma}$ the subgroup of G generated by the elements $x{}^{\sigma}yx^{-1}y^{-1}$, where $x \in G$, $y \in H$, $\sigma \in \Gamma$.

Let B be an Abelian group on which Γ acts trivially and $f: G \to B$ a homomorphism of Γ-groups. Then f factorizes uniquely through $G/[G, G]_{\Gamma}$. Consider the subgroup of G generated by the elements of the form $x{}^{\sigma}yx^{-1}y^{-1}$, $x, y \in G$, $\sigma \in \Gamma$. It is easily seen that this subgroup coincides with $[G, G]_{\Gamma}$. The elements $x{}^{\sigma}yx^{-1}y^{-1} = [x, y]_{\sigma}$ will be called Γ-commutators of G, the group $G/[G, G]_{\Gamma}$ the Γ-abelianization G_{Γ}^{ab} of G and the corresponding functor $G \mapsto G_{\Gamma}^{ab}$ the Γ-abelianization functor.

Proposition 8. *There is a functorial isomorphism*

$$I(G) \otimes_{G \rtimes \Gamma} A \cong G/[G, G]_{\Gamma} \otimes A,$$

where G and Γ act trivially on A.

Proof. It is enough to show that $I(G) \otimes_{G \rtimes \Gamma} Z \cong G/[G, G]_{\Gamma}$. This isomorphism is given by $(g - e) \otimes n \mapsto n[g]$. Its converse is defined by $[g] \mapsto (g - e) \otimes 1$. We have only to show the correctness of the converse map.

One has

$$\begin{aligned}
&\big(x{}^{\sigma}yx^{-1}y^{-1} - e\big) \otimes 1 \\
&\quad = \big(x{}^{\sigma}yx^{-1}y^{-1} - x + x - e\big) \otimes 1 \\
&\quad = x\big({}^{\sigma}yx^{-1}y^{-1} - e\big) \otimes 1 + (x - e) \otimes 1 \\
&\quad = \big({}^{\sigma}yx^{-1}y^{-1} - e\big) \otimes 1 + (x - e) \otimes 1 \\
&\quad = \big({}^{\sigma}yx^{-1}y^{-1} - {}^{\sigma}y + {}^{\sigma}y - e\big) \otimes 1 + (x - e) \otimes 1 \\
&\quad = {}^{\sigma}y\big(x^{-1}y^{-1} - e\big) \otimes 1 + \big({}^{\sigma}y - e\big) \otimes 1 + (x - e) \otimes 1 \\
&\quad = \big(x^{-1}y^{-1} - e\big) \otimes 1 + (y - e) \otimes 1 + (x - e) \otimes 1 \\
&\quad = \big(x^{-1}y^{-1} - x^{-1} + x^{-1} - e\big) \otimes 1 + (y - e) \otimes 1 + (x - e) \otimes 1 \\
&\quad = x^{-1}\big(y^{-1} - e\big) \otimes 1 + \big(x^{-1} - e\big) \otimes 1 + (y - e) \otimes 1 + (x - e) \otimes 1 \\
&\quad = \big(y^{-1} - e\big) \otimes 1 + \big(x^{-1} - e\big) \otimes 1 + (y - e) \otimes 1 + (x - e) \otimes 1.
\end{aligned}$$

But for any element $x \in G$ the following equalities hold:

$$0 = 0 \otimes 1 = \bigl(xx^{-1} - e\bigr) \otimes 1 = (x - e) \otimes 1 + \bigl(x^{-1} - e\bigr) \otimes 1.$$

It follows that under the afore defined converse map any Γ-commutator becomes 0 showing its correctness. □

The isomorphism of Proposition 8 holds only for trivial actions on A and it is natural in the sense that in this case it is functorial and in fact uniquely defined. For an arbitrary G-module in the classical case there exists another form of this isomorphism, where its right side is replaced by the non-Abelian tensor product of the groups G and A.

3. The equivariant integral homology

Denote the Γ-equivariant integral homology groups $H_n^\Gamma(G,\mathbb{Z}) = H_n^\Gamma(G)$, $n \geqslant 0$, the groups G and Γ acting trivially on $\mathbb{Z}$.

Corollary 9. *There is a functorial isomorphism*

$$H_1^\Gamma(G) \cong G/[G,G]_\Gamma.$$

Proof. The proof follows from Corollary 6 and Proposition 8. □

Note that the group $[G,G]_\Gamma/[G,G]$ shows the difference between the classical and the Γ-equivariant abelianization functors. We denote by $T\Gamma$ the functor assigning to any Γ-group G the Abelian group $[G,G]_\Gamma/[G,G]$. We also denote by $\Gamma \cdot G$ the subgroup of G generated by the elements ${}^\sigma g \cdot g^{-1}$, $g \in G$, $\sigma \in \Gamma$. One has a natural homomorphism

$$\beta_n : H_n(G) \to H_n^\Gamma(G), \quad n \geqslant 0,$$

induced by the morphism of Abelian simplicial groups

$$\bigl(I\bigl(F_*(G)\bigr) \otimes_G \mathbb{Z}\bigr) \to I\bigl(F_*(G)\bigr) \otimes_{G \rtimes \Gamma} \mathbb{Z}.$$

Theorem 10.

(i) *There is an isomorphism*

$$L_n^{\mathcal{F}}\bigl(G_\Gamma^{ab}\bigr) \cong H_{n+1}^\Gamma(G), \quad n \geqslant 0.$$

(ii) *There are a functorial short exact sequence*

$$0 \to \Gamma \cdot G/[G,G] \cap \Gamma \cdot G \to H_1(G) \to H_1^\Gamma(G) \to 0,$$

and a long exact homology sequence

$$\begin{aligned}
\cdots &\to H_{n+1}^\Gamma(G) \to L_{n-1}^{\mathcal{F}} T\Gamma(G) \to H_n(G) \to H_n^\Gamma(G) \\
&\to L_{n-2}^{\mathcal{F}} T\Gamma(G) \to \cdots \to H_3^\Gamma(G) \to L_1^{\mathcal{F}} T\Gamma(G) \to H_2(G) \\
&\to H_2^\Gamma(G) \to L_0^{\mathcal{F}} T\Gamma(G) \to H_1(G) \to H_1^\Gamma(G) \to 0.
\end{aligned}$$

Proof. (i) The proof follows from Proposition 7 and Corollary 9.

