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ONE THEOREM ON WEIGHTS IN THE SPACE $L^{p(\cdot)}$

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Abstract. Recently we proved one theorem on weights in the space L_p and used it for the solution of the boundary value problem. Now we prove the same theorem in the space $L^{p(\cdot)}$. This will make it possible to prove the boundary value problem for $g \in L^{p(\cdot)}$ not applying Stein's theorem.

რეზიუმე. წონის ერთი თეორემა, რომელიც ჩვენს მიერ ადრე დამტკიცებული იყო L_p სივრცეში და გამოყენებული იყო სასაზღვრო ამოცანის ამოსახსნელად, ამჟამად მტკიცდება $L^{p(\cdot)}$-სთვის. ეს საშუალებას მოგვცემს შემდგომში ამოვხსნათ სასაზღვრო ამოცანა $g \in L^{p(\cdot)}$-სთვის სტეინის თეორემის გამოყენების გარეშე.

To solve the boundary value problem of linear conjugation, as usual it, suffices to consider the product of two or several weight functions. Towards this end, we use Stein's theorem (see, for e.g., [1] and the works dealing with the boundary value problems).

In [2] and [3], in considering boundary value problem on an open Carleson arc, we used for consideration of the weight product another, more convenient for that particular case, way which can be easily extended to the space $L^{p(\cdot)}$.

Definitions and Some Information

The operator

$$S\varphi \equiv S_\Gamma \varphi \equiv \frac{1}{\pi i} \int\limits_G \frac{\varphi(t)}{t-\tau} dt, \tag{1}$$

is called singular integral operator. Here Γ is a rectifiable line, and integration is understood in the sense of the Cauchy principal value. As usual, $L_p(\Gamma)$, $p > 1$ denotes the Lebesgue space. The necesary and sufficient condition on Γ under which the operator (1) is bounded in $L_p(\Gamma)$, $p > 1$ is well-known (David's theorem). The lineas for which this condition is fulfilled are called Carleson lines, or regular lines, and in this case we write $\Gamma \in R$.

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Recently, singular integrals and related boundary value problems are being actively studied in the spaces $L^{p(\cdot)}$ (see [5], [6], etc.).

We say ([7]) that $\varphi \in L_p^{p(\cdot)}$, if

$$I_p(\varphi) \equiv \int\limits_{\Gamma} |\varphi(t)|^{p(t)} |dt| < \infty, \tag{2}$$

where $\varphi(t)$ and $p(t)$ are the measurable functions, and

$$1 < p_0 \leq p(t) \leq P < \infty. \tag{3}$$

The norm on the set with the condition (2) is defined as follows:

$$\|\varphi\|_{L_\Gamma^{p(\cdot)}} = \inf_{\lambda} \Big\{ \lambda > 0, \quad I_p\Big(\frac{\varphi}{\lambda}\Big) \leq 1 \Big\}.$$

By $W_{p(\cdot)}(\Gamma; S)$ we denote the set of positive measurable on Γ functions $\rho(t)$ for which the following conditions are fulfilled:

(i) if $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$, $p(t)$ and $q(t)$ satisfy (3), then $\rho \in L_\Gamma^{p(\cdot)}$, $\rho^{-1} \in L_\Gamma^{q(\cdot)}$;

(ii) $\|\rho S_\Gamma \rho^{-1} \varphi\|_{L_\Gamma^{p(\cdot)}} \leq M_p \|\varphi\|_{L_\Gamma^{p(\cdot)}}$.

In these cases $\rho(t)$ stands for the weight. It is well-known that under certain assumptions on $p(t)$, for $\Gamma \in R$ and $\rho = 1$ the condition (ii) is fulfilled, i.e., the operator S is bounded in $L_\Gamma^{p(\cdot)}$, although this fact will be neglected.

In the sequel, the use will be made of the well-known Hölder inequality

$$\int\limits_{\Gamma} \varphi(t) f(t) dt \leq c \|\varphi\|_{L_\Gamma^{p(\cdot)}} \cdot \|f\|_{L_\Gamma^{q(\cdot)}}, \tag{4}$$

where $p(t)$ and $q(t)$ satisfy the condition (i).

