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ON THE BOUNDARY VALUE PROBLEM OF LINEAR CONJUGATION FOR UNCLOSED CARLWSON ARCS IN THE SPACES $L_{p(\cdot)}$

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Abstract. On the condition (5) the boundary value problem (4) is solved when $g \in L_{p(.)}$.

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We say that Γ is a regular rectifiable line and write $\Gamma \in R$ if a singular integral

$$(S\varphi)(\tau) \equiv \frac{1}{\pi i} \int\limits_{\Gamma} \frac{\varphi(t)}{t-\tau}\, dt \tag{1}$$

forms a bounded operator in the Lebesgue space $L_p(\Gamma)$, $p > 1$. As is known, in [1], the necessary and sufficient condition for $\Gamma \in R$ is given.

In the sequel, we will need the space $L_{p(\cdot)}(\Gamma)$. We say that $f \in L_{p(\cdot)}(\Gamma)$, or $f \in L_{p(\cdot)}$, if

$$I_p(\Gamma) \equiv \int\limits_{\Gamma} |f(t)|^{p(t)}\, dt < \infty,$$

where $p(t) : \Gamma \to [1; \infty)$. The norm on the above set is defined as follows:

$$\|f\|_{L_{p(\cdot)}} = \inf \Big\{ \lambda > 0 : \ I_p\Big(\frac{f}{\lambda}\Big) \le 1 \Big\}.$$

In [2], it is shown that if $p(t)$ satisfies the condition

$$\big|p(t_1) - p(t_2)\big| \le \frac{A}{\ln \frac{1}{|t_1 - t_2|}}\,, \quad |t_1 - t_2| \le \frac{1}{2}\,, \quad t_1,\, t_2 \in \Gamma \tag{2}$$

and $\Gamma \in R$, then the operator S defined by formula (1) is bounded in $L_{p(\cdot)}$.

The basic properties cited in [3] and [4] for the integral (1) in $L_{p(\cdot)}$ made it possible to investigate various boundary value problems.

By $K\varphi$, or by $K_\Gamma\varphi$, we denote the Cauchy type integral

$$K\varphi \equiv \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(t)}{t-\tau}\, dt. \tag{3}$$

A class of functions representable by formula (3) for $\varphi \in L_{p(\cdot)}(\Gamma)$ we denote by $\{K_\Gamma^{p(\cdot)}\}$, and that of functions representable in the form $K_\Gamma\varphi + P$, where P is a polynomial, we denote by $\{K_\Gamma^{p(\cdot)} + P\}$.

The boundary value problem of linear conjugation is called the problem which is formulated as follows: find a function $\phi(z) \in \{K^{p(\cdot)} + P\}$, $\phi(\infty) = 0$ satisfying on Γ the boundary condition

$$\phi^+(t) + G(t)\phi^-(t) = g(t), \quad t \in \Gamma, \tag{4}$$

where G and g are the given functions.

We consider this problem on the unclosed simple arc with the ends a and b; the arc is directed from a to b and denote it by Γ_{ab}.

Assume that $\Gamma_{ab} \in R$, $G(t)$ is continuous on Γ_{ab}, $g(t) \in L_{p(\cdot)}$ and $p(t)$ satisfies the condition (2).

To solve the problem, we will need the result due to Seifullaev [5] which after [1] can be formulated as follows: if $\Gamma_{ab} \in R$, then there exist finite limits

$$\overline{\lim_{t\to c}} \frac{\arg(t-c)}{|\ln|t-c||} = \overline{\Delta}_c, \quad \underline{\lim_{t\to c}} \frac{\arg(t-c)}{|\ln|t-c||} = \overline{\Delta}_c, \quad c = a, b.$$

In the present work we assume that $\overline{\Delta}_c = \underline{\Delta}_c$, i.e., there exist the limits for $c = a$ and $c = b$:

$$\lim_{t\to c} \frac{\arg(t-c)}{|\ln|t-c||} = \Delta_c. \tag{5}$$

We represent the function $G(t)$ in the form $G = G_1 \cdot G_2$, where

$$\left.\begin{aligned} G_1(t) &\equiv \exp\left[\ln G(t) - \ln G(a) - \frac{\ln G(b) - \ln G(a)}{b-a}(b-a)\right] \equiv \exp\omega_1, \\ G_2(t) &\equiv \exp\left[\ln G(a) + \frac{\ln G(b) - \ln G(a)}{b-a}(b-a)\right] \equiv \exp\omega_2, \end{aligned}\right\} \quad t \in \Gamma_{ab}$$

and complement the arc Γ_{ab} to the closed Jordan line Γ of the class R (what is, as is known [6], always possible) and define $G_1(t) = G_2(t) = 1$ for $t \in \Gamma/\Gamma_{ab}$.