(ii) The commutative diagram

$$\begin{tikzcd} H_1(G) \arrow[r, "\beta_1"] \arrow[d, "\cong"'] & H_1^{\Gamma}(G) \arrow[d, "\cong"] \\ G/[G,G] \arrow[r] & G/[G,G]_{\Gamma} \end{tikzcd}$$

shows that $\operatorname{Ker}\beta_1$ is isomorphic to $[G,G]_{\Gamma}/[G,G]$ and it is clear that $[G,G]_{\Gamma} = [G,G]\cdot(T\Gamma(G))$. By applying the short exact sequence (ii) to the cotriple resolution $F_*(G) \to G$, we obtain a short exact sequence of simplicial Abelian groups

$$0 \to T\Gamma\big(F_*(G)\big) \to H_1\big(F_*(G)\big) \to H_1^{\Gamma}\big(F_*(G)\big) \to 0$$

inducing a long exact homology sequence and it remains to recall that the cotriple nth derived functor of the first integral homology gives the $(n+1)$th integral homology group, $n \geqslant 0$. $\square$

Let A be a $G \rtimes \Gamma$-module on which G and Γ act trivially. Then

$$H_1^{\Gamma}(G,A) \cong I(G) \otimes_{G\rtimes\Gamma} A \cong H_1^{\Gamma}(G) \otimes A \cong G/[G,G]_{\Gamma} \otimes A$$

and

$$\begin{aligned} H^1_{\Gamma}(G,A) &\cong \mathrm{Der}_{\Gamma}(G,A) \cong \mathrm{Hom}_{G\rtimes\Gamma}\big(I(G) \otimes_{G\rtimes\Gamma} \mathbb{Z}, A\big) \\ &\cong \mathrm{Hom}\big(G/[G,G]_{\Gamma}, A\big). \end{aligned}$$

On the other hand, if G is a Γ-free group with basis X and $\beta : G \to G/[G,G]_{\Gamma}$ is the canonical Γ-homomorphism, then $G/[G,G]_{\Gamma}$ is a free Abelian group with basis $\beta(X)$. Any map $\gamma : \beta(X) \to B$ to an Abelian group B induces a Γ-map $\gamma\beta : X \to B$ which is uniquely extended to a Γ-homomorphism $G \to B$ assuming Γ acts trivially on B and one gets a uniquely defined homomorphism $G/[G,G]_{\Gamma} \to B$ whose restriction on $\beta(X)$ is equal to γ.

We deduce that for any $G \rtimes \Gamma$-module A with trivial actions of G and Γ on A we obtain universal coefficient formulas for the equivariant (co)homology groups $H_n^{\Gamma}(G,A)$ and $H^n_{\Gamma}(G,A)$, $n \geqslant 0$.

Theorem 11. *There are short exact split* (*not naturally*) *sequences*

$$0 \to H_n^{\Gamma}(G) \otimes A \to H_n^{\Gamma}(G,A) \to \mathrm{Tor}_1\big(H_{n-1}^{\Gamma}(G), A\big) \to 0,$$
$$0 \to \mathrm{Ext}^1\big(H_{n-1}^{\Gamma}(G), A\big) \to H^n_{\Gamma}(G,A) \to \mathrm{Hom}\big(H_n^{\Gamma}(G), A\big) \to 0$$

for $n \geqslant 0$.

4. Universal central Γ-equivariant extensions and Hopf formula

Let G be a Γ-group and A a $G \rtimes \Gamma$-module.

Definition 12. A Γ-equivariant extension E of the Γ-group G by the Γ-equivariant G-module A is an extension of G by the G-module A

$$E : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} G \to 1$$

satisfying the following conditions:

(1) E is a sequence of Γ-groups,
(2) E is Γ-splitting, that is there is a Γ-map $\gamma : G \to B$ such that $\beta\gamma = 1_G$.

E is called a central Γ-equivariant extension of the Γ-group G, if $\alpha(A)$ belongs to the center of B and Γ acts trivially on A.

Note that Γ-equivariant extensions of Γ-groups investigated in [8] do not in general satisfy the condition (2).

A central Γ-equivariant extension (U, β) of G is called universal, if for any central Γ-equivariant extension (X, α) of G there is a unique Γ-homomorphism $U \to X$ over G.

Two Γ-equivariant extensions E and E' of G by A are called equivalent if there is a morphism $E \to E'$ which is the identity on A and G. We denote by $E_\Gamma(G, A)$ the set of equivalence classes of Γ-equivariant extensions of G by A.

Definition 13. A Γ-group G is called Γ-perfect, if G coincides with its Γ-commutator subgroup $[G, G]_\Gamma$ (see also [26]).

Below we give important examples of Γ-groups which are Γ-perfect but not perfect (see Section 6).

Proposition 14. *If (X, φ) is a central Γ-equivariant extension of a Γ-perfect group G, then the Γ-commutator subgroup $X' = [G, G]_\Gamma$ is Γ-perfect and maps onto G.*

Proof. Since G is Γ-perfect, it is clear that φ maps X' onto G. It follows that any element $x \in X$ can be written as a product $x'c$ with $x' \in X'$ and c belongs to $\operatorname{Ker}\varphi$. Therefore every generator of X' of the form $[x_1, x_2]$ is equal to $[x_1'c_1', x_2'c_2'] = [x_1', x_2']$ with $x_1', x_2' \in X'$ and of the form ${}^\sigma x \cdot x^{-1}$ is equal to ${}^\sigma(x'c) \cdot (x'c)^{-1} = {}^\sigma x' \cdot x'^{-1}$ with $x' \in X'$. Whence $X' = [X', X']_\Gamma$. □

Corollary 15. $(X', \varphi|_{X'})$ *is a central Γ-equivariant extension of G.*

Proof. We have only to show that $(X', \varphi|_{X'})$ is Γ-splitting. Let γ be the Γ-splitting map for (X, φ), that is $\varphi\gamma = 1_G$. Then a Γ-splitting map γ' for the extension $(X', \varphi|_{X'})$ is defined as follows: consider the decomposition $\{D_\eta\}$ of G into orbits with respect to the action of Γ on G. Choose a representative $z_\eta \in D_\eta$ for each D_η and choose an expression

$$z_\eta = [x_1, y_1]_{\sigma_1} \cdot [x_2, y_2]_{\sigma_2} \cdots [x_k, y_k]_{\sigma_k}$$

of z_η in terms of Γ-commutators. The equalities

$$\begin{aligned}{}^{\sigma'}\big([x,y]_\sigma\big) &= {}^{\sigma'}\big(x^\sigma y x^{-1} y^{-1}\big) = {}^{\sigma'}x^{\sigma'\sigma} y^{\sigma'}\big(x^{-1}\big)^{\sigma'}\big(y^{-1}\big) \\ &= {}^{\sigma'}x^{\sigma'\sigma\sigma'-1}\big({}^{\sigma'}y\big)^{\sigma'}\big(x^{-1}\big)^{\sigma'}\big(y^{-1}\big) = \big[{}^{\sigma'}x, {}^{\sigma'}y\big]_{\sigma'\sigma\sigma'^{-1}},\end{aligned}$$

$x, y \in G$, $\sigma, \sigma' \in \Gamma$, imply the expression

$${}^{\sigma}z_\eta = \big[{}^{\sigma}x_1, {}^{\sigma}y_1\big]_{\sigma\sigma_1\sigma^{-1}} \cdot \big[{}^{\sigma}x_2, {}^{\sigma}y_2\big]_{\sigma\sigma_2\sigma^{-1}} \cdots \big[{}^{\sigma}x_k, {}^{\sigma}y_k\big]_{\sigma\sigma_k\sigma^{-1}}$$

for ${}^{\sigma}z_\eta$, $\sigma \in \Gamma$.