In what follows, we will need the following notation. By Γ_{ab} we denote a continuous arc with the ends a and b which is directed from a to b. If we are to emphasize whether the points a and b belong to that arc, we write $\Gamma_{(a,b)}$, $\Gamma_{[a,b)}$, $\Gamma_{(a,b]}$ $\Gamma_{[a,b]}$. If $\Gamma_{ab} \subset \Gamma$, and Γ is a closed line, then we assume that the direction from a to b coincides with the positive direction on Γ. By $\chi(\Gamma_{ab})$ we denote the characteristic function of the set Γ_{ab}.

Next, by γ_t we denote the arc on Γ, such that $t \in \Gamma$, $\gamma_t = \Gamma_{(\alpha,\beta)}$, if t is an interior point on Γ. Moreover, $\gamma_a = \Gamma_{[a,a_1)}$, $\gamma_b = \Gamma_{(b_1 b]}$ if $\Gamma = \Gamma_{ab}$.

The Main Result

Theorem. *Let Γ be a simple finite rectifiable line* (*closed, or unclosed*), *and let $\rho(t) > 0$ be a measurable function on Γ. If for every point $t \in \Gamma$ we can choose γ_t, such that $\rho \in W_{p(\cdot)}(\gamma_t, S_{\gamma_t})$, then $\rho \in W_{p(\cdot)}(\Gamma; S_\Gamma)$.*

The proof of the theorem is based on the following two lemmas.

Lemma 1. *If the set of points of the continuous arc Γ_{ab} is represented in the form $\gamma_{ab} = \Gamma_{[a,c]} \cup \Gamma_{[d,b]}$, $c \neq d$ and $\rho \in W_{p(\cdot)}(\Gamma_{ac}; S_{ac}) \cap W_{p(\cdot)}(\Gamma_{db} S_{db})$, then $\rho \in W_{p(\cdot)}(\Gamma; S_\Gamma)$.*

Here and in the sequel, instead of $S_{\Gamma_{ac}}$ we will write S_{ac}.

Lemma 2. *If Γ is a simple closed contour, a, b, c, d are different points on Γ, and the direction of the arcs Γ_{ab}, Γ_{bc}, Γ_{cd} corresponds to the positive direction on Γ, then it follows from the conditions $\rho \in W_{p(\cdot)}(\Gamma_{dc}, S_{\Gamma_{dc}})$ and $\rho \in W_{p(\cdot)}(\Gamma_{ba}, S_{\Gamma_{ba}})$ that $\rho \in W_{p(\cdot)}(\Gamma; S_\Gamma)$.*

Proof of Lemma 1. In what follows, we will instead of $S_{\Gamma_{ab}}$ and S_{ab} write $\|\cdot\|_{L_\Gamma^{p(\cdot)}}$ and $\|\cdot\|_{L_{ab}^{p(\cdot)}}$, respectively.

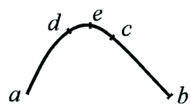

Fig. 1

Take the point $e \in \Gamma_{dc}$. Obviously, we have

$$\begin{aligned}\|\rho S_{ab}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} &\le \|\rho S_{ae}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} + \|\rho S_{eb}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} \le \\ \le \|\chi_{ac}\rho S_{ae}\rho^{-1}\varphi\|_{L_{ab}{}^{p(\cdot)}} &+ \|\chi_{cb}\rho S_{ae}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} + \|\chi_{ad}\rho S_{eb}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} + \\ &+\|\chi_{db}\rho S_{eb}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} = I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{5}$$

where

$$\begin{aligned} I_1 \equiv \|\chi_{ac}\rho S_{ae}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} &= \|\rho S_{ae}\rho^{-1}\varphi\|_{L_{ac}^{p(\cdot)}} \le \\ \le C_1\|\varphi\|_{L_{ac}^{p(\cdot)}} &= C_1\|\chi_{ac}\varphi\|_{L_{ab}^{p(\cdot)}} \\ I_2 \equiv \|\chi_{cb}\rho S_{ae}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}}&. \end{aligned} \tag{6}$$

If D stands for the distance between the sets Γ_{cb} and Γ_{ae}, then we obtain

$$|\chi_{cb}\rho S_{ae}\rho^{-1}\varphi| \le \frac{1}{D}|\chi_{cb}\rho\frac{1}{\pi i}\int\limits_{\Gamma_{ae}} \rho^{-1}\varphi|.$$