Using equality (5), just in the same way as in [7], in the neighborhood of the points a and b we obtain

$$\exp(K\omega_2)(z) = \phi_2(z)\exp|z-a|^{\beta(a)}|z-b|^{\beta(b)},$$

where $0 \neq m < \phi_c(z) < M$,

$$\beta(a) = \frac{\ln|G(a)|}{2\pi}\Delta_a - \frac{\arg G(a)}{2\pi}$$

and

$$\beta(b) = -\frac{\ln|G(b)|}{2\pi}\Delta_b - \frac{\arg G(b)}{2\pi}.$$

Assume that $\beta(a) \neq \frac{2\pi}{p(a)} \pmod{2\pi}$, $\beta(b) \neq \frac{2\pi}{p(b)} \pmod{2\pi}$. We choose integers $\varkappa_a$ and $\varkappa_b$ such that

$$\begin{gathered} \beta(a) = \varkappa_a + \alpha_a, \quad \beta(b) = \varkappa_b + \alpha_b, \\ -\frac{1}{p(a)} < \alpha_a < \frac{1}{q(a)}, \quad -\frac{1}{p(b)} < \alpha_b < \frac{1}{q(b)}, \\ q(c) = p(c)\cdot(p(c)-1)^{-1}, \quad c = a, b. \end{gathered} \tag{6}$$

Denote

$$\varkappa = -\varkappa_a - \varkappa_b \tag{7}$$

and $X_2(z) \equiv (z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}\exp(K\omega_2)(z)$.

Since the function $p(t)$ is continuous, we find that $X_2^{\pm}(t) \in L_{p(\cdot)}(\Gamma)$, $(X_2^{-1})^{\pm} \in L_{q(\cdot)}(\Gamma)$. Next, it can be shown that $X_2(z) \in \{K_{p(\cdot)} + P\}$, $X_2^{-1}(z) \in \{K_{q(\cdot)} + P\}$. Thus $X_2(z)$ is a factor function of G_2.

Consider the operator

$$A \equiv P + GQ, \quad P \equiv I + S, \quad Q \equiv I - S.$$

We choose a rational function $r(t)$ such that

$$\Big|\frac{G_1(t) - r(t)}{r(t)}\Big| < \frac{1}{\|Q\|} \ \text{ for } \ t \in \Gamma.$$

Let $r^{\pm}$ be factorization of r. For $\varkappa = 0$, the operator A can be represented as

$$A = r^+X_2^+\Big(I + \frac{G-r}{r}\,Q\Big)\Big(\frac{1}{r^+X_2^+}\,P + \frac{1}{r^-X_2^-}\,Q\Big)$$

and its inverse as

$$A^{-1} = (r^+X_2^+P + r^-X_2^-Q)\Big(I + \frac{G_1 - r}{r}\,Q\Big)^{-1}(r^+X_2^+)^{-1}.$$

This implies that the boundary value problem (4) for $g \in L_{p(\cdot)}(\Gamma_{ab})$ and $\varkappa = 0$ has a unique, vanishing at infinity solution. If we denote

$$X(z) \equiv (z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}\exp K\ln G,$$

then this solution can be written by the formula

$$\phi(z) = X(z)\Big(K_{\Gamma_{ab}}\,\frac{g}{X^+}\Big)(z). \tag{8}$$

Taking into account (8), it is not difficult to conclude that $X^+(t)$ is the weighted function for S, now for any $\varkappa$.

From the above reasoning we arrive at the following

Theorem 1. *Let Γ_{ab} be the simple unclosed arc and $\Gamma_{ab} \in R$. Moreover, let $G(t)$ be the continuous function on Γ_{ab}. Then:*

(a) *the index $\varkappa$ of the boundary value problem* (4) *is defined by formulas* (6) *and* (7)*;*

(b) *solutions* (*if any*) *are given by the formula,*

$$\phi(z) = X(z)K_{\Gamma_{ab}}\Big(\frac{g}{X^+}\Big)(z) + X(z)P_{\varkappa-1}(z),$$

where $P_n(z)$ for $n > 0$ is an arbitrary polynomial of the n-th degree, and $P_n(z) \equiv 0$ for $n \le 0$;

(c) *if $\varkappa = 0$, then the problem has a unique solution. If $\varkappa > 0$, then the problem has $\varkappa$ linear independent solutions, while if $\varkappa < 0$, then for the solvability of the problem it is necessary and sufficient that*

$$\int\limits_{\Gamma_{ab}} \frac{t^k g(t)}{X^+(t)}\,dt = 0, \quad k = 0, 1, \dots, -\varkappa - 1.$$

Remark. The results of Theorem 1 differ from the classical ones by especially the formula for the index.

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