Clearly any element $g \in G$ has the form ${}^{\sigma}z_\eta$ for some $z_\eta \in D_\eta$ and $\sigma \in \Gamma$. The required Γ-splitting map $\gamma' : G \to X'$ is given by setting

$$\begin{aligned}\gamma'(g) = \gamma'\big({}^{\sigma}z_\eta\big) &= \big[{}^{\sigma}\gamma(x_1), {}^{\sigma}\gamma(y_1)\big]_{\sigma\sigma_1\sigma^{-1}} \cdot \big[{}^{\sigma}\gamma(x_2), {}^{\sigma}\gamma(y_2)\big]_{\sigma\sigma_2\sigma^{-1}} \cdots \\ &\times \big[{}^{\sigma}\gamma(x_k), {}^{\sigma}\gamma(y_k)\big]_{\sigma\sigma_k\sigma^{-1}}. \qquad \square\end{aligned}$$

Theorem 16. *A central Γ-equivariant extension (U, β) of a Γ-group G is universal if and only if U is Γ-perfect and every central equivariant Γ-extension (W, α) of U splits.*

Proof. We will follow Milnor's proof of the classical case [28]. Let U be Γ-perfect and every central Γ-equivariant extension of U splits. Let (X, φ) be an arbitrary central Γ-equivariant extension of G. Take the following diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & X \times U \arrow[r, "p"] \arrow[d, "q"'] & U \arrow[r] \arrow[d, "\beta"] & 1 \\ 0 \arrow[r] & C \arrow[r] & X \arrow[r, "\varphi"] & G \arrow[r] & 1 \end{tikzcd}$$

where $X \times U$ is the fiber product of $X \to G \leftarrow U$. It is easy to check that the top row is a central Γ-equivariant extension of U. Therefore one has a Γ-section $s : U \to X \times U$. Thus the Γ-homomorphism $f = qs : U \to X$ is over G and the diagram

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d] & U \arrow[r, "\beta"] \arrow[d, "f"'] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & C \arrow[r] & X \arrow[r, "\varphi"] & G \arrow[r] & 1 \end{tikzcd}$$

is commutative. To prove that (U, β) is universal, it remains to show the uniqueness of such an f. Let $f_1, f_2 : U \to X$ be two Γ-homomorphisms over G. Then one gets a Γ-homomorphism $h : U \to C$ given by $h(u) = f_1(u) \cdot f_2(u)^{-1}$, $u \in U$, which is trivial, since U is Γ-perfect and Γ acts trivially on C.

Let (X, φ) and (Y, ψ) be central Γ-equivariant extensions of G. Then, as we have seen, if Y is Γ-perfect there exists at most one Γ-homomorphism from Y to X over G. If Y is not Γ-perfect, then there is a suitable central Γ-equivariant (X, φ) of G such that there exists more than one Γ-homomorphism from Y to X over G. Indeed in this case there exists a nontrivial Γ-homomorphism f from Y to some Abelian group A on which Γ acts trivially. Take the central Γ-equivariant split extension

$$0 \to A \to A \times G \to G \to 1.$$

Setting $f_1(y) = (0, \psi(y))$ and $f_2(y) = (f(y), \psi(y))$ one gets two distinct Γ-homomorphisms f_1 and f_2 from Y to $A \times G$ over G.

Now let (U, β) be a universal central Γ-equivariant extension of G and (W, α) a central Γ-equivariant extension of U. Since (U, β) is a universal central Γ-equivariant extension of G, the group U is Γ-perfect. We will show that $(W, \beta\alpha)$ is a central Γ-equivariant extension of G. Take $x_0 \in \operatorname{Ker} \beta\alpha$. Then $\alpha(x_0)$ belongs to the center of U and Γ acts trivially on $\operatorname{Ker} \beta\alpha$. In effect, if $\gamma : U \to W$ is a Γ-section of (W, α), one has ${}^{\sigma}(x_0 - \gamma\alpha(x_0)) = x_0 - \gamma\alpha(x_0)$; on the other hand, ${}^{\sigma}(x_0 - \gamma\alpha(x_0)) = {}^{\sigma}x_0 - \gamma^{\sigma}\alpha(x_0) = {}^{\sigma}x_0 - \gamma\alpha(x_0)$. Whence ${}^{\sigma}x_0 = x_0$, $x_0 \in \operatorname{Ker} \beta\alpha$, $\sigma \in \Gamma$. We obtain a Γ-homomorphism $h : W \to W$ over U given by $h(x) = x_0 x x_0^{-1}$, $x \in W$. Since $[W, W]_\Gamma$ is Γ-perfect, the restriction of h to the Γ-commutator subgroup $[W, W]_\Gamma$ of W is the identity map. Thus x_0 commutes with elements of $[W, W]_\Gamma$ implying x_0 belongs to the center of W, since W is generated by $[W, W]_\Gamma$ and $\operatorname{Ker} \alpha$.

We deduce that there is a unique morphism $(U, \beta) \to (W, \beta\alpha)$ over G, since (U, β) is universal. Therefore the composite αk of the induced Γ-homomorphism $k : U \to W$ over G with α is equal to the identity showing that (W, α) splits. $\square$

It should be noted that central Γ-equivariant extensions without Γ-splitting property were used in [26] to characterize universal central relative extensions of an epimorphism $\nu : \Gamma \to Q$ of groups.

Let $\tau : P \to G$ be a projective presentation of the Γ-group G and R denotes the kernel of τ. Then the Γ-homomorphism τ sends the normal Γ-subgroup $[P, R]_\Gamma$ of P to 1 and therefore induces a Γ-homomorphism $\tau' : [P, P]_\Gamma/[P, R]_\Gamma \to [G, G]_\Gamma$ which is surjective.

Theorem 17.

(i) *If G is Γ-perfect, then $([P, P]_\Gamma/[P, R]_\Gamma, \tau')$ is a universal central Γ-equivariant extension of G.*

(ii) *For any Γ-group G the group $(R \cap [P, P]_\Gamma)/[P, R]_\Gamma$ is isomorphic to $H_2^\Gamma(G)$.*

Proof. (i) The extension $(P/[P, R]_\Gamma, \tau)$ is a central Γ-equivariant extension of G. Clearly this extension is central. The group Γ acts trivially on $\operatorname{Ker} \tau = R/[P, R]_\Gamma$. Indeed the element ${}^{\sigma}x \cdot x^{-1}$, $x \in R$, belongs to $[P, R]_\Gamma$ for any $\sigma \in \Gamma$.

