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ON THE BOUNDEDNESS OF THE ERGODIC HILBERT TRANSFORM IN LORENTZ SPACES

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Let $(X, \mathbb{S}, \mu)$ be a σ-finite measure space and $(T_\tau)_{\tau\in\mathbb{R}}$ be an ergodic group of measure-preserving transformations on $(X, \mathbb{S}, \mu)$. If $\mu(X) < \infty$, we will assume that $\mu(X) = 2\pi$ which makes corresponding constants simpler below.

For an integrable function f, $f \in L^1(X)$, its ergodic Hilbert transform is defined by

$$\mathbb{H}f(x) = \lim_{\delta\to 0+} \frac{1}{\pi} \int\limits_{\{\delta\le|\tau|\le 1/\delta\}} \frac{f(T_{-\tau}x)}{\tau}\, d\tau. \tag{1}$$

The limit in (1) exists and consequently $\mathbb{H}f(x)$ is well defined for a.a. $x \in X$ (see, e.g., [4], [5]).

It was proved in [2], [3] that for any measurable set $E \subset X$

$$\mu\{x \in X : \mathbb{H}(\mathbf{1}_E)(x) > \lambda\} = \mu\{x \in X : \mathbb{H}(\mathbf{1}_E)(x) < -\lambda\} = \tag{2}$$
$$= \begin{cases} \frac{\mu(E)}{\sinh(\pi\lambda)} & \text{if } \mu(X) = \infty, \\ 2\arctan\frac{\sin(\mu(E)/2)}{\sinh(\pi\lambda)} & \text{if } \mu(X) = 2\pi \end{cases}.$$

This is a generalization of the well known Stein-Weiss theorem for classical Hilbert transform and the conjugate operator (see [6]).

Let S be the Calderón operator

$$S\psi(t) = \frac{1}{t}\int_0^t \psi(s)\, ds + \int_t^\infty \psi(s)\frac{ds}{s}, \qquad \psi \in L^1(0,\infty), \tag{3}$$

and, for any measurable f on X, let f^* be its decreasing rearrangement

$$f^*(t) = \inf\big\{\lambda : \mu(|f| > \lambda) \le t\big\}.$$

As in the classical case, equality (2) allows us to estimate the decreasing rearrangement of $\mathbb{H}f$ by the Calderón operator.

Theorem 1. (cf. [1], Theorem 3.4.7.) *Let $(T_\tau)_{\tau\in\mathbb{R}}$ be an ergodic group of measure-preserving transformations on $(X, \mathbb{S}, \mu)$ and let $f \in L(X)$. Then*

$$(\mathbb{H}f)^*(t) \le cS(f^*)(t), \qquad 0 < t < \mu(X), \tag{4}$$

where c is a constant independent of f and t.

The Calderón operator is the sum of the Hardy operator and its dual,

$$S\psi = P\psi + Q\psi, \tag{5}$$

where

$$P\psi(t) = \frac{1}{t}\int_0^t \psi(s)\, ds, \qquad Q\psi(t) = \int_t^\infty \psi(s)\frac{ds}{s}$$

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(see (3)). With this notation, the Hardy inequalities (see [1], Lemma 3.3.9) are expressed as (for $-\infty < \lambda < 1$, $1 \le q < \infty$ and $\psi \ge 0$)

$$\left[\int_0^\infty \left(t^\lambda P\psi(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \le \frac{1}{1-\lambda}\left[\int_0^\infty \left(t^\lambda \psi(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, \tag{6}$$

$$\sup_{0<t<\infty} t^\lambda P\psi(t) \le \frac{1}{1-\lambda} \sup_{0<t<\infty} t^\lambda \psi(t),$$

$$\left[\int_0^\infty \left(t^{1-\lambda} Q\psi(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \le \frac{1}{1-\lambda}\left[\int_0^\infty \left(t^{1-\lambda} \psi(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, \tag{7}$$

$$\sup_{0<t<\infty} t^{1-\lambda} Q\psi(t) \le \frac{1}{1-\lambda} \sup_{0<t<\infty} t^{1-\lambda} \psi(t),$$

The classical Lorentz spaces $L^{p,q}(X)$, $0 < p, q \le \infty$, are defined as a set of measurable functions on X for which the quantity

$$\|f\|_{p,q} = \begin{cases} \left[\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, & (0 < q < \infty), \\ \sup_{0<t<\infty} t^{1/p} f^*(t), & (q = \infty), \end{cases} \tag{8}$$

is finite. $L^{p,p}$ coincides with usual Lebesgue space L^P, by definition (8). $\| \cdot \|_{p,q}$ is not the norm always, but it is equivalent to some norm when $1 < p \le \infty$ and $0 < q \le \infty$ (see [1], Lemma 4.4.5).

If we now take $\lambda = \frac{1}{p}$ in (6) and $\lambda = 1 - \frac{1}{p}$ in (7), then it follows from (5), (6) and (7) that

$$\|S(f^*)\|_{p,q} \le C_p \|f\|_{p,q}, \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{9}$$

Thus, inequalities (4) and (9), imply the boundedness of the ergodic Hilbert transform in the Lorentz spaces.

Theorem 2. *Let $(T_\tau)_{\tau\in\mathbb{R}}$ be an ergodic group of measure-preserving transformations on $(X, \mathbb{S}, \mu)$ and let $f \in L(X)$. If $1 < p < \infty$, and $1 \le q \le \infty$, then*

$$\|S(f^*)\|_{p,q} \le C_p \|f\|_{p,q}.$$

We emphasize that obtained proof is without any application of interpolation theory.

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