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MATHEMATICS

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On Generalization of One Quasivariational Inequality

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ABSTRACT. **The unilateral quasivariational inequality (the so-called Implicit Signorini Problem) for a second order elliptic coercive form is stated in more general terms, also with bilateral boundary restrictions. The similar problem is also considered with domain unilateral and bilateral restrictions with Neumann condition on boundary. The unique solvability of these problems is proved. For the unilateral problems the same is proved in the noncoercive case.**

Key words: implicit Signorini problem, quasivariational inequality, unilateral and bilateral restrictions, supersolution, coercivity property.

Let $\Omega\subset R^n$ be a bounded domain with the boundary Γ, ν be the outward normal to Ω, $\nu\in C_1$; $H^s(\Omega)$ and $H^s(\Gamma)$ be the real Sobolev spaces. The norm in $H^1(\Omega)$ we denote as $\|\bullet\|_1$. Let us define the bilinear form on the space $H^1(\Omega)\times H^1(\Omega)$ as follows:

$$a(u,v)=\int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx+\int_\Omega a_i\frac{\partial u}{\partial x_i}v\,dx+\int_\Omega a_0 u\,v\,dx,$$

$$a_{ij},a_i,a_0\in L^\infty\left(R^n\right),\ \ a_{ij}\xi_i\xi_j\geq\beta|\xi|^2,\ \ \beta=const>0,\ \ \forall\xi\in R^n, \qquad (1)$$

$$a_0\geq a^0,\ \ a^0=const>0.$$

Suppose that the form (1) is coercive, i. e.,

$$a(u,u)\geq\alpha\,\|u\|_1^2,\qquad \alpha=const>0,\ \forall\, u\in H^1(\Omega). \qquad (2)$$

Define the following operators:

$$A(x,\partial)=-\sum_{i,j=1}^n\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)+\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}+a_0;\qquad \frac{\partial}{\partial\nu_A}=\sum_{i,j=1}^n a_{ij}\nu_j\frac{\partial}{\partial x_i}. \qquad (3)$$

As it is known, if $u\in H^1(\Omega)$ and $Au\in L_2(\Omega)$ then $\frac{\partial u}{\partial\nu_A}\in H^{-\frac{1}{2}}(\Gamma)$ and the following Green formula is true [1]:

$$a(u,v)=\left\langle\frac{\partial u}{\partial\nu_A},v\right\rangle_\Gamma+\int_\Omega Au\,v\,dx,\qquad \forall v\in H^1(\Omega). \qquad (4)$$

Here $\langle\cdot,\cdot\rangle_\Gamma$ is the relation of duality between the spaces $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

Let us state an implicit Signorini problem as a quasivariational inequality, considered by Lions, Bensoussan and Mosco [1].

Find u such that

$$u\in K(u);\ \ a(u,v-u)>\int_\Omega f(v-u)\,dx,\qquad f\in L_2(\Omega),\ \forall\ v\in K(u). \qquad (5)$$

$$K(u)=\left\{ v\in H^1(\Omega),\ v|_\Gamma \geq h-\left\langle \frac{\partial u}{\partial \nu_A},\varphi \right\rangle_\Gamma \right\},\qquad h,\varphi\in H^{\frac{1}{2}}(\Gamma),\ \varphi\geq 0. \tag{6}$$

Remark 1. Here and in what follows, the inequalities between $L_2(\Omega)$ or $L_2(\Gamma)$ functions are understood almost everywhere.

For problem (5), (6) the solvability was known [1], the uniqueness is proved in [2]. Further we generalize this problem and afterwards we set it with domain restrictions. First we give one definition:

Definition. Let $f\in L_2(\Omega)$;

1) we say that $v\in H^1(\Omega)$ is f-supersolution, if

$$a(v,\phi)\geq \int_\Omega f\phi\,dx, \qquad \forall\ \phi\in H^1(\Omega),\ \phi\geq 0.$$

2) we say that $w\in H^{\frac{1}{2}}(\Gamma)$ is f-supersolution, if

$$AW=f; \qquad W|_\Gamma = w,$$

$$a(W,\phi)\geq \int_\Omega f\phi\,dx, \qquad \forall\ \phi\in H^1(\Omega),\ \phi|_\Gamma \geq 0.$$

If $f=0$, we simply write "supersolution".

Let us state the implicit Signorini problem (5), (6) in more general terms, i.e. consider the variational inequality (5) on the set

$$K(u)=\left\{ v\in H^1(\Omega),\ v|_\Gamma \geq h-\left\langle \frac{\partial u}{\partial \nu_A},\varphi \right\rangle_\Gamma \xi \right\},\qquad h,\xi,\varphi\in H^{\frac{1}{2}}(\Gamma),\ \varphi\geq 0, \tag{7}$$

where ξ is a supersolution.

It can be proved, that $\xi=1$ is a supersolution, i.e., the problem (5), (7) represents the generalization of the problem (5), (6).

Set the problem (5), (6) with domain restrictions.

$$u\in K(u),\quad a(u,v-u)\geq \int_\Omega f(v-u)\,dx, \qquad f\in L_2(\Omega),\ \forall\, v\in K(u). \tag{8}$$

$$K(u)=\left\{ v\in H^1(\Omega),\ v\geq g-a(u,\phi)\ \ in\ \Omega \right\} \quad g,\phi\in H^1(\Omega),\ \phi\geq 0. \tag{9}$$

For the problem (5), (6) the equality $Au=f$ in the distributional sense is proved. Thus, in view of Green formula (4), the boundary restriction in the set (6) is similar to the domain restriction involved in the definition of the set (9).