By Proposition 14 the group $[P, P]_\Gamma/[P, R]_\Gamma$ is Γ-perfect and maps onto G. By Corollary 15 it follows that the extension $[P, P]_\Gamma/[P, R]_\Gamma \to G$ has a Γ-equivariant splitting map. Let (X, ψ) be a central Γ-equivariant extension of G. Then there is a Γ-homomorphism $f : P \to X$ over G. Since (X, ψ) is a central Γ-equivariant extension of G, it is easy to see that $f([P, R]_\Gamma) = 1$. Therefore the restriction of f on $[P, P]_\Gamma$ induces a unique Γ-homomorphism $[P, P]_\Gamma/[P, R]_\Gamma \to X$ over G, since $[P, P]_\Gamma/[P, R]_\Gamma$ is a Γ-perfect group. We deduce that the short exact sequence

$$0 \to \big(R \cap [P, P]_\Gamma\big)/[P, R]_\Gamma \to [P, P]_\Gamma/[P, R]_\Gamma \to G \to 1$$

is a universal central Γ-extension of the Γ-perfect group G.

(ii) Let $P_* \to G$ be the $\mathcal{P}_{\mathcal{F}}$-projective resolution of the Γ-group G induced by the projective presentation $\tau : P \to G$, which we have defined in Section 2. The long exact homotopy sequence induced by the short exact sequence

$$1 \to \big([P_*, P_*]_\Gamma \to [G, G]_\Gamma\big) \to (P_* \to G) \to \big((P_*)^{ab}_\Gamma \to G^{ab}_\Gamma\big) \to 1$$

of augmented pseudosimplicial groups yields, according to Theorem 10(i), the exact sequence

$$0 \to H_2^\Gamma(G) \to \pi_0\big([P_*, P_*]_\Gamma\big) \xrightarrow{\tau'} G, \tag{6}$$

where the image of τ' is $[G, G]_\Gamma$. It is clear that $P_* \to G$ is simplicially exact. For the calculation of $\pi_0([P_*, P_*]_\Gamma)$ we will prove the equality

$$\operatorname{Ker} \partial_0^1 \cap [P_1, P_1]_\Gamma = \big[P_1, \operatorname{Ker} \partial_0^1\big]_\Gamma. \tag{7}$$

It suffices to show the inclusion $\operatorname{Ker} \partial_0^1 \cap [P_1, P_1]_\Gamma \subset [P_1, \operatorname{Ker} \partial_0^1]_\Gamma$, since the converse inclusion is obvious. First we will prove the equality

$$[P_1, P_1]_\Gamma = \big[P_1, \operatorname{Ker} \partial_0^1\big]_\Gamma \cdot \big([P_1, P_1]_\Gamma \cap s_0(P_0)\big). \tag{8}$$

In effect, if $[x, y]_\sigma \in [P_1, P_1]_\Gamma$, then for $x = d \cdot s_0(c)$, $y = b \cdot s_0(a)$ with $\partial_0^1(x) = c$, $\partial_0^1(y) = a$ and $b, d \in \operatorname{Ker} \partial_0^1$, one has

$$\begin{aligned}
[x, y]_\sigma &= \big[x, b \cdot s_0(a)\big] = x^\sigma b^\sigma\big(s_0(a)\big)x^{-1} \cdot s_0(a)^{-1} \cdot b^{-1}\\
&= [x, b]_\sigma \cdot bx^\sigma\big(s_0(a)\big) \cdot x^{-1} \cdot s_0(a)^{-1} \cdot b^{-1}\\
&= [x, b]_\sigma \cdot bds_0(c)^\sigma\big(s_0(a)\big)s_0(c)^{-1} \cdot d^{-1}s_0(a)^{-1} \cdot b^{-1}\\
&= [x, b]_\sigma \cdot bd\big[s_0(c), s_0(a)\big]_\sigma \cdot s_0(a)d^{-1}s_0\big(a^{-1}\big) \cdot b^{-1}\\
&= [x, b]_\sigma \cdot z \cdot \big[s_0(c), s_0(a)\big]_\sigma \cdot bd \cdot s_0(a)d^{-1}s_0(a)^{-1}b^{-1}\\
&= [x, b]_\sigma \cdot z \cdot \big[s_0(c), s_0(a)\big]_\sigma \cdot b \cdot \big[d, s_0(a)\big] \cdot b^{-1}
\end{aligned}$$

with $z \in [P_1, \operatorname{Ker} \partial_0^1]_\Gamma$, $b[d, s_0(a)]b^{-1} \in [P_1, \operatorname{Ker} \partial_0^1]$ and $[s_0(c), s_0(a)]_\sigma \in s_0(P_0)$.

It follows that $[x, y]_\Gamma$ belongs to $[P_1, \operatorname{Ker} \partial_0^1]_\Gamma \cdot ([P_1, P_1]_\Gamma \cap s_0(P_0))$ that proves the required equality (8).

Let $w \in [P_1, P_1]_\Gamma \cap \operatorname{Ker} \partial_0^1$. Then by (8) $w \in [P_1, \operatorname{Ker} \partial_0^1] \cdot ([P_1, P_1]_\Gamma \cap s_0(P_0))$, that is $w = w' \cdot x'$ with $w' \in [P_1, \operatorname{Ker} \partial_0^1]_\Gamma$ and $x' \in [P_1, P_1]_\Gamma \cap s_0(P_0)$. It follows that $x' = w'^{-1}w$ belongs to $\operatorname{Ker} \partial_0^1 \cap s_0(P_0) = 1$. Whence $w = w'$ and the equality (7) is proved.

Since $\partial_1^1(\operatorname{Ker} \partial_0^1) = R$, using the equality (7) one gets

$$\partial_1^1\big([P_1, P_1]_\Gamma \cap \operatorname{Ker} \partial_0^1\big) = \partial_1^1\big(\big[P_1, \operatorname{Ker} \partial_0^1\big]_\Gamma\big) = [P_0, R]_\Gamma$$

showing that $\pi_0([P_*, P_*]_\Gamma) = [P, P]_\Gamma / [P, R]_\Gamma$. The required isomorphism of the theorem follows now from the exact sequence (6). □

The result of Theorem 1.17(ii) will be called the equivariant Hopf formula. In a forthcoming paper equivariant higher Hopf type formulas will be established for the Γ-equivariant integral homology $H_*^\Gamma(G)$.