By virtue of (4), we find that

$$\int\limits_{\Gamma_{ae}} \rho^{-1}\varphi dt \le \|\varphi\|_{L_{ae}^{p(\cdot)}} \cdot \|\rho^{-1}\|_{L_{ae}^{q(\cdot)}},$$

where p and q satisfy the condition (i). Then we obviously obtain

$$|\chi_{cb}\rho S_{\Gamma_{ae}\rho^{-1}}\varphi| \le \frac{1}{\pi D}|\chi_{cb}\rho| \cdot \|\rho^{-1}\|_{L_{ae}^{p(\cdot)}}\|\varphi\|_{L_{ae}^{p(\cdot)}},$$

whence

$$I_2 \le C_2 \|\varphi\|_{L_{ab}^{p(\cdot)}}. \tag{7}$$

Analogously to the estimate I_2, we have

$$I_3 \equiv \|\chi_{ad}\rho S_{eb}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} \le C_3 \|\varphi\|_{L_{ab}^{p(\cdot)}}, \tag{8}$$

and analogously to the estimate I_1,

$$I_4 \equiv \|\chi_{db}\rho S_{eb}\rho^{-1}\varphi\| \le C_4 \|\varphi\|_{L_{ab}^{p(\cdot)}}. \tag{9}$$

Inserting (6), (7), (8) and (9) into (5), we get

$$\|\rho S_{ab}\rho^{-1}\varphi\|_{L_{ab}^{p(\cdot)}} \le C\|\varphi\|_{L_{ab}^{p(\cdot)}}.$$

Thus the lemma is complete. □

Proof of Lemma 2. Take the points $e_1 \in \Gamma_{da}$ and $e_2 \in \Gamma_{bc}$.

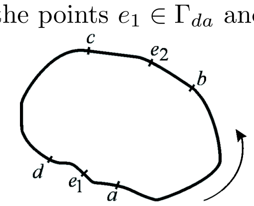

Fig. 2

For the brevity, by $S_{e_1ae_2}$ we denote the integral along $\gamma_{e_1e_2}$ when $a \in \Gamma_{e_1e_2}$.

$$\begin{aligned}\|\rho S_\Gamma \rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}} = \|\rho S_{e_1ae_2}\rho^{-1}\varphi + \rho S_{e_2ce_1}\rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}} \le \\ \le I_1' + I_2' + I_3' + I_4',\end{aligned}$$

where

$$\begin{aligned} I_1' &\equiv \|\chi_{dac}\rho S_{e_1ae_2}\rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}} \\ I_2' &\equiv \|\chi_{cd}\rho S_{e_1ae_2}\rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}} \\ I_3' &\equiv \|\chi_{bca}\rho S_{e_2ce_1}\rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}} \\ I_4' &\equiv \|\chi_{ab}\rho S_{e_2cc_1}\rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}}. \end{aligned}$$

Here I_1' and I_3' are the expressions of the same type as I_1 and I_4, and I_2' and I_4' are the same as I_2. Therefore we have

$$\|\rho S_\Gamma \rho^{-1}\varphi\|_{L_\Gamma^{p(\cdot)}} \le C\|\varphi\|_{L_\Gamma^{p(c)}}.$$

Thus the lemma is complete. □

Proof of Theorem. By the condition of the theorem, to every interior point $t \in \Gamma$ there correspond open (but in the case of endpoints, if Γ is unclosed, there correspond semi-open) arc-wise intervals covering the contour Γ. Obviously, from that covering we can distinguish a finite covering of Γ by the arcs $\Gamma_{a_k b_k}$, $k = 1, 2, \dots, n$ for which we have $\rho \in W_{p(\cdot)}(\Gamma_{a_k b_k, S_{a_k b_k}})$. Of these arcs we take two arbitrary ones, satisfying Lemma 1. According to our lemma, the number of intervals reduces by one. Thus, after a finite number of steps, if the arc Γ_{ab} is unclosed, we arrive at the whole contour, but if the contour is closed, we obtain the sutuation appearing in Lemma 2, which proves the theorem. □

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