Problem (8), (9) can be generalized as the problem (5), (6), i. e., variational inequality (8) can be considered on the set

$$K(u)=\left\{ v\in H^1(\Omega),\ v\geq g-a(u,\phi)\zeta \ \ in\ \Omega \right\} \quad g,\zeta,\phi\in H^1(\Omega),\ \phi\geq 0. \tag{10}$$

Here ζ is a supersolution too.

Problems (5),(6); (5),(7); (8),(9) and (8),(10) have physical meanings in Thermostatics and Hydrostatics. They express control of temperature (fluid pressure) regulated in domain or on boundary.

In the proof of the existence and uniqueness theorems for problems (5), (7) and (8), (10) the following general lemma is very essential.

Lemma. *Let* $f_i \in L_2(\Omega)$, $h_i \in H^1(\Omega)$, $\left(h_i \in H^{\frac{1}{2}}(\Gamma) \right)$, *and* u_i, $i=1,2$ *be solutions of the following variational inequalities:*

$$u_i \in K_i, \quad a(u_i, v-u_i) \geq \int_\Omega f(v-u_i)\,dx, \qquad \forall v \in K_i,$$

$$K_i = \left\{ v \in H^1(\Omega), \quad v \geq h_i \quad in\ \Omega \right\} \qquad i=1,2,$$

$$\left(K_i = \left\{ v \in H^1(\Omega), \quad v|_\Gamma \geq h_i \right\} \right).$$

If h_1*-*h_2 *is an* f_1*-*f_2*-supersolution, then* u_1*-*u_2 *is an* f_1*-*f_2*-supersolution as well.*

Remark 2. Since the form (1) is coercive, then the variational inequalities in Lemma have a unique solution (see e. g. [3]) .

This lemma gives us opportunity to prove the following

Theorem 1. *Problems (5), (6); (5), (7); (8), (9) and (8), (10) have unique solutions which are stable with respect to the data and to the coefficients of the form (1).*

We can consider the problems (5), (7) and (8), (10) in the noncoercive case, i. e., for the form (1) without condition (2). If the essential supremums of the data f,h,g are finite, then the unique solvability of the problems (5), (7) and (8), (10) are proved.

Set the problems (5), (7) and (8), (10) with the bilateral restrictions:

$$u \in K(u), \quad a(u, v-u) \geq \int_\Omega f(v-u)\,dx, \qquad f \in L_2(\Omega), \forall v \in K(u), \tag{11}$$

$$K(u) = \left\{ v \in H^1(\Omega), \ g + \left\langle \frac{\partial u}{\partial \nu_A}, \varphi \right\rangle_\Gamma \eta \geq v|_\Gamma \geq h - \left\langle \frac{\partial u}{\partial \nu_A}, \varphi \right\rangle_\Gamma \xi \right\}, \tag{12}$$

$$\left\langle \frac{\partial u}{\partial \nu_A}, \varphi \right\rangle_\Gamma \geq 0, \tag{13}$$

$h, g, \varphi, \xi, \eta \in H^{\frac{1}{2}}(\Gamma)$, $g \geq \varphi + u_0 \geq h$, ξ and η are supersolutions, where u_0 is the solution of the following variational inequality

$$u_0 \in H^1(\Omega),\ g \geq u_0|_\Gamma \geq h, \quad a(u_0, v-u_0) \geq \int_\Omega f(v-u_0)\,dx, \qquad \forall v \in H^1(\Omega),\ g \geq v|_\Gamma \geq h.$$

State the same problem with domain restrictions:

$$u \in K(u), \quad a(u, v-u) \geq \int_\Omega f(v-u)\,dx, \qquad f \in L_2(\Omega), \forall v \in K(u). \tag{14}$$

$$K(u) = \left\{ v \in H^1(\Omega), \ \mathrm{g} + a(u,\Phi)\varsigma \geq v \geq \mathrm{h} - a(u,\Phi)\zeta \quad in\ \Omega \right\} \tag{15}$$

$$a(u,\Phi) \geq \int_\Omega f\Phi\,dx, \tag{16}$$

$$\mathrm{h}, \mathrm{g}, \Phi, \Psi, \zeta, \varsigma \in H^1(\Omega)$$

$\mathrm{g} + \int_\Omega f\Phi\,dx\,\varsigma \geq \Phi + u_0 \geq \mathrm{h} - \int_\Omega f\Phi\,dx\,\zeta$ in Ω, ζ and ς are supersolutions.

where u_0 is the solution of the following variational inequality

$$a(u_0, v-u_0) \geq \int_\Omega f(v-u_0)\,dx, \quad \forall v \in H^1(\Omega),\ g+\int_\Omega f\Phi\,dx\,\varsigma \geq v \geq h-\int_\Omega f\Phi\,dx\,\zeta,$$

$$u_0 \in H^1(\Omega),\ g+\int_\Omega f\Phi\,dx\,\varsigma \geq u_0 \geq h-\int_\Omega f\Phi\,dx\,\zeta.$$

There is also proved a lemma similar to the above one, which gives us an opportunity to prove the following

Theorem 2. *The problems (11)-(13) and (14)-(16) have unique solutions which are stable with respect to the data and to the coefficients of the form (1).*

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REFERENCES

1. *A. Bensoussan, J. L. Lions.* Controle impulsionnel et inequations quasivariationnelles. Dunod, Paris, 1982.
2. *A. Gachechiladze.* Bull. Georg. Acad. Sci., **164**, 3, 2001, 436-439.
3. *A. Fridman.* Variation principles and problems with free boundaries. M., 1990 (Russian).

მათემატიკა

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