Corollary 18. *The group $R \cap [P, P]_\Gamma/[P, R]_\Gamma$ does not depend on the projective presentation $P \to G$ of the Γ-group G. A Γ-group has a universal central Γ-equivariant extension if and only if it is Γ-perfect.*

Theorem 19. *Let $1 \to N \to E \xrightarrow{\alpha} G \to 1$ be a short exact sequence of Γ-groups such that the Γ-homomorphism α has a Γ-excision and $\tau : P \to E$ a projective presentation of the Γ-group E. Then there is an exact sequence*

$$0 \to U \to H_2^\Gamma(E) \to H_2^\Gamma(G) \xrightarrow{\delta} N/[E, N]_\Gamma \to H_1^\Gamma(E) \to H_1^\Gamma(G) \to 0,$$

where U is the kernel of the Γ-homomorphism $[P, S]_\Gamma/[P, R]_\Gamma \to [E, N]_\Gamma$, $R = \operatorname{Ker} \tau$, $S = \operatorname{Ker} \alpha\tau$, induced by τ.

Proof. Using the exact sequence (6) the Γ-homomorphism α induces the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\gamma' \arrow[r] \arrow[d] & \operatorname{Ker}\gamma \arrow[r, "\eta"] \arrow[d] & N \arrow[r] \arrow[d] & N/\eta(\operatorname{Ker}\gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2^\Gamma(E) \arrow[r] \arrow[d, "\gamma'"'] & \pi_0([P_*, P_*]_\Gamma) \arrow[r] \arrow[d, "\gamma"'] & E \arrow[r] \arrow[d, "\alpha"] & H_1^\Gamma(E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2^\Gamma(G) \arrow[r] & \pi_0([P_*^G, P_*^G]_\Gamma) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H_1^\Gamma(G) \arrow[r] \arrow[d] & 0 \\
 & & 1 & 1 & 0 &
\end{tikzcd}$$

where $P_* \to E$ and $P_*^G \to G$ are $\mathcal{P}_\mathcal{F}$-projective resolutions of E and G induced by τ and $\alpha\tau$, respectively. Clearly $\operatorname{Ker}\gamma = [P, S]_\Gamma/[P, R]_\Gamma$. It follows that the image $(\eta(\operatorname{Ker}\gamma))$ of $\operatorname{Ker}\gamma$ is equal to $[E, N]_\Gamma$ and $\operatorname{Ker}\eta$ is isomorphic to $\operatorname{Ker}\gamma'$. Therefore this diagram yields the required exact sequence, where the connecting homomorphism δ is defined in a natural way. □

Theorem 1.19 is a generalized equivariant version of the well-known Stallings–Stammbach exact sequence in integral homology [31].

Theorem 20. *If A is a Γ-equivariant G-module, then there is a bijection*

$$E_\Gamma(G, A) \cong H_\Gamma^2(G, A).$$

Proof. We will use the isomorphism $H_\Gamma^2(G, A) \cong R_\mathcal{F}^1 \operatorname{Der}_\Gamma(G, A)$ (see Theorem 3) and show the bijection $E_\Gamma(G, A) \cong R_\mathcal{F}^1 \operatorname{Der}(G, A)$. Take the free cotriple resolution $F_* \to G$ of the Γ-group G which is simplicially exact. Then $F_1 \to F_0$ factors through $F_1 \to$

$M \to F_0$, $l_0\tau_1 = \partial_0^1$, $l_1\tau_1 = \partial_1^1$, where M is the simplicial kernel of τ and τ_1 is surjective. If $f \in \mathrm{Der}_\Gamma(F_1, A)$ such that $\sum f(-1)^i\partial_i^2 = 0$, then there is $f' \in \mathrm{Der}_\Gamma(M, A)$ such that $f'\tau_1 = f$ and $f'(\Delta) = 0$, $\Delta = \{(x, x),\ x \in F_0\}$. Denote by $\widetilde{\mathrm{Der}}_\Gamma(M, A)$ the subgroup of $\mathrm{Der}_\Gamma(M, A)$ consisting of Γ-derivations f with $f(\Delta) = 0$. Conversely, if $f' \in \widetilde{\mathrm{Der}}_\Gamma(M, A)$, then $\sum_i f'\tau_1(-1)^i\partial_i^2 = 0$. It follows that it is sufficient to establish a bijection with $\mathrm{Coker}\,\eta$, where $\eta : \mathrm{Der}_\Gamma(F_0, A) \to \widetilde{\mathrm{Der}}_\Gamma(M, A)$, $\eta = \mathrm{Der}_\Gamma(l_0, A) - \mathrm{Der}_\Gamma(l_1, A)$.

Define a map $\vartheta : E_\Gamma(G, A) \to H_\Gamma^2(G, A)$ as follows. If $[E] \in E_\Gamma(G, A)$, $E : 0 \to A \to B \to G \to 1$, since E is Γ-splitting, there is a commutative diagram

$$\begin{tikzcd}
M \arrow[r, shift left, "l_0^1"] \arrow[r, shift right, "l_1^1"'] \arrow[d, "f"'] & F_0 \arrow[r, "\tau"] \arrow[d, "\varphi"] & G \arrow[d, equal] \\
A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & G
\end{tikzcd}$$

where $F_0 = F(G)$, φ is a Γ-homomorphism and $f(x) = \varphi l_0(x) \cdot \varphi l_1^{-1}(x)$, $x \in M$. It is easily seen that f is a Γ-derivation such that $f(\Delta) = 0$ and define $\vartheta([E]) = [f]$. Conversely, if $[f] \in H_\Gamma^2(G, A)$, then take the semidirect product $A \rtimes G$ and introduce a relation

$$(a, x) \sim (a', x') \iff \tau(x) = \tau(x')$$

and $a \cdot f(x, x') = a'$. It is easy to check that this relation is a congruence and let C be the quotient $A \rtimes G/{\sim}$ which is a Γ-group. One gets a commutative diagram

$$\begin{tikzcd}
M \arrow[r, shift left, "l_0^1"] \arrow[r, shift right, "l_1^1"'] \arrow[d, "f"'] & F(G) \arrow[r, "\tau"] \arrow[d, "\psi"] & G \arrow[d, equal] \\
A \arrow[r, "\sigma"] & C \arrow[r, "\mu"] & G
\end{tikzcd}$$

where $\sigma(a) = [(a, x)]$, $\mu([(a, x)]) = \tau(x)$, $\psi(x) = [(o, x)]$. The extension $E : 0 \to A \xrightarrow{\sigma} C \xrightarrow{\mu} G \to 1$ is a Γ-equivariant extension of G by A, the splitting Γ-map is given by $\gamma(g) = \psi(0, |g|)$, $g \in G$. Define $\vartheta' : H_\Gamma^2(G, A) \to E_\Gamma(G, A)$ by $\vartheta'([f]) = [E]$. It is standard to show that ϑ and ϑ' are well defined and inverse to each other. $\square$

