Proc. A. Razmadze Math. Inst. 145(2007), 93—94

L. Ephremidze and N. Fujii

THE JOHN-NIRENBERG INEQUALITY FOR ERGODIC SYSTEMS

(Reported on 05.12.2007)

The John-Nirenberg's classical theorem [4] asserts that for any locally integrable function $F \in L_{\mathrm{loc}}\mathbb{R}^d$, every cube $Q \subset \mathbb{R}^d$ and $\lambda > 0$, the following inequality holds

$$m\{x \in Q : |F(x) - F_Q| > \lambda\} \leq C_1 m(Q) \exp\left(\frac{-\lambda C_2}{\|F\|_{\mathrm{BMO}}}\right),$$

where m is the Lebesgue measure on $\mathbb{R}^d$, $F_Q = (1/m(Q)) \int\limits_Q F\, dm$, and $\|F\|_{\mathrm{BMO}} = \sup\limits_Q \frac{1}{m(Q)} \int\limits_Q |F - F_Q|\, dm$. The constants C_1 and C_2 are independent of F and Q.

Garsia [3] formulated and proved the John-Nirenberg inequality for martingales and L. D. Pitt [6] generalized this inequality for submartingales.

We generalize the theorem to the ergodic systems.

Let $(X, \mathbb{S}, \mu)$ be a finite measure space, $\mu(X) < \infty$, and $T : X \to X$ be a measure-preserving ergodic invertible transformation (see, e.g., [5] for definitions). For an integrable function $f : X \to \mathbb{R}$, $f \in L(X)$, the ergodic sharp maximal function is defined as

$$f^\sharp(x) = \sup_{m,n \geq 0} \frac{1}{m+n+1} \sum_{k=-m}^{n} \left|f(T^k x) - E_{m,n}(f, x)\right|,$$

where $E_{m,n}(f,x) = \frac{1}{m+n+1} \sum\limits_{k=-m}^{n} f(T^k x)$, and the ergodic BMO norm of f is defined as (see [1])

$$\|f\|_{\mathrm{BMO}} = \operatorname{ess\,sup} f^\sharp.$$

Theorem. *There exist universal constants C_1 and C_2 such that for any finite measure space $(X, \mathbb{S}, \mu)$, measure-preserving ergodic invertible transformation T and $f \in L(X)$, we have*

$$\mu\{x \in X : |f(x) - E(f)| > \lambda\} \leq C_1 \mu(X) \exp\left(\frac{-\lambda C_2}{\|f\|_{\mathrm{BMO}}}\right),$$

where $E(f) = (1/\mu(X)) \int_X f\, d\mu$ and $\lambda \geq 0$.

The proof depends on the discrete version of the John-Nirenberg theorem and on a new method of transferring results on the real line to the general ergodic setting developed in [2].

2000 *Mathematics Subject Classification:* 37A05.

Key words and phrases. Measure-preserving ergodic transformations, the John-Nirenberg Inequality.

Acknowledgement

The authors had a grant of the Japan Society of Promotion of Science during the above research. Besides the first author had also the INTAS grant Nr 06-1000017-8792.

References

1. R. R. Coifman and G. Weiss, Maximal functions and H^P spaces defined by ergodic transformations. *Proc. Nat. Acad. Sci. U.S.A.* **70**(1973), 1761–1763.
2. L. Ephremidze, A new proof of the ergodic maximal equality. *Real Anal. Exchange* **29**(2003/04), No. 1, 409–411.
3. A. M. Garsia, Martingale inequalities: Seminar notes on recent progress. *Mathematics Lecture Notes Series. W. A. Benjamin, Inc., Reading, Mass.-London-Amsterdam*, 1973.
4. F. John and L. Nirenberg, On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* **14**(1961), 415–426.
5. K. E. Petersen, Ergodic theory. Cambridge Studies in Advanced Mathematics, 2. *Cambridge University Press, Cambridge*, 1983.
6. L. D. Pitt, A one sided John-Nirenberg inequality for submartingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34**(1976), No. 1, 29–32.

Authors' addresses:

L. Ephremidze
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: lasha@rmi.acnet.ge

Current address:
Department of Mathematics
Tokai University
3-20-1, Orido Shimizu-ku, Shizuoka-shi, 424-8610
Japan
E-mail: 05k054@scc.u-tokai.ac.jp

N. Fujii
Department of Mathematics, Tokai University
3-20-1, Orido Shimizu-ku, Shizuoka-shi, 424-8610
Japan
E-mail: nfujii@scc.u-tokai.ac.jp