Note that Theorem 20 could also be proved using the corresponding factor set theory for Γ-groups and the bijection ϑ is in fact an isomorphism with respect to the "Baer sum" which could be introduced on $E_\Gamma(G, A)$. The description of higher Γ-equivariant group cohomology $H_\Gamma^{n+1}(G, A)$, $n \geqslant 2$, by extensions is also realizable using n-fold Γ-equivariant extensions of G by A, that means extensions of the form

$$E : 0 \to A \to X_1 \to X_2 \to \cdots \to X_n \to G \to 1,$$

where $0 \to A \to X_1 \to \mathrm{Im}\,\alpha_1 \to 0$, $0 \to \mathrm{Im}\,\alpha_{i-1} \to X_i \to \mathrm{Im}\,\alpha_i \to 0$, for $1 \leqslant i \leqslant n-1$, are proper short exact sequences of $G \rtimes \Gamma$-modules and $0 \to \mathrm{Im}\,\alpha_{n-1} \to X_n \to G \to 1$ is a Γ-equivariant extension of G by $\mathrm{Im}\,\alpha_{n-1}$, and by introducing the Γ-equivariant characteristic class $\chi(E)$ of a Γ-equivariant extension E of G by A, which means by constructing for any Γ-equivariant extension E of G by A:

$$E: 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} G \to 1$$

the exact sequence of $G \rtimes \Gamma$-modules

$$\chi(E) = 0 \to A \xrightarrow{\alpha'} F(B')/L \xrightarrow{\beta'} \mathbb{Z}(G) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $F(B')$ is the relatively free $G \rtimes \Gamma$-module generated by the Γ-set $B' = \{[b],\ b \in B,\ b \neq 0\}$ with $[0] = 0$, ${}^{\sigma}(x[b]) = {}^{\sigma}x[{}^{\sigma}b]$ for $x \in G$, $\sigma \in \Gamma$, $b \in B'$; L is its $G \rtimes \Gamma$-submodule generated by the elements $[b_1 + b_2] - \beta(b_1)[b_2] - [b_1]$, where $b_1, b_2 \in B$, and the $G \rtimes \Gamma$-homomorphisms α', β' are induced in a natural way by α and β, respectively.

Using the cochain description (see Section 1) of the Γ-equivariant cohomology of groups $H^*_\Gamma(G, A)$ the cup product can be defined, since the tensor product [1] of Γ-cochains is again a Γ-cochain. Therefore there is a cup product

$$H^p_\Gamma(G, A) \otimes H^q_\Gamma(G, B) \to H^{p+q}_\Gamma(G, A \otimes B)$$

for $p, q \geqslant 1$, endowing on $H^*_\Gamma(G, A)$ a structure of $H^*_\Gamma(G)$-module.

For any short exact sequence of $G \rtimes \Gamma$-modules

$$0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$$

such that β is Γ-splitting the sequences

$$0 \to P \otimes_{G \rtimes \Gamma} A' \to P \otimes_{G \rtimes \Gamma} A \to P \otimes_{G \rtimes \Gamma} A'' \to 0$$

and

$$0 \to \mathrm{Hom}_{G \rtimes \Gamma}(P, A') \to \mathrm{Hom}_{G \rtimes \Gamma}(P, A) \to \mathrm{Hom}_{G \rtimes \Gamma}(P, A'') \to 0$$

are exact for any relatively projective $G \rtimes \Gamma$-module P implying exact Γ-equivariant homology and cohomology sequences, respectively.

Let G be finite and let us consider the homomorphism $N_G : A \to A$, $N_G(a) = \sum_{s \in G} {}^{s}a$, where A is a $G \rtimes \Gamma$-module. Assume that Γ acts trivially on $N_G(A)$. Therefore one has the inclusion $N_G(A) \subset A^{G \rtimes \Gamma}$ and N_G induces a homomorphism $N^*_G : H^\Gamma_0(G, A) \to H^0_\Gamma(G, A)$. For this we have only to show that $N_G({}^{\sigma}a - a) = 0$. In effect $N_G(a) = {}^{\sigma}N_G(a) = {}^{\sigma}(\sum_{s \in G} {}^{s}a) = \sum^{\sigma}_{s \in G}({}^{s}a) = \sum_{s \in G} {}^{\sigma}s({}^{\sigma}a) = N_G({}^{\sigma}a)$. Under the afore given assumption we can define Γ-equivariant Tate cohomology groups $\check{H}^n_\Gamma(G, A)$, $n \in \mathbb{Z}$, by setting

$$\check{H}^n_\Gamma(G, A) = H^n_\Gamma(G, A), \quad n \geqslant 1,$$
$$\check{H}^0_\Gamma(G, A) = \mathrm{Ker}\, N^*_G, \qquad \check{H}^{-1}_\Gamma(G, A) = \mathrm{Coker}\, N^*_G,$$
$$\check{H}^{-n}_\Gamma(G, A) = H^\Gamma_{n-1}(G, A), \quad n \geqslant 2.$$

Proposition 21. *For any short exact sequence of $G \rtimes \Gamma$-modules*

$$E: 0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$$

such that β is Γ-splitting there is a long exact sequence of Γ-equivariant Tate cohomology groups

$$\begin{aligned}\cdots &\to \widehat{H}_{\Gamma}^{n-1}(G, A'') \to \widehat{H}_{\Gamma}^{n}(G, A') \to \widehat{H}_{\Gamma}^{n}(G, A)\\ &\to \widehat{H}_{\Gamma}^{n}(G, A'') \to \widehat{H}_{\Gamma}^{n+1}(G, A') \to \cdots.\end{aligned}$$

Proof. By using for E the exact Γ-equivariant homology and cohomology sequences the proof is similar to the classical case (see [1, Chapter IV, Theorem 6.1]). □

Note that it would be interesting to construct the Γ-equivariant versions of Farell cohomology theory of groups [5] and Vogel (co)homology theory of groups [18,35] generalizing the above defined Γ-equivariant Tate cohomology theory of groups.

5. Relationship with equivariant cohomology of topological spaces

Let X be a topological space. If a group G acts on X, then this action induces an action of G on the singular complex $S(X)$ of X given by $gf,\ f : \Delta_n \to X,\ f \in S(X)$, making $S(X)$ a chain complex of G-modules, where $g : X \to X$ is the homeomorphism of X induced by the action of $g \in G$.

Throughout out this section X is a G-space the group G acting on X properly. That means each point x of X belongs to some proper open subset of X. Recall that an open subset U of X is called proper with respect to the action of G, if ${}^{g}U \cap U$ is empty for all elements $g \neq 1$ of G [27].

We will assume also that a separate group Γ acts on G and X such that the following condition holds:

$$^{\sigma}\left({}^{g}x\right) = {}^{\sigma}g\left({}^{\sigma}x\right), \tag{9}$$

for $x \in X,\ g \in G,\ \sigma \in \Gamma$.

For example if X is a G-space, take $\Gamma = G$ with the actions of Γ on G by conjugation and on X as G is acting.

Then the augmented singular complex $S(X) \to Z$:

$$\cdots \to S_n(X) \to S_{n-1}(X) \to \cdots \to S_1(X) \to S_0(X) \to \mathbb{Z} \to 0 \tag{10}$$

is a chain complex of $G \rtimes \Gamma$-modules, where the groups G and Γ act trivially on $\mathbb{Z}$.

It will be said that the topological space X has the property (c), if the singular complex (10) is exact and any induced short exact sequence

$$0 \to \operatorname{Ker} \partial_k \to S_k(X) \to \operatorname{Im} \partial_k \to 0$$

is Γ-splitting for $k \geqslant 0$.

For instance X satisfies the condition (c) if either X is acyclic and Γ acts trivially on X or X is Γ-contractible, that is the identity map $1_X : X \to X$ is Γ-homotopic to a constant map $f_0 : X \to x_0 \in X$.

Theorem 22. *If a topological space X satisfies the condition* (c)*, then there is an isomorphism*

$$H_\Gamma^n(G, A) \cong H_\Gamma^n(X/G, A)$$

for $n \geqslant 0$, where A is an Abelian group on which G and Γ act trivially and $H_\Gamma^*(X/G, A)$ is the equivariant cohomology of topological spaces [4].

Proof. Since G acts properly on X, by Lemma 11.2 [27, Chapter IV] the sequence (10) is a chain complex of free G-modules. Therefore each $S_n(X)$, $n \geqslant 0$, is a $G \rtimes \Gamma$-module which is free as G-module and its basis consisting of singular nth simplexes is a Γ-subset. It follows that (10) is a relatively free $G \rtimes \Gamma$-resolution of $\mathbb{Z}$.

By Proposition 11.4 [27, Chapter IV] the canonical map $p : X \to X/G$ induces an isomorphism

$$p^*\colon \operatorname{Hom}_Z\bigl(S(X/G), A\bigr) \cong \operatorname{Hom}_G\bigl(S(X), A\bigr)$$

of chain complexes. Notice that the group Γ acts naturally on X/G and the map p is a Γ-map. Indeed, the action of Γ given by ${}^\sigma([x]) = [{}^\sigma x]$, $x \in X$, $\sigma \in \Gamma$, is well defined, thanks to the equality (9); if ${}^g x = y$ for some $g \in G$, then ${}^\sigma y = {}^\sigma({}^g x) = {}^{{}^\sigma g}({}^\sigma x)$ for any $\sigma \in \Gamma$. It is obvious that under so defined action of Γ the map p is a Γ-map. This implies that the isomorphism p^* induces an isomorphism

$$\operatorname{Hom}_\Gamma\bigl(S(X/G), A\bigr) \cong \operatorname{Hom}_{G \rtimes \Gamma}\bigl(S(X), A\bigr)$$

of cochain complexes giving the required isomorphism of the equivariant group cohomology of the space X/G with the equivariant group cohomology of the group G with coefficients in A with trivial actions of G and Γ. $\square$

Property (c) holds whenever Γ acts trivially on the group G and on the acyclic space X. Therefore Theorem 22 is an equivariant version of the classical Theorem 11.5 [27, Chapter IV].

6. Applications to algebraic K-theory

Let A be a unital ring and I its ideal. The group $E_n(A, I)$ is the normal subgroup of the group $E_n(A)$ of elementary n-matrices generated by I-elementary n-matrices ε_n of the form $\varepsilon_n = I_n + ae_{ij}$ with $a \in I$ and $i \neq j$ (see [2]). The group $E(A, I)$ is defined as $\varinjlim_n E_n(A, I)$. It is known [2] that

$$E_n(A, I) = \bigl[E_n(A), E_n(A, I)\bigr]$$

for $n \geqslant 3$.

It follows that the group $E_n(A, I)$, $n \geqslant 3$, which is not perfect in general, is a $E_n(A)$-perfect group, the group $E_n(A)$ acting on $E_n(A, I)$ by conjugation. Clearly the same is true for the relative elementary group $E(A, I)$, that is $E(A, I)$ is a $E(A)$-perfect group.

Now let A be a ring not necessarily with identity. Denote by A^+ the unital ring given by $A^+ = \{(a, n),\ a \in A,\ n \in \mathbb{Z}\}$ with usual sum and product

$$(a, n) \cdot (a', n') = (aa' + na' + n'a, nn').$$

One has a short split exact sequence of rings

$$0 \to A \xrightarrow{\sigma} A^+ \xrightarrow{\tau} \mathbb{Z} \to 0, \tag{11}$$

where $\sigma(a) = (a, 0)$, $\tau(a, n) = n$ and with splitting map $\gamma : Z \to A^+,\ \gamma(n) = (0, n)$.

By definition $E(A) = \operatorname{Ker} E(\tau)$, $\mathrm{St}(A) = \operatorname{Ker} \mathrm{St}(\tau)$ and $K_2(A) = \operatorname{Ker} K_2(\tau)$. Clearly $E(A) = E(A^+, A)$. Whence we have the following short exact sequence of short exact sequences induced by (11):

$$0 \to \bigl(0 \to K_2(A) \to \mathrm{St}(A) \xrightarrow{\beta} E(A) \to 1\bigr) \to \bigl(0 \to K_2\bigl(A^+\bigr)$$
$$\to \mathrm{St}\bigl(A^+\bigr) \xrightarrow{\beta^+} E\bigl(A^+\bigr) \to 1\bigr) \to \bigl(0 \to K_2(\mathbb{Z}) \to \mathrm{St}(\mathbb{Z}) \to E(\mathbb{Z}) \to 1\bigr) \to 0.$$

One gets an action of $\mathrm{St}(\mathbb{Z})$ on $\mathrm{St}(A)$ by conjugation using the splitting map $\mathrm{St}(\gamma)$. The group $E(A)$ and the general linear group $GL(A)$ also become $\mathrm{St}(\mathbb{Z})$-groups by conjugation via the map $\beta^+ \cdot \mathrm{St}(\gamma)$. Clearly β is a $\mathrm{St}(\mathbb{Z})$-homomorphism and $\mathrm{St}(\mathbb{Z})$ acts trivially on $K_2(A)$. Therefore the central extension of the group $E(A)$

$$0 \to K_2(A) \to \mathrm{St}(A) \xrightarrow{\beta} E(A) \to 1 \tag{12}$$

is a sequence of $\mathrm{St}(\mathbb{Z})$-groups.

There is a presentation of the Steinberg group $\mathrm{St}(A)$ as a $\mathrm{St}(\mathbb{Z})$-group as follows [32]. The generators x_{ij}^a for $i, j \geqslant 1,\ i \neq j,\ a \in A$, satisfy the relations

(1) $x_{ij}^a x_{ij}^b = x_{ij}^{a+b}$.
(2) $[x_{ij}^a, x_{km}^b] = 1,\ j \neq k,\ i \neq m$.
(3) $[x_{ij}^a, x_{jk}^b] = x_{ik}^{ab}$.
(4) $x_{ij}^z x_{ij}^a (x_{ij}^z)^{-1} = x_{ij}^a,\ z \in Z$.
(5) $x_{ij}^z x_{km}^b (x_{ij}^z)^{-1} = x_{km}^b,\ z \in Z,\ i \neq m,\ j \neq k$.
(6) $x_{ij}^z x_{jk}^b (x_{ij}^z)^{-1} = x_{ik}^{zb} x_{jk}^b,\ z \in Z,\ i \neq k$.
(7) $x_{ij}^z x_{ki}^b (x_{ij}^z)^{-1} = x_{kj}^{-zb} x_{ki}^b,\ z \in Z,\ j \neq k$.

Theorem 23. *The sequence* (12) *is a universal central* $\mathrm{St}(\mathbb{Z})$*-equivariant extension of* $E(A)$.

Proof. The map $\kappa^+ : \{e_{ij}^{a^+}\} \to \mathrm{St}(A^+)$ of elementary matrices given by $\kappa^+(e_{ij}^{a^+}) = x_{ij}^{a^+}$, $a^+ \in A^+$, $i \neq j$, induces a $\mathrm{St}(\mathbb{Z})$-map $\kappa : E(A) \to \mathrm{St}(A)$ such that $\beta\kappa = 1_{E(A)}$ by using the proof of a similar fact in the case of Corollary 15 and taking into account the action of $\mathrm{St}(\mathbb{Z})$ on the generators e_{ij}^a. We conclude that (12) is a central $\mathrm{St}(\mathbb{Z})$-equivariant extension of $E(A)$.

From the above condition (6) on generators one has $x_{ij}^1 x_{jk}^b (x_{ij}^1)^{-1} = x_{ik}^b x_{jk}^b$, $i \neq k$, $b \in A$. It follows that $[\mathrm{St}(A), \mathrm{St}(\mathbb{Z})] = \mathrm{St}(A)$ showing the groups $\mathrm{St}(A)$ and $E(A)$ are $\mathrm{St}(\mathbb{Z})$-perfect groups. Following Theorem 16, it remains to show that any central $\mathrm{St}(\mathbb{Z})$-equivariant extension of $\mathrm{St}(A)$ splits.

Let

$$o \to C \to Y \xrightarrow{\theta} \mathrm{St}(A) \to 1 \tag{13}$$

be a central St($\mathbb{Z}$)-equivariant extension of St(A) with splitting St($\mathbb{Z}$)-map $\bar{\gamma}:\mathrm{St}(A)\to Y$. Consider the following exact sequence

$$0\to C\to Y\rtimes \mathrm{St}(\mathbb{Z})\to \mathrm{St}(A)\rtimes \mathrm{St}(\mathbb{Z})\to 1 \tag{14}$$

induced by (13) and by the given action of St($\mathbb{Z}$) on this sequence. Clearly $\mathrm{St}(A^+)\cong \mathrm{St}(A)\rtimes \mathrm{St}(\mathbb{Z})$ and (14) is a central extension of $\mathrm{St}(A^+)$. Therefore (14) splits, since $\beta^+:\mathrm{St}(A^+)\to E(A^+)$ is a universal central extension of $E(A^+)$. Obviously $\mathrm{St}(A^+)$ is generated by x_{ij}^a and x_{kl}^z, $a\in A$, $z\in\mathbb{Z}$.

Following the proof of Theorem 5.10 [28], the section for (14) can be constructed as follows. For $i\neq j$ chose an index h distinct from i and j. Take the elements $y=\bar{\gamma}(x_{ih}^1)$ and $y'=\bar{\gamma}(x_{hj}^a)$. Then the needed section is defined by $x_{ij}^a\mapsto [y,y']=s_{ij}^a$, $x_{jk}^z\mapsto x_{jk}^z$. In [28] it is shown that this section does not depend on h and the elements s_{ij}^a, x_{jk}^z satisfy all the Steinberg relations. This implies that the elements s_{ij}^a, x_{jk}^z satisfy the relations (1)–(7) of the St($\mathbb{Z}$)-presentation of St(A). It follows that by sending x_{ij}^a to s_{ij}^a, $a\in A$, $i\neq j$, this map gives rise to the required splitting St($\mathbb{Z}$)-homomorphism $s:\mathrm{St}(A)\to Y$. □

Corollary 24. *There is an isomorphism*

$$K_2(A)\cong H_2^{\mathrm{St}(\mathbb{Z})}\big(E(A)\big)\quad \textit{and}\quad H_2^{\mathrm{St}(\mathbb{Z})}\big(\mathrm{St}(A)\big)=0$$

for any ring A.

Proof. The proof follows from Theorems 17, 23 and 20. □

Finally we provide the construction of an alternative equivariant algebraic K-theory K_*^Γ by using Γ-equivariant commutators.

Let $\Gamma_i(G)$, $i\geqslant 0$, be the lower Γ-equivariant central series of a Γ-group G [11], where $\Gamma_0(G)=G$, $\Gamma_1(G)=[G,G]_\Gamma$ and $\Gamma_{i+1}(G)=[G,\Gamma_i(G)]_\Gamma$, $i\geqslant 0$. First we give the equivariant version of the Z-completion functor $Z_\infty:\mathcal{G}\to\mathcal{G}$ defined on the category of groups [3], by setting

$$Z_\infty^\Gamma(G)=\varprojlim_i G/\Gamma_i(G),$$

where $\{\Gamma_i(G)\}$ is the Γ-equivariant lower central series of the Γ-group G. We obtain a covariant functor $Z_\infty^\Gamma:\mathcal{G}_\Gamma\to\mathcal{G}_\Gamma$.

Let A be a ring and Γ a group acting on the general linear group $GL(A)$. Define the Γ-equivariant algebraic K-functors by

$$K_n^\Gamma(A)=L_{n-1}^{\mathcal{P}_\mathcal{F}}Z_\infty^\Gamma\big(GL(A)\big),\quad n\geqslant 1, \tag{15}$$

where $\mathcal{P}_\mathcal{F}$ is the projective class induced by the free cotriple $\mathcal{F}$ in the category $\mathcal{G}_\Gamma$ of Γ-groups. This definition could actually be considered as an equivariant version of Quillen's algebraic K-theory, since in the case of the trivial action of Γ on $GL(A)$ it is proved that the left derived functors of the functor Z_∞ with respect to the projective class induced by the free cotriple in the category of groups are isomorphic to Quillen's K-groups up to dimension shift [23,20]. It would be interesting to establish the relationship of the afore defined equivariant algebraic K-theory (15) with the equivariant algebraic K-theory given in [13].

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